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A generalization of iteration-free search vectors of ABS methods

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Abstract

Recently, we introduced iteration-free search vectors of the ABS methods and showed how they can be used to compute the search directions of primal--dual interior point methods, when the coefficient matrix of the constraints of the linear programming problem is square. Here, we generalize those results for the general case when, the coefficient matrix is non-square.

Keywords: Interior point methods, Infeasible interior pointmethods, Primal--dual algorithms, ABS algorithms, Searchdirection.

1. Introduction

Assume that A is an $m \times n$ matrix with rank(A)=m. Let c, x and s be n-vectors and b be an m-vector. Then, the primal linear programming problem [5,7] is defined to be the minimization of the objective function$c^T x$ subject to the functional constraints $Ax = b$ and the non-negativity constraints $x \geq 0$. The dual of this problem is then the maximization of $b^T y$ subject to $A^T y + s = c$ and $s \leq 0$. In the kth iteration of primal--dual infeasibleinterior point algorithms the search direction is computed by solving the following system of linear equations [3,8,9]:

$$\begin{pmatrix} 0 & A^T & I_n \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{pmatrix} \qquad (1)$$

where r_c^k, r_b^k and r_{xs}^k are given by $r_b^k = Ax^k - b, r_c^k = A^T \lambda^k + s^k - c, r_{xs}^k = -X^k S^k e + \sigma_k \mu_k \mathbf{1}$. and X^k and S^k denote the diagonal matrices whose diagonal elements are the components ofthe vectors x^k

and s^k, respectively, and $\mathbf{1} = (1, \dots, 1)^T \in R^n$. Moreover, $\sigma_k \in (0,1)$ and $\mu_k = \frac{(x^k)^T s^k}{n}$ are centering parameterand duality gap, respectively.
The ABS class of algorithms was first introduced by Abaffy, Broyden and Spedicato [1,2,6] for solving linear systems. For simplicity in notation, assume$R^{m\times n}$ has full row rank. An ABS method for solving the linear system $Ax = b$, with $b \in R^m$, starts with anarbitrary initial vector $x_1 \in R^n$ and an arbitrary nonsingular matrix$H_1 \in R^{n\times n}$, the so-called Spedicato's parameter. In the ith iteration, having computed x_i, asolution of the first $i-1$ equations of $Ax = b$, and H_i a matrix with rowsgenerating the null space of the first $i-1$ rows of A, an ABS algorithm computes x_{i+1} asa solution of the first i equations of $Ax = b$ and H_{i+1}, with rows generating the null space of the first i rows of A as explained below [1,2]. To compute the search vector, z_i (Broyden's parameter) is determined so that $z_i^T H_i a_i \neq 0$ and the search vector is set to be $q_i = H_i^T z_i$. Then, the solution is updated by $x_{i+1} = x_i + \alpha_i q_i$, where the step size α_i is given by $\alpha_i = (b_i - a_i^T x_i)/a_i^T q_i$. Next, H_{i+1}\$ is computed so that $H_{i+1}a_j = 0$, $1 \le j \le i$. This can be accomplished by updating H_i (the so called Abaffian) by

$$H_{i+1} = H_i - H_i a_i w_i^T H_i / w_i^T H_i a_i \qquad (2)$$

with $w_1 \in R^n$ (Abaffy's parameter) satisfying $w_i^T H_i a_i \neq 0$. It can be shown that [1] in an ABS algorithm, we have $s_i = H_i a_i \neq 0$\$ if and only ifa_i is linearly independent of$a_1, a_2, \dots, a_{i-1}$ (orequivalently, $a_i = 0$ if and only if a_i is linearly dependent on$a_1, a_2, \dots, a_{i-1}$. The rows of H_{i+1} generate thenull space of the first $i$rows of A. If rank(A)=$m$then every solution of the first i equations of the system can bewritten as $x_{i+1} + H_{i+1}^T s$, for some choice of $s \in R^n$. Let x^* be the special solution of the linear system $Ax = b$, then there exist a vector $s^* \in R^n$ such that $x^* = x_{i+1} + H_{i+1}^T s^*$. Indeed let $r_{i+1} = b - Ax_{i+1}$, $Z = (H_{i+1}a_{i+1}, \dots, H_{i+1}a_m)^T$ and d be the solution of the linear system $ZZ^T d = (r_{i+1})_{m-i}$ where $(r_{i+1})_{m-i}$ denotes the last $m-i$ components of the vector r_{i+1}. Then, we have $s^* = Z^T d$. If $x_1 = 0$, then the solution of the system is $P\tau$, where P=$(p_1, \dots, p_m)$ and $\tau = (\tau_1, \dots, \tau_m)$.
In Section 2, we describe the ideas of iteration-free search vectors of the ABS algorithm for solving (1). In Section 3, we show how we can use these iteration-free search vectors, to characterize the ABS solution of (1), in case $m < n$. Section 4 is devoted to the concluding remarks.

2. Iteration-free search vectors

In the kth iteration of primal-dual IIPMs to solve linear optimization problems, the search direction $(\Delta x^k, \Delta\lambda^k, \Delta s^k)$ is computed by solving the linear system (1). We start the ABS algorithm with $x_1 = 0 \in R^{2n+m}$ and $H_1 = I_{2n+m}$, where I_{2n+m} is the identity matrix. Then, it can be easily verified that for $1 \le i \le n$, if we let $z_i = w_i = (0,0,e_i^T) \in R^{2n+m}$, where e_i is the ith column of the identity matrix I_n, then, in the ith iteration of the ABS algorithm forsolving (1), we have

$$p_i = \begin{pmatrix} 0 \\ 0 \\ e_i \end{pmatrix}, \quad H_{i+1} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & -\sum_{j=1}^{i} \hat{a}_j\, e_j^T \\ 0 & 0 & I_n - \sum_{j=1}^{i} e_j\, e_j^T \end{pmatrix}, \qquad (3)$$

where $\hat{a}_j$ is the jth column of the matrix A. Now let a_i's, $1 \le i \le m$, denote the ith row of the matrix A. By applying the ABS algorithm to the system $A\Delta x^k = -r_b^k$, starting with $\bar{H}_1 = I_n$ and $\bar{x}_1 = 0 \in R^n$, we obtain the parameters $\bar{z}_i \in R^n$, $\bar{p}_i \in R^n$ and $H_{i+1} \in R^{n\times n}$, for$1 \le i \le m$ so that$\bar{z}_i^T \bar{H}_i a_i \neq 0$, $\bar{w}_i^T \bar{H}_i a_i \neq 0$, $p_i = \bar{H}_i^T \bar{z}_i$, $\bar{H}_{i+1} = \bar{H}_i - \bar{H}_i a_i \bar{w}_i^T \bar{H}_i / \bar{w}_i^T \bar{H}_i a_i$. It can be easily verified that for $1 \le i \le m$, if we let $z_{n+i} = (\bar{z}_i, 0,0)^T \in R^{2n+m}$,$w_{n+i} = (\bar{w}_i, 0,0)^T \in R^{2n+m}$, where $\bar{w}_i$ and $\bar{z}_i$ are defined as above, then, in the $(n+i)$th iteration of the ABS algorithm applied to solve (1), we have

$$p_{n+i} = \begin{pmatrix} \bar{p}_i \\ 0 \\ 0 \end{pmatrix}, \quad H_{n+i+1} = \begin{pmatrix} \bar{H}_{i+1} & 0 & 0 \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4)$$

Let $\tilde{A} = (\tilde{a}_1, \dots, \tilde{a}_n)$, where the ith column of this matrix is constructed asfollows. Let$|x|^2 = \sum_{i=1}^{n} x_i^2$ and $\check{a}_1 = \hat{a}_1$, $\tilde{a}_1 =$. Now, we define $\check{a}_i = \hat{a}_i - \sum_{j=1}^{i-1} (\tilde{a}_j^T \hat{a}_i)\, \tilde{a}_j$, for $2 \le i \le n$ and $\tilde{a}_i = \check{a}_i / ||\check{a}_i||$, for $2 \le i \le n$. The vectors $\tilde{a}_j$, $1 \le j \le m$, are orthonormal. Using $\tilde{A}$, we define the matrices $N_j \in R^{n\times m}$, $M_k^j \in R^{n\times m}$, $C_j \in R^{m\times m}$ and $B_k^j \in R^{n\times m}$, for$1 \le j \le m$, according to the followingrelations:

$$N_j = -\frac{1}{||\check{a}_i||} e_j e_j^T \tilde{A}^T, \;\; M_k^j = \frac{s_j^k}{x_j^k} N_j \, , \; C_j = I_m - \sum_{j=1}^{i} \tilde{A} e_j \, e_j^T \tilde{A}^T \qquad (5)$$

and

$$B_k^1 = H_{m+1} M_k^1, \qquad B_k^j = B_k^{j-1} + B_k^{j-1} A N_j C_{j-1} - \bar{H}_{m+1} M_k^j C_{j-1} \qquad (6)$$

In the following Theorem, we provide the ABS parameters forthe $(n+m+i)$th, $1 \le i \le m$, iteration of the algorithmapplied to solve (1). The proof can be found in [4].

Theorem: For $1 \le i \le m$, let

$$w_{n+m+i} = z_{n+m+i} = \begin{pmatrix} 0 \\ \tilde{A} e_i \\ 0 \end{pmatrix} \in R^{2n+m}$$

Then, in the $(n+m+i)$th iteration of the ABS algorithmapplied to solve (1), we have

$$p_{n+m+i} = \begin{pmatrix} 0 \\ \tilde{A} e_i \\ -A^T \tilde{A} e_i \end{pmatrix}, \quad H_{n+m+i+1} = \begin{pmatrix} \bar{H}_{m+1} & B_k^i & -B_k^i A \\ 0 & C_i & -C_i A \\ 0 & 0 & 0 \end{pmatrix}, \qquad (7)$$

where the matrices $C_i$and B_k^i are defined by (5) and(6), respectively.

2. Computing the search directions

In this section, we provide the search directions of primal-dual IIPMs using the search vectors obtained in Section 2 for the Newton system (1) for the general case where, $m \le n$. In this case, we first derive an efficient formula to compute B_k^m. Then, using properties of the ABS algorithm, we characterize the solution of system (1) from the solution of the first $2n + m$ equations. Consider the case in which $m \le n$ and B_k^m is not zero matrix. For $1 \le i \le m$, we define

$$U_i^T = \left(u_1^i, \dots, u_i^i\right) \in R^{m+i}, \quad D_i^k = \left(\varepsilon_1^i e_1, \dots, \varepsilon_i^i e_i\right) \in R^{n+i} \qquad (8)$$

where,

$$\delta_j = \frac{-1}{\left\|\check{a}_j\right\|}, \quad \varepsilon_j^k = -\frac{s_j^k \delta_j}{x_j^k}, \quad 1 \le j \le i \qquad (9)$$

and $u_1^1 = \tilde{A} e_1$, $u_j^i = u_j^{i-1} + \delta_i \tilde{A} e_i e_i^T A^T u_j^{i-1}$, for $1 \le j \le i-1$ and $u_i^i = \tilde{A} e_i$. The following theorem provides an efficient formula to computeB_k^i, $1 \le i \le m$.

Theorem: Let $1 \le i \le m$ and the matrices D_i^k and U_i be defined as in (8). Then,

$$B_k^i = \bar{H}_{m+1} D_i^k U_i = \bar{H}_{m+1} \sum_{j=1}^{i} \varepsilon_j^k e_j \,(u_j^i)^T. \qquad (10)$$

Proof: We proceed by induction. For $i = 1$, we have

$$B_k^1 = \bar{H}_{m+1} D_1^k U_1 = \bar{H}_{m+1} \varepsilon_1^k e_1 (u_1^1)^T$$

which is true by the definition of B_k^1 in (6). Suppose that (10}) is true up to $i = 1,2, \dots, t-1$. For $i = t$, from (5), (6) and (9) we have

$$B_k^t = B_k^{t-1} + B_k^{t-1} A N_t C_{t-1} - \bar{H}_{m+1} M_k^t C_{t-1} \qquad (11)$$

$$M_k^t C_{t-1} = \frac{-s_t^k}{\left|\left|\check{a}_j\right|\right| x_t^k} e_t e_t^T \tilde{A}^T \left(I_m - \sum_{j=1}^{t-1} \tilde{A} e_j\, e_j^T \tilde{A}^T\right) = -\varepsilon_t^k e_t e_t^T \tilde{A}^T \quad (12)$$

$$N_t C_{t-1} = -\frac{e_t e_t^T \tilde{A}^T}{\left|\left|\check{a}_t\right|\right|}\left(I_m - \sum_{j=1}^{t-1} \tilde{A} e_j\, e_j^T \tilde{A}^T\right) = \delta_t^k e_t e_t^T \tilde{A}^T \quad (13)$$

Using the hypothesis of the induction and (11), (12) and (13), we can write

$$B_k^t = B_k^{t-1} + \delta_t^k B_k^{t-1} A e_t e_t^T \tilde{A}^T + \varepsilon_t^k \overline{H}_{m+1} e_t e_t^T \tilde{A}^T$$

$$= \overline{H}_{m+1} \sum_{j=1}^{t-1} \varepsilon_j^k e_j \left(u_j^{t-1}\right)^T + \varepsilon_t^k \overline{H}_{m+1} e_t e_t^T \tilde{A}^T + \delta_t^k \sum_{j=1}^{t-1} \varepsilon_j^k e_j \left(u_j^{t-1}\right)^T A e_t e_t^T \tilde{A}^T$$

$$= \overline{H}_{m+1} (\varepsilon_t^k e_t e_t^T \tilde{A}^T + \sum_{j=1}^{t-1} \varepsilon_j^k e_j \{\left(u_j^{t-1}\right)^T + \delta_t^k \left(u_j^{t-1}\right)^T A e_t e_t^T \tilde{A}^T)\}$$

$$= \overline{H}_{m+1} \sum_{j=1}^{t} \varepsilon_j^k e_j \left(u_j^t\right)^T = \overline{H}_{m+1} D_t^k U_t$$

This completes the induction.

It is worth mentioning that the matrices U_i and $\overline{H}_{m+1}$need to be computed only once in the first iteration of the IIPMs, and D_j^k is a diagonal matrix. Here, using properties of the ABS algorithm, we construct the solution of system (1) for the case $m \le n$. Let $S_{(n-m)}^k = \sum_{j=m+1}^n s_j^k e_j \bar{e}_{j-m}^T$, $X_{(n-m)}^k = \sum_{j=m+1}^n x_j^k e_j \bar{e}_{j-m}^T$, $(\bar{Z}^k)^T = \overline{H}_{m+1} S_{(n-m)}^k - B_k^t A X_{(n-m)}^k$, where $\bar{e}_i$ is the ith column of the identity matrix I_{n-m}. Assume that

$$(Z^k)^T = \left(H_{2n+m+1}\begin{pmatrix} s_{m+1}^k e_{m+1} \\ 0 \\ x_{m+1}^k e_{m+1}\end{pmatrix}, \dots, H_{2n+m+1}\begin{pmatrix} s_n^k e_n \\ 0 \\ x_n^k e_n\end{pmatrix}\right) = \begin{pmatrix} (\bar{Z}^k)^T \\ 0 \\ 0 \end{pmatrix}.$$

We note that

$$Z^k \left(Z^k\right)^T = \left(\bar{Z}^k, 0,0\right)\begin{pmatrix} \left(\bar{Z}^k\right)^T \\ 0 \\ 0 \end{pmatrix} \bar{Z}^k \left(\bar{Z}^k\right)^T.$$

The residual vector of system (1) in thesolution of the first $2m + n$ equations is:

$$r^k = \begin{pmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{pmatrix} - \begin{pmatrix} 0 & A^T & I_n \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix}\begin{pmatrix} P\lambda^k \\ \tilde{A}\beta^k \\ -r_c^k - A^T \tilde{A}\beta^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -r_{xs}^k - S^k P\lambda^k + X^k r_c^k + X^k A^T \tilde{A}\beta^k \end{pmatrix},$$

where the last equality obtains from the fact that $\left((\lambda^k)^T P^T, (\beta^k)^T \tilde{A}^T, (-r_c^k - A^T \tilde{A}\beta^k)^T\right)$is the solution of thefirst $2m + n$ equations. Let $\left(r^k\right)_{n-m}$ denotes the last $n - m$components of the vector r^k, i.e.,$\left(r^k\right)_{n-m} = \left(-r_{xs}^k - S^k P\lambda^k + X^k r_c^k + X^k A^T \tilde{A}\beta^k\right)_{n-m}$.Now, let d^k satisfies$Z^k \left(Z^k\right)^T d^k$=$\left(r^k\right)_{n-m}$.Thus, using properties of the ABS algorithms, the solution of (1) is as follows:

$$\begin{pmatrix} \Delta x^k \\ \Delta\lambda^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} P\lambda^k \\ -\tilde{A}\beta^k \\ -r_c^k - A^T \tilde{A}\beta^k \end{pmatrix} + \begin{pmatrix} \overline{H}_{m+1}^T & 0 & 0 \\ (B_k^m)^T & 0 & 0 \\ -A^T (B_k^m)^T & 0 & 0 \end{pmatrix}\begin{pmatrix} \left(\bar{Z}^k\right)^T d^k \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} P\lambda^k + \overline{H}_{m+1}^T \left(\bar{Z}^k\right)^T d^k \\ -\tilde{A}\beta^k + (B_k^m)^T \left(\bar{Z}^k\right)^T d^k \\ -r_c^k - A^T \tilde{A}\beta^k - A^T (B_k^m)^T \left(\bar{Z}^k\right)^T d^k \end{pmatrix}$$

2. Conclusions

We generalized iteration free search vectors of the ABS algorithms. Then we used these iteration free search vectors to characterize the solution of the Newton systems of primal-dual infeasible interior point methods.

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