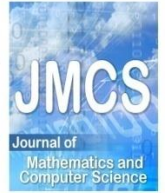Contents list available at JMCS

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Integrated High Accuracy Multiquadric Quasi-interpolation Scheme for Solving the Nonlinear Klein-Gordon Equation

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Abstract

A collocation scheme based on the use of the multiquadric quasi-interpolation operator L_{W_2}, integrated radial basis function networks (IRBFNs) method and three order finite difference method is applied to the nonlinear Klein-Gordon equation. In the present scheme, the three order finite difference method is used to discretize the temporal derivative and the integrated form of the multiquadric quasi-interpolation scheme is used to approximate the unknown function and its spatial derivatives. Several numerical experiments are provided to show the efficiency and the accuracy of the given method.

1. Introduction

The Klein-Gordon equation is a relativistic version of the Schrodinger equation, which describes scalar spineless particles. It appears in various physical applications such as the propagation of fluxions in the Josephson junctions, the motion of rigid pendula attached a stretched wire, and dislocations in crystals [21].

In this paper, we consider the one-dimensional nonlinear Klein-Gordon equation of the form:

$$u_{tt} + \mu u_{xx} + F(u) = f(x,t), \quad x \in \Omega = [a,b] \subset \mathbb{R}, 0 < t \leqslant T, \tag{1}$$

with the initial conditions

$$u(x,0) = g_1(x), \quad x \in \Omega,$$
$$u_t(x,0) = g_2(x), \quad x \in \Omega, \tag{2}$$

and the boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \tag{3}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t, μ is a known constant, $F(u)$ is the nonlinear force such that $\dfrac{\partial F}{\partial u} \geqslant 0$ and $g_1(x)$, $g_2(x)$ and $g(x,t)$ are known functions.

The general form of (1) covers many different generalized Klein-Gordon equations arising in various physical applications. For example, when $F(u) = \sin(u)$, $F(u) = \sin(u) + \sin(2u)$ and $F(u) = \sinh(u) + \sinh(2u)$, (1) becomes the well-known sine-Gordon equation, the double sine-Gordon equation and the double sinh-Gordon equation, respectively. The above nonlinear Klein-Gordon equations are Hamiltonian partial differential equations (PDEs), and for a wide class of force $F(u)$, it has the conserved Hamiltonian quantity (or energy) [18]

$$H = \int \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + G(u)\right)dx,$$

where $G'(u) = F(u)$.

Up to now, many authors have studied the numerical and approximation solutions of the nonlinear Klein-Gordon equation by using various techniques that some of them are the finite difference method, the spectral method, the adomian decomposition method, the inverse scattering method, Bäcklund transformation, the auxiliary equation method, the Wadatitrace method, Hirota bilinear forms, pseudo-spectral method, the tanh-sech method, the sine-cosine method, Jacobi elliptic functions and the Riccati equation expansion method and so on (see [10, 15, 25, 26, 30, 31] and references therein).

Finite difference methods are known as the first technique for solving PDEs. Even though these methods have been widely used but they require the construction and update of a mesh and hence bring inconvenience during computation.

In 1990, Kansa [13, 14] modified Hardy's multiquadric (MQ) RBF method to solve PDEs. Since then, using of the RBFs collocation method for solving of PDEs have attracted the attention of researchers because no tedious mesh generation is required. Recently, the Kansa's method has been developed to simulate nonlinear PDEs [2] such as Burger's equation [11], sine-Gordon equation [6], Klein-Gordon equation [7], and Korteweg-de Vries (KdV) equation [5] etc. In all these works, the solution function is decomposed into RBFs and its derivatives are then obtained through differentiation that caused the reduction in convergence rate. In order to avoid from this problem, Mai-Duy and Tran-Cong introduced an integrated MQ-RBFNs scheme for the approximation of function and its derivatives [20]. Numerical

experiments and theoretical analysis indicate that for solving PDEs integrated RBF (IRBF) procedure is more accurate in comparison with direct RBF (DRBF) procedure. Also, IRBF scheme is more stable than DRBF for a range of PDEs [20, 27].

In both DRBF and IRBF schemes, one must resolve a linear system of equations at each time step. Hon and Wu [12], Wu [8, 9] and others have provided some successful examples using MQ quasi-interpolation scheme for solving differential equations. Beatson and Powell [1] and Wu and Schaback [29] proposed other univariate MQ quasi-interpolations. In [3, 4], Chen and Wu used MQ quasi-interpolation to solve Burgers' equation and hyperbolic conservation laws. Also, Xiao et al. [25] presented the numerical method based on Chen and Wu's method for solving the third-order KdV equation. Recently, Jiang et al. [16] have introduced a new multi-level univariate MQ quasi-interpolation approach with high approximation order compared with initial MQ quasi-interpolation scheme. This approach is based on inverse multiquadric (IMQ) RBF interpolation, and Wu and Schaback's MQ quasi-interpolation operator L_D that have the advantages of high approximation order.

The Jiang et al. MQ quasi-interpolation one-dimensional operator L_{W_2} is summation of two series that the second series coefficients are combined of first series coefficients. By giving relation between two series coefficients based on function values, we can convert it to a compact form that is based on one series and can use it in integrated form for the numerical solution of PDEs.

This paper presents a novel numerical scheme to solve the nonlinear Klein-Gordon equation that is based on integrated MQ quasi-interpolation scheme. The rest of present paper is arranged as follows. Brief information of the MQ quasi-interpolation scheme is given in Section 2. Section 3 describes our method on the nonlinear Klein-Gordon equation. Several numerical experiments are presented in Section 4, followed by a conclusion summary in Section 5.

2. The MQ quasi-interpolation scheme

In this section, some elementary knowledge about of three univariate MQ quasi-interpolation schemes, namely, L_D, L_W and L_{W_2} is presented. For more information about MQ quasi-interpolation operators see [1,3,4, 29].

For a given interval $\Omega = [a,b]$ and a finite set of distinct points

$$a = x_0 < x_1 < \ldots < x_N = b, \quad h = \max_{1 \leqslant i \leqslant N} (x_i - x_{i-1}),$$

quasi-interpolation of a univariate function $f : [a,b] \to \mathbb{R}$ is given by

$$L(f) = \sum_{i=0}^{N} f(x_i)\phi_i(x),$$

where function $\phi_i(x)$ is a linear combination of the MQs

$$\psi_i(x) = \sqrt{c^2 + (x - x_i)^2},$$

and $c \in \mathbb{R}^+$ is a shape parameter. In [29], Wu and Scheback presented the univariate MQ quasi-interpolation operator L_D that is defined as

$$L_D f(x) = \sum_{i=0}^{N} f(x_i) \psi_i(x),$$

(4)

where

$$\psi_0(x) = \frac{1}{2} + \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\psi_1(x) = \frac{\psi_2(x) - \psi_1(x)}{2(x_2 - x_1)} - \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

(5)

$$\psi_i(x) = \frac{\psi_{i+1}(x) - \psi_i(x)}{2(x_{i+1} - x_i)} - \frac{\psi_i(x) - \psi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 2 \leqslant i \leqslant N - 2,$$

$$\psi_{N-1}(x) = \frac{(x_N - x) - \psi_{N-1}(x)}{2(x_N - x_{N-1})} - \frac{\psi_{N-1}(x) - \psi_{N-2}(x)}{2(x_{N-1} - x_{N-2})},$$

and

$$\psi_N(x) = \frac{1}{2} + \frac{\psi_{N-1}(x) - (x_N - x)}{2(x_N - x_{N-1})}.$$

In RBFs interpolation, high approximation order can be gotten by increasing the number of interpolation centers but we have to solve unstable linear system of equations. By using MQ quasi-interpolation scheme, we can avoid this problem, whereas the approximation order is not good. Therefore, Jiang et al. [16] defined two MQ quasi-interpolation operators denoted as L_W and L_{W_2} which pose the advantages of RBFs interpolation and MQ quasi-interpolation scheme. The process of MQ quasi-interpolation of L_W and L_{W_2} are as follows that is described in [16].

Suppose that $\{x_{k_j}\}_{j=1}^{\bar{N}}$ is a smaller set from the given points $\{x_i\}_{i=0}^{N}$ where $\bar{N}$ is a positive integer satisfying $\bar{N} < N$ and $0 = k_0 < k_1 < \ldots < k_{\bar{N}+1} = N$. Using the IMQ-RBF, the second derivative of $f(x)$ can be approximated by RBF interpolant $S_{f''}$ as

$$S_{f''}(x) = \sum_{j=1}^{\bar{N}} \alpha_j \bar{\varphi}_j(x),$$

(6)

where

$$\bar{\varphi}_j(x) = \frac{s^2}{(s^2 + (x - x_{k_j})^2)^{3/2}},$$

and $s \in \mathbb{R}^+$ is a shape parameter. The coefficients $\{\alpha_j\}_{j=1}^{\bar{N}}$ are uniquely determined by the interpolation condition

$$S_{f''}(x_{k_i}) = \sum_{j=1}^{\bar{N}} \alpha_j \bar{\varphi}_j(x_{k_i}) = f''(x_{k_i}), \quad 1 \leqslant i \leqslant \bar{N}.$$

(7)

Since, the Eq. (7) is solvable [19], so

$$\alpha = A_X^{-1} f_X'',$$

(8)

where

$$X = \{x_{k_1}, \ldots, x_{k_{\bar{N}}}\}, \qquad \alpha = [\alpha_1, \ldots, \alpha_{\bar{N}}]^T, \qquad A_X = [\bar{\varphi}_j(x_{k_i})], \qquad f_X'' = [f''(x_{k_1}), \ldots, f''(x_{k_{\bar{N}}})]^T.$$

By using f and the coefficient α defined in Eq. (8), a function $e(x)$ is constructed in the form

$$e(x) = f(x) - \sum_{i=1}^{\bar{N}} \alpha_i \sqrt{s^2 + (x - x_{k_i})^2}.$$

(9)

Then the MQ quasi-interpolation operator L_W by using L_D defined by Eqs. (4) and (5) on the data $(x_i, e(x_i))_{0 \leqslant i \leqslant N}$ with the shape parameter c is defined as follows:

$$L_W f(x) = \sum_{i=1}^{\bar{N}} \alpha_i \sqrt{s^2 + (x - x_{k_i})^2} + L_D e(x).$$

(10)

The shape parameters c and s should not be the same constants in the Eq. (10).

In Eq. (7), $f''_{x_{k_j}}$ can be replaced by

$$f''_{x_{k_j}} = \frac{2[(x_{k_j} - x_{k_{j-1}})f(x_{k_{j+1}}) - (x_{k_{j+1}} - x_{k_{j-1}})f(x_{k_j}) + (x_{k_{j+1}} - x_{k_j})f(x_{k_{j-1}})]}{(x_{k_j} - x_{k_{j-1}})(x_{k_{j+1}} - x_{k_j})(x_{k_{j+1}} - x_{k_{j-1}})},$$

when the data $(x_{k_i}, f(x_{k_i}))_{0 \leqslant i \leqslant \bar{N}}$ are given. So, if f_X'' in Eq. (8) replace by

$$F_X'' = (f''_{x_{k_1}}, \ldots, f''_{x_{k_{\bar{N}}}})^T,$$

(11)

the quasi-interpolation operator defined by Eqs. (9) and (10) is denoted by L_{W_2}. For more details about

the properties and accuracy of L_W and L_{W_2}, one can see [26]. In this paper, we use the MQ quasi-interpolation operator L_{W_2} with equally spaced points and $N = 2\overline{N}$.

The operator L_{W_2} can be written in the compact form

$$L_{W_2} f(x) = \sum_{i=0}^{N} f(x_i) \psi_i(x), \tag{12}$$

where the basis functions $\psi_i(x)$ are obtained by substituting Equations (8), (9) and (11) into (10), see [23].

By writing operator L_{W_2} in the compact form (12), we can use it in the indirect form for the numerical solution of PDEs. Also, by converting operator L_{W_2} to form (12), we do not require to solve a linear system of equations for getting of the coefficients α_i at each time step, see [17].

3. The numerical method

In this section, the numerical scheme is presented for solving the Klein-Gordon equation (1) by using the MQ quasi-interpolation L_{W_2}. In our approach, the fourth order finite difference approximation is first employed for discretizing of the temporal derivative similar to work that Rashidinia did in [22]. Then, the highest order derivatives (second order in this paper) of the solution function are approximated by Eq. (12), and their lower order derivatives and the solution function are then obtained by symbolic integration. At the end, the collocation scheme is applied.

3.1. The discretization of time

According of the fourth order finite difference, the term $u_{tt}^n = u_{tt}(x, t_n), t_n = n\Delta t$ can be arranged as

$$u_{tt}^n \cong \frac{\delta_t^2}{(\Delta t)^2 (1 + \delta_t^2)} u^n + O((\Delta t)^4), \tag{13}$$

where $\delta_t^2 = u^{n+1} - 2u^n + u^{n-1}$.

Substituting Eq. (13) into Eq. (1) yields the following time discretized form of Klein-Gordon equation:

$$\delta_t^2 u^n + \mu(\Delta t)^2 (1 + \gamma\delta_t^2) u_{xx}^n + (\Delta t)^2 (1 + \gamma\delta_t^2) F^n = (\Delta t)^2 (1 + \gamma\delta_t^2) f^n, \tag{14}$$

where $f^n = f(x, t_n)$ and $F^n = F(u^n)$.

In this paper, we consider the nonlinear force $F(u) = \alpha u + \beta u^k$ wherein $k = 2$ or $k = 3$ and α and β are known constants. After some arrangements, Eq. (14) can be written in the following form:

$$(\frac{1}{\gamma(\Delta t)^2} + \alpha) u^{n+1} + \mu u_{xx}^{n+1} + \beta (u^{n+1})^k = \chi(x),$$ (15)

where

$$\chi(x) = (\frac{2}{\gamma(\Delta t)^2} - \alpha\kappa)u^n - (\frac{1}{\gamma(\Delta t)^2} + \alpha)u^{n-1} - \mu[u_{xx}^n + u_{xx}^{n-1}] - \beta[\kappa(u^n)^k + (u^{n-1})^k],$$

and $\kappa = \dfrac{1 - 2\gamma}{\gamma}$.

3.2. The indirect MQ quasi-interpolation scheme

In this scheme, the highest order derivatives (second order in this paper) of the solution function, u_{xx}^n ,   is approximated by MQ quasi-interpolation L_{W_2} on data $\{x_j\}_{j=1}^{N-1}$ as follows:

$$u_{xx}^n(x) = \sum_{j=1}^{N-1} u_{xx}^n(x_j)\hat{\psi}_j(x).$$ (16)

Now, symbolic integrating of Eq. (16) yields

$$u_x^n(x) = \sum_{j=1}^{N-1} u_{xx}^n(x_j)\int \hat{\psi}_j(x)dx + C_1,$$ (17)

$$u^n(x) = \sum_{j=1}^{N-1} u_{xx}^n(x_j)\int\int \hat{\psi}_j(x)dxdx + C_1 x + C_2.$$ (18)

The Eqs. (16)- (18) can be rewritten in the compact form as follows:

$$u^n(x) = \sum_{j=0}^{N} w_j^n \varphi_j(x), \quad u_x^n(x) = \sum_{j=0}^{N} w_j^n \dot{\varphi}_j(x), \quad u_{xx}^n(x) = \sum_{j=0}^{N} w_j^n \ddot{\varphi}_j(x),$$ (19)

where

$$\varphi_j(x) = \int\int \hat{\psi}_j(x)dxdx, \quad 1 \leqslant j \leqslant N-1, \quad \varphi_0(x) = x, \qquad \varphi_N(x) = 1,$$

$$\dot{\varphi}_j(x) = \int \hat{\psi}_j(x)dx, \qquad 1 \leqslant j \leqslant N-1, \quad \dot{\varphi}_0(x) = 1, \qquad \dot{\varphi}_N(x) = 0,$$

$$\ddot{\varphi}_j(x) = \hat{\psi}_j(x), \qquad 1 \leqslant j \leqslant N-1, \quad \ddot{\varphi}_0(x) = 0, \qquad \ddot{\varphi}_N(x) = 0,$$

and

$$w_j^n = u_{xx}^n(x_j), \qquad\qquad 1 \leqslant j \leqslant N-1, \ w_0^n = C_1, \ w_N^n = C_2.$$

Now, substituting Eq. (19) into Eqs. (15) and (3) and applying collocation method yield the following equations:

$$\left(\frac{1}{\gamma(\Delta t)^2} + \alpha\right)\sum_{j=0}^{N} w_j^{n+1}\varphi_{ij} + \mu \sum_{j=0}^{N} w_j^{n+1}\ddot{\varphi}_{ij} + \beta \left(\sum_{j=0}^{N} \tilde{w}_j^{n+1}\varphi_{ij}\right)^k = \chi(x_i), \qquad i = 1,\ldots,N-1, \tag{20}$$

and

$$\sum_{j=0}^{N} w_j^{n+1}\varphi_{ij} = g^{n+1}(x_i), \quad i = 0,N, \tag{21}$$

where $\varphi_{ij} = \varphi_j(x_i)$, $\ddot{\varphi}_{ij} = \ddot{\varphi}_j(x_i)$, $g^n(x_i) = g(x_i, t_n)$ and

$$\chi(x_i) = \left(\frac{2}{\gamma(\Delta t)^2} - \alpha\kappa\right)u^n(x_i) - \left(\frac{1}{\gamma(\Delta t)^2} + \alpha\right)u^{n-1}(x_i) - \mu[u_{xx}^n(x_i) + u_{xx}^{n-1}(x_i)] - \beta[\kappa(u^n(x_i))^k + (u^{n-1}(x_i))^k].$$

At $n = 1$, according to the initial conditions that was introduced in (2) and approach that Rashidinia did in [30] on based Taylor series, we apply the following assumptions

$$u^0(x) = u(x,0) = g_1(x),$$

and

$$u^1(x) = g_1(x) + \Delta t g_2(x) - \frac{(\Delta t)^2}{2!}[\mu u_{xx} + F - f]^0 - \frac{(\Delta t)^3}{3!}[\mu g_2''(x) + F_t - f_t + F_u u_t]^0 + \mu \frac{(\Delta t)^4}{4!}[\mu u_{xxxx}$$
$$+ F_x u_x + F_{xx} - f_{xx} + F_{xu} u_x + u_x(F_{xu} + F_{uu} u_x)]^0 + O((\Delta t)^5),$$

In each time step (for example time step $n + 1$), at first we set $\tilde{w}_j^{n+1} = w_j^n$. Having this, Eqs. (20) and (21) are solved as a system of linear algebraic equations for unknowns $w_j^{n+1}; j = 0,1,\ldots,N$. Then, we recomputed $\tilde{w}_j^{n+1} = w_j^{n+1}$ where w_j^{n+1} as we illustrate, can be obtained by solving Eqs. (20) and (21). Now, at each time level, we iterate calculating $\tilde{w}_j^{n+1}$ and solving the approximation values of the unknown, until the tolerance of any two latest iterations is not bigger than 10^{-8}, i.e. a predictor-corrector scheme is adopted in each time level, then we can move on to the next time level.

4. The numerical experiments

Four experiments are studied to investigate the robustness and the accuracy of the proposed method. The numerical results of the Klein-Gordon equation by using this scheme is compared with the analytical solutions and solutions in [7, 22]. These methods include Thin Plate Splines (TPS) RBF collocation method [7] and cubic B-spline collocation method (CBS) [22]. Our scheme is denoted by IMQQI. The L_2, L_∞ and RMS error norms which are defined by

$$L_2 = \| u^{*n} - u^n \|_2 = \sqrt{h \sum_{j=0}^{N} (u^{*n}(x_j) - u^n(x_j))^2},$$

$$L_\infty = \| u^{*n} - u^n \|_\infty = \max_{0 \leqslant j \leqslant N} | u^{*n}(x_j) - u^n(x_j)|,$$

$$RMS = \sqrt{(\sum_{j=0}^{N} (u^{*n}(x_j) - u^n(x_j))^2)/(N+1)},$$

are used to measure the accuracy of our scheme where u^* is the approximation solution. In all experiments, the shape parameter s is considered twice of the shape parameter c.

The computations associated with our experiments are performed in Maple 14 on a PC with a CPU of 2.4 GHZ.

Experiment 1. In this experiment, the Klein-Gordon equation (1) is considered with $\mu = 1$, $f(x,t) = -x \cos(t) + x^2 \cos(t)$ in interval $-1 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u^2$ so the values of constants in (15) are $\alpha = 0$, $\beta = 1$ and $k = 2$. The initial conditions are given by

$$u(x,0) = x, \quad -1 \leqslant x \leqslant 1,$$

$$u_t(x,0) = 0, \quad -1 \leqslant x \leqslant 1.$$

The exact solution is given in [28] as

$$u(x,t) = x \cos(t).$$

The boundary function $g(x,t)$ can be extracted from the exact solution. The L_2, L_∞ and RMS errors in the solutions with $\Delta t = 0.0001$, $N = 10$, $c = 1.63 \times 10^{-2}$ and $\gamma = \dfrac{1}{12}$ that calculated in 100 points are listed in Table 1 and compared with the results in [7, 22].

The space-time graphs of the estimated solution are drawn in Fig. 1. Table 1 indicates that the proposed method requires more less nodes to attain the accuracy of the CBSM [22] and TPSM [7]. Also, it shows that this scheme performs better than TPSM.
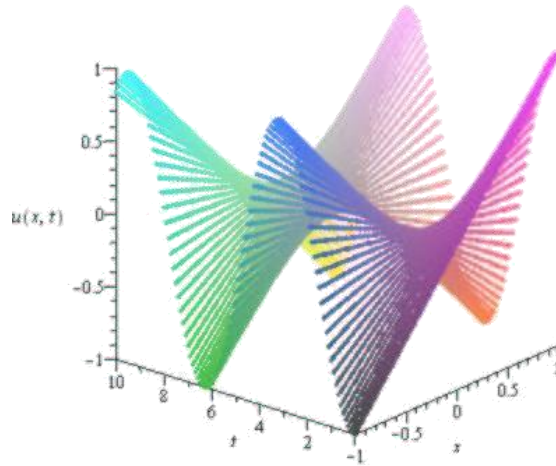
Figure 1: The graph of the estimated solution up to $t = 10$ with $\Delta t = 0.0001$ and $N = 10$ of experiment 1.

Table 1: The comparison of the L_1; L_2 and RMS errors of our method with the results of [7, 22] at different times of experiment 1.

Time	1	3	5	7	10
IMQQI; N = 10					
L_1	9.6719E-11	1.4351E-10	1.7096E-10	1.4199E-10	1.0761E-10
L_2	6.6578E-11	1.1603E-10	1.5434E-10	1.4325E-10	7.6498E-11
RMS	4.6771E-11	8.1926E-11	1.0882E-10	1.0164E-10	5.3802E-11
Time	1	3	5	10	20
CBSM [22]; N = 100					
L_1	4.7698E-13	2.8899E-13	4.3667E-13	8.4593E-13	3.1344E-12
L_2	1.8986E-12	1.0680E-12	1.5686E-12	5.3244E-12	4.4324E-11
RMS	2.6850E-14	1.5105E-13	2.2184E-12	5.9738E-12	2.2332E-12
Time	1	3	5	7	10
TPSM [7]; N = 100					
L1	1.2540E-05	1.5554E-05	3.3792E-05	3.7753E-05	1.3086E-05
L_2	6.5422E-05	1.1717E-04	2.2011E-04	2.5892E-04	7.9854E-05
RMS	6.5097E-06	1.1659E-05	2.1902E-05	2.5763E-05	7.9458E-06

Experiment 2. Consider the Klein-Gordon equation (1) with $\mu = 1$, $f(x,t) = 6xt(x^2 - t^2) + x^6 t^6$ in interval $0 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u^2$ wherein α, β and k are considered 0, 1 and 2, respectively. The initial conditions are given by

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant 1,$$

$$u_t(x,0) = 0, \quad 0 \leqslant x \leqslant 1.$$

The exact solution is given in [28] as
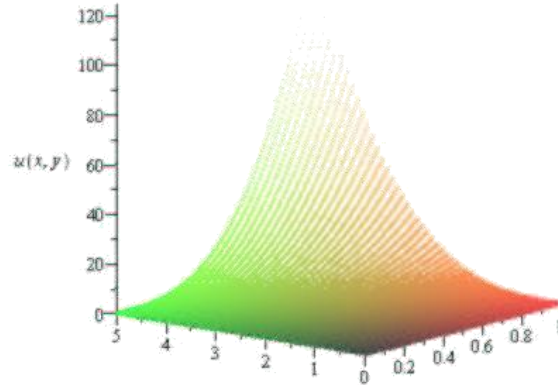
Figure 2: The graph of the estimated solution up to $t = 5$ with $\Delta t = 0.001$ and $N = 50$ of experiment 2.
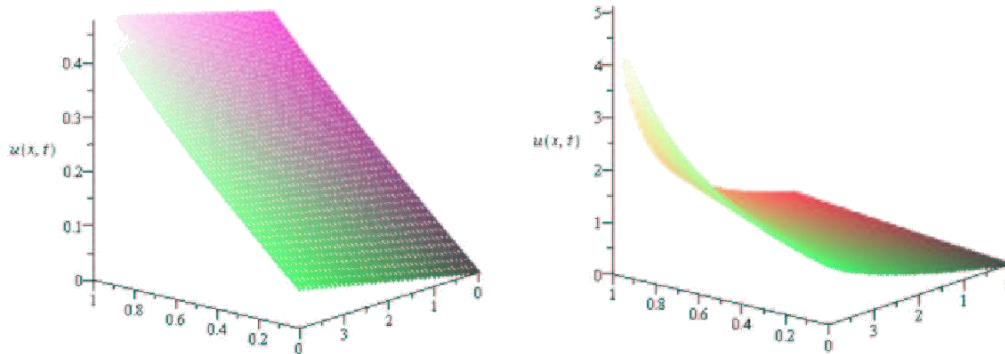
Table 2: The comparison of the L_1; L_2 and RMS errors of our method with the results of [7, 22] at different times of experiment 2.

Time	1	2	3	4	5
IMQQI; N = 20					
L_1	6.1753E-11	2.3374E-10	7.2934E-10	6.3525E-10	1.0866E-09
L_2	7.7595E-11	6.0731E-10	1.7852E-09	1.9430E-09	2.1023E-09
RMS	1.6933E-11	1.3253E-10	3.8956E-10	4.2399E-10	4.5876E-10
IMQQI; N = 5					
L_1	1.9531E-11	2.0935E-10	6.9190E-10	5.5946E-10	1.0916E-09
L_2	5.0991E-11	1.3371E-09	3.9844E-09	4.3492E-09	4.7010E-09
RMS	5.0738E-12	1.3305E-10	3.9646E-10	4.3276E-10	4.6777E-10
CBSM [22]; N = 50					
L_1	5.5733E-14	3.0198E-13	3.5829E-12	5.1088E-12	7.2456E-11
L_2	1.4257E-13	8.7463E-13	1.0177E-11	1.7568E-11	3.0183E-10
RMS	2.0162E-14	1.2369E-13	1.4392E-12	2.4846E-12	4.2685E-12
TPSM [7]; N = 50					
L_1	1.1012E-05	1.6496E-04	5.9728E-04	1.8264E-03	3.6915E-03
L_2	5.4998E-05	1.1522E-03	3.2588E-03	9.8191E-03	1.9139E-02
RMS	5.4725E-06	1.1465E-04	3.2426E-04	9.7704E-04	1.9044E-03

$u(x,t) = x^3 t^3.$

The boundary function $g(x,t)$ can be extracted from the exact solution. Table 2 shows the L_2, L_∞ and RMS errors in the solutions with $\Delta t = 0.001$, $N = 50$, $c = 1.63 \times 10^{-2}$ and $\gamma = \dfrac{1}{12}$. Our numerical results are compared with the results in [7, 22]. Moreover, the space-time graph of the estimated solution is drawn in Fig. 2.

Table 2 indicates that the proposed method requires more less nodes to attain the accuracy of the TPSM [7] and has better accuracy than TPSM [7].

Figure 3: The graph of the estimated solution up to $t = 4$ with $\Delta t = 0.001$ and $N = 30$ of experiment 3, (Left) $C = 0.05$, (Right) $C = 0.5$.

Table 3: The comparison of the L_∞, L_2 and RMS errors of our method with the results of [7, 22] with $C = 0.05$ at different times of experiment 3.
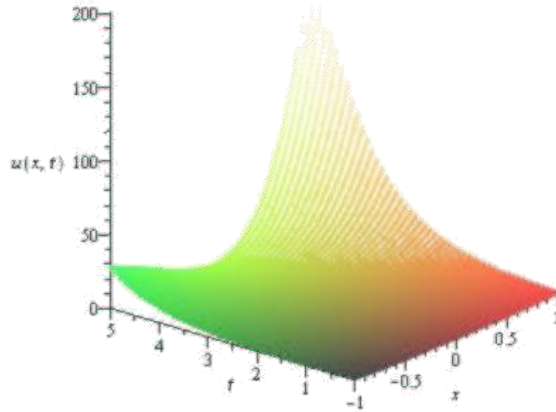
Time	1	2	3	4
IMQQI; N = 50				
L_1	3.2112E-09	3.5349E-09	4.0395E-09	4.7216E-09
L_2	1.1963E-08	1.6133E-08	1.8570E-08	1.7156E-08
RMS	1.6752E-09	2.2591E-09	2.6003E-09	2.4023E-09
IMQQI; N = 100				
L_1	1.9972E-10	2.2253E-10	2.4427E-10	2.8106E-10
L_2	8.6479E-10	8.2885E-10	9.3908E-10	1.1417E-09
RMS	8.6050E-11	8.2474E-11	9.3442E-11	1.1360E-10
CBSM [22]; N = 100				
L_1	1.1986E-08	2.4733E-08	2.8958E-08	1.9916E-08
L_2	7.5619E-08	1.7997E-07	2.0797E-07	1.4058E-07
RMS	7.5619E-08	1.7997E-08	2.0797E-08	1.4058E-08
TPSM [7]; N = 100				
L_1	3.6497E-07	3.8952E-07	4.2123E-07	4.5928E-07
L_2	1.7861E-06	1.5383E-06	1.7275E-06	2.0097E-06
RMS	1.7772E-07	1.5306E-07	1.7190E-07	1.9997E-07

Experiment 3. In this experiment, the Klein-Gordon equation (1) is considered with $\mu = 2.5$, $f(x,t) = 0$ in interval $0 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u + 1.5u^3$ wherein α, β and k are 1, 1.5 and 3, respectively. The initial conditions are given by

$$u(x,0) = B \tan(Kx), \quad 0 \leqslant x \leqslant 1,$$

$$u_t(x,0) = BCK \sec^2(Kx), \quad 0 \leqslant x \leqslant 1.$$

The exact solution is given in [32] as

Figure 4: The graph of the estimated solution up to $t = 5$ with $\Delta t = 0.001$ and $N = 50$ of experiment 4.

Table 4: The comparison of the L_∞, L_2 and RMS errors of our method with the results of [7, 22] with $C = 0.5$ at different times of experiment 3.

Time	1	2	3	4
IMQQI; N = 50				
L_1	1.3646E-08	6.6291E-08	6.7475E-07	5.6663E-05
L_2	4.8756E-08	2.1550E-07	1.8868E-06	1.1334E-04
RMS	6.8272E-09	3.0176E-08	2.6420E-07	1.5871E-05
IMQQI; N = 100				
L_1	8.7470E-10	4.2448E-09	4.3933E-08	4.0592E-06
L_2	6.1403E-09	1.7262E-08	1.4987E-07	1.0671E-05
RMS	6.1099E-10	1.7177E-09	1.4913E-08	1.0618E-06
CBSM [22]; N = 100				
L_1	2.6949E-08	8.7462E-08	3.0903E-07	1.9394E-06
L_2	1.9015E-07	6.3813E-07	2.2341E-06	1.3439E-05
RMS	1.9015E-08	6.3813E-08	2.2344E-07	1.3439E-06
TPSM [7]; N = 100				
L1	5.9964E-06	2.1973E-05	9.0893E-05	8.2945E-04
L_2	4.0761E-05	1.5769E-04	6.4792E-04	5.3572E-03
RMS	4.0559E-06	1.5691E-05	6.4470E-05	5.3306E-04

$$u(x,t) = B \tan(K(x + Ct),$$

where $B = \sqrt{\dfrac{\alpha}{\beta}}$ and $K = \sqrt{\dfrac{\alpha}{-2\mu + C^2}}$. The boundary function $g(x,t)$ can be extracted from the exact solution. In Tables 3 and 4, the L_2, L_∞ and RMS errors in the solutions are listed with $\Delta t = 0.0001$, $N = 100$, $N = 50$, $\gamma = \dfrac{1}{12}$ and $C = 0.05$ and $C = 0.5$. The shape parameter c is choose 8.15×10^{-2} and

Table 5: The comparison of the L_1, L_2 and RMS errors of our method with the results of [7, 22] at different times of experiment 4.

Time	1	2	3	4	5
IMQQI; N = 100					
L_1	4.9908E-07	1.2551E-06	1.7749E-06	1.4689E-06	7.3491E-07
L2	1.3310E-06	1.9435E-06	3.3137E-06	2.1256E-06	2.2948E-06
RMS	1.3244E-07	1.9339E-07	3.2972E-07	2.1150E-07	2.9342E-07
CBSM [22]; N = 200					
L_1	3.5666E-06	3.1949E-06	3.9619E-06	5.6889E-06	6.3356E-06
L_2	2.5993E-05	2.2013E-05	2.3990E-05	2.9542E-05	3.2638E-05
RMS	2.5930E-06	2.2013E-06	2.3909E-06	2.9542E-06	2.9092E-06
TPSM [7]; N = 200					
L_1	5.0705E-05	5.0260E-04	2.0612E-03	6.5720E-03	1.9067E-02
L_2	2.9474E-04	2.7082E-03	9.7246E-03	2.7881E-02	7.7337E-02
RMS	2.0789E-05	1.9102E-04	6.8592E-04	1.9666E-03	5.4549E-03

1.63×10^{-2} for $N = 100$ and $N = 50$, respectively. Tables 3 and 4 indicate that the proposed method requires more less nodes to attain the accuracy of the CBSM [22] and TPSM [7].

Experiment 4. Consider the nonlinear Klein-Gordon equation (1) with $\mu = 1$ and the nonlinear force $F(u) = u + u^3$ in interval $-1 \leqslant x \leqslant 1$. In this case, the constants α, β and k are considered 1, 1 and 3, respectively. The initial conditions are given by

$u(x,0) = x^2 \cosh(x), \quad -1 \leqslant x \leqslant 1$

$u_t(x,0) = x^2 \sinh(x), \quad -1 \leqslant x \leqslant 1.$

The exact solution is given as

$u(x,t) = x^2 \cosh(x + t).$

The boundary function $g(x,t)$ can be extracted from the exact solution. Table 5 shows the L_2, L_∞ and RMS error norms in the solutions with $\Delta t = 0.0001$, $c = 8.15 \times 10^{-2}$, $N = 100$ and $\gamma = \dfrac{1}{12}$. We compare our results with the results in [7, 22]. Also, the space-time graph of the estimated solution is drawn in Fig. 4. Table 5 shows that our scheme has better accuracy than CBSM [22] and TPSM [7].

5. Conclusion

In this paper, a numerical scheme based on high accuracy MQ quasi-interpolation scheme and IRBFN method has been applied to solve the nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity. The numerical results which are given in the previous section demonstrate the good accuracy of the present scheme. Also, the Tables show that this scheme performs better than TPS method and requires more less nodes to attain accuracy. Moreover, we have used bigger time step Δt , in comparison with [13]. Therewith, we would like to emphasize that, the scheme introduced in this paper can be well studied for any other nonlinear PDEs.

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