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Fixed Point Theorems for Multi-valued Weakly C -contractive Mappings in Quasi-ordered Metric Spaces

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Abstract

The goal of this paper is to present some common fixed point theorems for multivalued weakly C-contractive mappings in quasi-ordered complete metric space. These results generalizes the existing fixed point results in the literature.

Keywords: Multivalued mapping, Hausdorff distance, Weakly C-contractive mapping, Common fixed point.

1. Introduction

Fixed point theory for contractive mapping first studied by Banach [1]. He proved that every contraction defined on a complete metric space has a unique fixed point. Since then the fixed point theory for single valued and multivalued mappings in metric space has been rapidly developed. In 1972, Chatterjea [2] introduce the concept of C -contraction as follows.

Definition1.1. A mapping $T:\mathrm{X}\to\mathrm{X}$ where (X, d) is a metric space is said to be a C-contraction if there exists $k\in(0,0.5)$ such that for all $x,y\in X$ the following inequality holds:

$$d(Tx,Ty)\le k((d(x,Ty)+d(y,Tx)).$$

Chatterjea [2] proved the following theorem:

Theorem1.1. Every C-contraction in a complete metric space has a unique fixed point.

Choudhury [3] introduce the concept of weakly C -contractive mapping as a generalization of C -contractive mapping and prove that every weakly C -contractive mapping in a complete metric space has a unique fixed point.

Definition1.2. Let (X, d) be a metric space. A mapping $T: X \to X$, is said to be weakly C-contractive if for all $x, y \in X$,

$$d(Tx,Ty) \leq \frac{1}{2}(d(x,Ty)+d(y,Tx)) - \varphi(d(x,Ty), d(y,Tx)),$$

Where $\varphi:[0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only $x=y=0$.

Harjani et al. [5] announced some fixed point results for weakly C -contractive mappings in a complete metric space endowed with a partial order. Meanwhile, Shatanawi [9] proved some fixed point theorems for a nonlinear weakly C -contraction type mapping in metric and ordered metric spaces. In this paper, we introduce the concept of multivalued weakly C -contractive mappings in quasi-ordered partial metric spaces and we prove some existence theorems of common fixed point for such mappings in the context of complete quasi-partial metric spaces under certain conditions.

2. Preliminaries

Let $(X, d, \leq)$ be a quasi-ordered metric space, with an order $\leq$ as a quasi-order (that is, a reflexive and transitive relation) and a metric d. Assume that X having the following properties which appears in [8]:

(H1): if $\{x_n\}$ is a non-decreasing (resp. non-increasing) sequence in X such that $x_n \to x$, then $x_n \leq x$ (resp. $x_n \geq x$) for all $n \in N$.

Let 2^X denote the family consisting of all nonempty subsets of X we define the Hausdorff-Pseude metric in $H_d: 2^X \times 2^X \to \Re_+ \cup \{\infty\}$ given by

$$H_d(C,D) = \max\{\sup_{a\in C} d(a,D), \sup_{b\in D} d(C,b)\},$$

where $d(a,D) = \inf_{b\in D} d(a,b)$, $d(C,b) = \inf_{a\in C} d(a,b)$.

Definition2.1. Let $(X, d, \leq)$ be a quasi-ordered metric space. We say that X is sequentially complete if every Cauchy sequence whose consecutive terms are comparable in X converges.

Definition2.2. [6,7] Let X be a quasi-ordered metric space. A subset $D \subseteq X$ is said to be approximative if the multivalued mapping

$$P_D(x)=\{y\in D: d(x,y)=d(D,x)\},\quad \forall x\in X$$

has nonempty values.

The multivalued mapping $T: X\to 2^X$ is said to have approximative values, AV for short, if Tx is approximative for each $x\in X$.

The multivalued mapping $T: X\to 2^X$ is said to have comparable approximative values, CAV for short, if T has approximative values and, for each $z\in X$, there exists $y\in P_{Tz}(x)$ such that y is comparable to z.

The multivalued mapping $T: X\to 2^X$ is said to haveu pper comparable approximative values, UCAV, for short (resp: lower comparable approximative values, LCAV for short) if T has approximative values and, for each $z\in X$, there exists $y\in P_{Tz}(x)$ such that $y\geq z$ (resp: $y\leq z$). It is clear that T has approximative values if it has compact values. In addition, if T is single-valued, Then UCAV (LCAV) means that $Tx\geq x$ $(Tx\leq x)$ for $x\in X$.

Definition2.3. The multivalued mappings T,S are said to have a common fixed point if there is $x\in X$ such that $x\in Tx$ and $x\in Sx$.

In what follows, we give an analogy of the contraction which called multivalued C-weakly contraction mapping will play an important role in this sequel. To this end, we first introduce the following function.

Let $a\in(0,\infty], R_a^+=[0,a)$. let $f:\Re_a^+\to\Re$ satisfy,

(i) $f(0)=0$ and $f(t)>0$ for each $t\in(0,a)$

(ii) f is non-decreasing on $\Re_a^+$

(iii) f is continuous

(iv) $f(t+s)\leq f(t)+f(s)$ for $s,t\in\Re_a^+$.

For examples of such function f we refer to (6).

Define

$$\mathfrak{F}[0,a)=\{f \mid f \quad \text{satisfies (i)-(iv) above}\}.$$

Let $a\in(0,\infty]$, $\varphi: R_a^+\times R_a^+\to R^+$ satisfy

(i) $\varphi(t,s)=0$ if and only if $s=t=0$.

(ii) φ is continuous.

(iii) For any sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n=0$, there exist $a\in(0,\frac{1}{2})$ and $n_0\in N$ such that $\varphi(r_n,0)\geq(1-a)r_n$ (or $\varphi(0,r_n)\geq(1-a)r_n$) for each $n\geq n_0$. Define $\Phi([0,a)\times[0,a))=\{\varphi:\phi \quad \text{satisfies(i)-(iii)above}\}$.

Definition2.3. Let X be a metric space and $d = sup\{d(x,y) : x, y \in X\}$. Set $a = d$ if $d = \infty$ and $a > d$ if $d < \infty$. Suppose the multivalued mappings $T, S : X \to 2^X$, $f \in \mathfrak{F}[0,a)$ and $\varphi \in \Phi([0, f(a-0)) \times [0, f(a-0)))$ satisfy

$$f(H_d(Tx,Sy)) \le f(\frac{1}{2}(d(x,Sy)+d(y,Tx))) - \varphi(f(d(x,Sy)), f(d(y,Tx)))$$

For all $x, y \in X$ with x and y comparable. Then we say T and S satisfy weakly C -contraction with respect to f and φ.

Definition2.4. For two subsets A , B of X , we say that $A \le_r B$ if, for each $a \in A$, there exists $b \in B$ such that $a \le b$, and $A \le B$ if each $a \in A$ and each $b \in B$ imply that $a \le b$. A multi-valued mapping $T : X \to 2^X$ is said to be r -non-decreasing (r -non-increasing) if $x \le y$ implies that $Tx \le_r Ty$ $(Ty \le_r Tx)$ for all $x, y \in X$. T is said to be r -monotone if T is r -non-decreasing or r -non-increasing. The notion of non-decreasing (non-increasing) is similarly defined by writing $\le$ instead of the notation $\le_r$

3. Main Result

In this section we established common fixed point theorems for multivalued mappings on quasi-ordered complete metric spaces. The idea of the present theorem3.1 originate from the study of ananalogous problem for single-valued mappings in [4] and [9], and multivalued mappings in [6], [7] and [10].

Theorem3.1. Let X be a quasi-ordered sequentially complete metric space and satisfy (H1). Suppose that the multivalued mappings T and S have UCAV and satisfy the weakly C-contraction with respect to f and φ, then T and S have a common fixed point. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T and S .

Proof: First we show that, if T or S has a fixed point it is a common fixed point of T and S. Indeed, let x be a fixed point of T then we have,

$$\begin{aligned} f(d(x,Sx)) &\le f(H_d(Tx,Sx)) \\ &\le f(0.5(d(x,Sx)+d(x,Tx))) - \varphi(f(d(x,Sx)), f(d(x,Tx))) \\ &= f(0.5d(x,Sx)) - \varphi(f(d(x,Sx),0) \\ &\le f(d(x,Sx)) - \varphi(f(d(x,Sx),0) \end{aligned}$$

This implies that, $\varphi(f(d(x,Sx)),0) = 0$ and hence $f(d(x,Sx)) = 0$ therefore $d(x,Sx) = 0$.
Since x is AV, therefore there exist $y \in P_{Sx}(x)$ such that $d(y,x) = 0$ i.e, $y = x$. Thus

$x \in Sx$. Let $x_0 \in X$, if $x_0 \in Tx_0$ the proof is finished. Otherwise, from the fact that Tx_0 has UCAV it follows there exists $x_1 \in Tx_0$ with $x_1 \neq x_0$ and $x_1 \geq x_0$ such that

$$d(x_0, x_1) = \inf_{x \in Tx_0} d(x, x_0) = d(Tx_0, x_0).$$

Again since Sx_1 has UCAV it follows there exist $x_2 \in Sx_1$ with $x_2 \neq x_1$ and $x_2 \geq x_1$ such that

$$d(x_1, x_2) = \inf_{x \in Sx_1} d(x, x_1) = d(Sx_1, x_1).$$

By induction and using UCAV, we can find in this way a sequence $\{x_n\}$ in X with $x_{n+1} \geq x_n$ such that $x_{2n+1} \in Tx_{2n}$ and

$$d(x_{2n+1}, x_{2n}) = d(Tx_{2n}, x_{2n})$$

and $x_{2n+2} \in Sx_{2n+1}$ with

$$d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n+1}, x_{2n+1}).$$

On the other hand

$$\begin{aligned} d(Tx_{2n}, x_{2n}) &\leq \sup_{x \in Sx_{2n-1}} d(Tx_{2n}, x) \\ &\leq H_d(Tx_{2n}, Sx_{2n-1}). \end{aligned}$$

Therefore

$$d(x_{2n+1}, x_{2n}) \leq H_d(Tx_{2n}, Sx_{2n-1}). \qquad (1)$$

Similarly we can show that

$$d(x_{2n+2}, x_{2n+1}) \leq H_d(Sx_{2n+1}, Tx_{2n}). \qquad (2)$$

Now we show that $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$. By using (2) and since f is non-decreasing, we have

$$\begin{aligned} & f\big(d(x_{2n+1}, x_{2n+2})\big) \leq f\big(H_d(Tx_{2n}, Sx_{2n+1})\big) \\ & \leq f\Big(0.5\big(d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})\big)\Big) - \varphi\Big(f\big(d(x_{2n}, Sx_{2n+1})\big), f\big(d(x_{2n+1}, Tx_{2n})\big)\Big) \\ & \leq f\big(0.5(d(x_{2n}, x_{2n+2})\big) - \varphi\Big(f\big(d(x_{2n}, x_{2n+2})\big), 0\Big) \leq f\big(0.5d(x_{2n}, x_{2n+2})\big). \end{aligned} \qquad (3)$$

As f is a non-decreasing function, we get

$$d(x_{2n+1}, x_{2n+2}) \leq 0.5d(x_{2n}, x_{2n+2}). \qquad (3)$$

Since

$$d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}).$$

We have

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \qquad (4)$$

Similarly, by using (1) one can show that

$$d(x_{2n}, x_{2n+1}) \leq 0.5d(x_{2n-1}, x_{2n+1}). \quad (5)$$

Thus

$$d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}). \qquad (6)$$

From (4) and (6), we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad \forall n \in N. \qquad (7)$$

So, by (7) we get that $\{d(x_n, x_{n+1}) : n \in N\}$ is a non-increasing sequence. Hence there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

By (3) and (5) we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq 0.5d(x_{n-1}, x_{n+1}) \\ &\leq 0.5\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big). \qquad (8) \end{aligned}$$

Letting $n \to \infty$ and using (8), we get that

$$r \leq \lim_{n\to\infty} 0.5d(x_{n-1}, x_{n+1}) \leq 0.5(r + r).$$

Hence

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r.$$

Using the continuity f , φ and (3), we get that

$$f(r) \leq f\big(0.5(2r)\big) - \varphi\big(f(2r), 0)\big),$$

which implies that $\varphi\big(f(2r), 0\big) = 0$ and hence $r = 0$.

Next we show that (x_n) is a Cauchy sequence in X . Since $\lim_{n\to\infty} f\big(d(x_{n-1}, x_{n+1})\big) = 0$, from assumption (iii) of φ there exists $0 < a < \frac{1}{2}$ and $n_0 \in N$ such that

$$\varphi\big(f\big((d(x_{n-1}, x_{n+1})\big), 0\big) \geq af\big(d(x_{n-1}, x_{n+1})\big) \quad for \quad all \quad n \geq n_0.$$

On the other hand, for any given $\epsilon > 0$, we choose $\delta > 0$ to be small enough such that $f(\delta) < \frac{a}{1-2a} f(\epsilon)$. Moreover, there exists n_1 such that $d(x_{n+1}, x_n) \le \delta$, for each $n \ge n_1$.

Now for any numbers $m > n \ge \max\{n_0, n_1\}$, from the inequality (1) and (2) we have

$$\begin{aligned}
f\big(d(x_{n+1}, x_n)\big) &\le f(H_d(Tx_n, Sx_{n-1})) \quad (\quad \text{or} \quad f(H_d(Tx_{n-1}, Sx_n)) \\
&\le f\big(0.5\big(d(x_n, Sx_{n-1}) + d(x_{n-1}, Tx_n)\big)\big) \\
&\quad -\varphi\big(f\big(d(x_n, Sx_{n-1})\big), f\big(d(x_{n-1}, Tx_n)\big)\big) \\
&\le f\big(0.5\big(d(x_{n-1}, x_{n+1})\big)\big) - \varphi\big(0, f\big(d(x_{n-1}, x_{n+1})\big)\big) \\
&\le f\big(d(x_{n-1}, x_{n+1})\big) - (1-a) f\big(d(x_{n-1}, x_{n+1})\big) \\
&\le af\big(d(x_{n-1}, x_{n+1})\big) \\
&\le a(f\big(d(x_{n-1}, x_n)\big) + f\big(d(x_n, x_{n+1})\big)).
\end{aligned}$$

Therefore

$$f\big(d(x_n, x_{n+1})\big) \le (a/(1-a)) f\big(d(x_{n-1}, x_n)\big).$$

Set $\alpha = \frac{a}{1-a} < 1.$ By repeating this procedure, for any $k > n$ we obtain

$$f\big(d(x_k, x_{k-1})\big) \le \alpha f\big(d(x_{k-1}, x_{k-2})\big) \le \ldots \le a^{k-n} f\big(d(x_n, x_{n-1})\big).$$

Therefore, from the assumption of f we have,

$$\begin{aligned}
f\big(d(x_m, x_n)\big) &\le f\big(d(x_m, x_{m-1})\big) + f\big(d(x_{m-1}, x_{m-2})\big) + \ldots + f\big(d(x_{n+1}, x_n)\big) \\
&\le \alpha^{m-n} f\big(d(x_n, x_{n-1})\big) + \alpha^{m-n-1} f\big(d(x_n, x_{n-1})\big) + \ldots \\
&\quad + \alpha f(\big(d(x_n, x_{n-1})\big) \\
&= (\alpha - \alpha^{m-n+1}/(1-\alpha)) f\big(d(x_n, x_{n-1})\big) \\
&< (\alpha/(1-\alpha)) f\big(d(x_n, x_{n-1})\big) < (\alpha/(1-\alpha)) f(\delta) \\
&= (a/(1-2a)) f(\delta) < f(\epsilon).
\end{aligned}$$

This shows that $d(x_m, x_n) < \epsilon$, so $\{x_n\}$ is a $\le -$non-decreasing Cauchy sequence. Since X is a sequentially complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Finally, we prove that x^* is a common fixed point of T and S. For every $n \in N$, $(H1)$ guarantees that x_n is comparable to x^* , so for $n \in N$ we have,

$$f(d(x_{2n+2},Sx^*)) \le f(\sup_{x\in Tx_{2n+1}} d(x,Sx^*)) \le f(H_d(Tx_{2n+1},Sx^*))$$
$$\le f(0.5(d(x_{2n+1},Sx^*)+d(x^*,Tx_{2n+1}))) - \varphi(f(d(x_{2n+1},Sx^*)), f(d(x^*,Tx_{2n+1})))$$
$$\le f(0.5(d(x_{2n+1},Sx^*)+d(x^*,x_{2n+2}))) - \varphi(f(d(x_{2n+1},Sx^*)), f(d(x^*,x_{2n+2}))),$$

(9)

Since φ is l.s.c, letting $n\to\infty$ in (9) we get

$$f(d(x^*,Sx^*)) \le f(0.5d(x^*,Sx^*)) - \varphi(f(d(x^*,Sx^*)),0).$$

Which implies $\varphi(f(d(x^*,Sx^*)),0)=0$ and hence $d(x^*,Sx^*)=0$. Since Sx^* is AV, there exist $y\in P_{Sx^*}$. such that $d(y,x^*)=0$ i.e, $y=x^*$, therefore $x^*\in Sx^*$, i.e x^* is a fixed point of S, and so it is a common fixed point. This completes the proof.

Similar to the proof of Theorem 3.1 we have the following Theorem.

Theorem.3.2. Let X be a sequentially complete quasi-ordered metric space and satisfy (H1). Suppose that $T,S: X\to 2^X$ be two mappings that satisfy weakly C -contraction with respect to f and φ, and have LCAV. Then T and S have a common fixed point. Further, for each $x_0\in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ converges to the common fixed point of T and S .

Theorem3.3. Let X be an totally ordered sequentially complete metric space and satisfy (H1) and the following

(H2) $x\le y\le z$ implies that $d(z,x)\ge d(y,x)$ for all $x,y,z\in X$.

Suppose that T and S satisfy all conditions given in Theorem 3.1 (resp. in Theorem 3.2), then T, S have a unique common fixed point $x\in X$ and the iterated convergence of Theorem 3.1 holds.

Proof: Theorem 3.1 (resp. Theorem 3.2) ensures existence of common fixed points. To prove the uniqueness, let both x and y be common fixed point of T and S. Since $(X,\le)$ is a totally ordered space, we have either $x>y$ or $y>x$. Without loss of generality, we assume that the former is true. If T has UCAV, we have $x^*\in Tx$, with $x\le x^*$ and $d(x^*,y)=d(Tx,y)$. From our assumption it follows that $d(x^*,y)\ge d(x,y)$. On the other hand, $x \in Tx$ implies that $d(x^*, y) \le d(x, y)$. Hence, $d(x^*, y) = d(x, y) = d(Tx, y)$. If $x \ne y$, then $d(x, y) > 0$. Thus

$$d(x, y) = d(Tx, y) \le H_d(Tx, Sy). \qquad (10)$$

If T has LCAV, so does S, we have $y^* \in Sy$ with $y^* \le y$ and $d(y^*, x) = d(Sy, x)$. From (H2) it follows that $d(y^*, x) \ge d(x, y)$. On the other hand, $y \in Sy$ implies that $d(y^*, x) \le d(x, y)$. Hence, $d(y^*, x) = d(x, y) = d(x, Sy)$. At all events, (10) holds if $x \ne y$.

$$\begin{aligned} f(d(x,y)) \le f(H_d(Tx,Sy)) &\le f(\frac{1}{2}(d(y,Tx) + d(x,Sy))) - \varphi(d(y,Tx), d(x,Sy)) \\ &= f(d(x,y)) - \varphi(d(x,y), d(x,y)) < f(d(x,y)) \end{aligned}$$

This is a contradiction. Consequently, the inequality $x < y$ is not true. By the same methods we can verify that $y < x$ is also not true. Thus $x = y$.

Theorem.3.3. Let X be a sequentially complete quasi-ordered metric space and satisfy (H1). Suppose that $T, S : X \to 2^X$ be two mappings have AV, are non-decreasing, and weak C-contraction with respect to f and φ. If there exists $x_0 \in X$ such that $\{x_0\} \le Sx_0 \le Tx_0$. Then T and S have a common fixed point. Further, the iterated convergence of Theorem 3.1 holds.

Proof: let $x_0 \in X$, if $x_0 \in Sx_0$ then is a common fixed point of T and S thus the proof is complete. Otherwise, since Sx has AV, there exist $x_1 \in Sx_0$ with $x_1 \ge x_0$ and $d(x_0, x_1) = d(Sx_0, x_0)$. Since $x \ge x_1$ for all $x \in Tx_1$. If $x_1 \in Tx_1$, the proof is finished, otherwise, by means of Tx is AV, there exist $x_2 \in Tx_1$ with $x_2 \ge x_1$ and $d(x_1, x_2) = d(Tx_1, x_1)$. Inductively, we can construct a sequence x_n in X as $x_n \ne x_{n-1}$ and $x_n \ge x_{n-1}$ such that $x_{2n+1} \in T_{2n}$, $x_{2n+2} \in Sx_{2n+1}$ and (1), (2) hold. Now the rest of the proof is the same as theorem 3.1.

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