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Compact Topological Semigroups associated with Oids

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Abstract

The known theory for a discrete oid T shows that how to find a subset T^{∞} of βT which is a compact right topological semigroup (see section 2 for details).In this paper we try to find an analogue of almost periodic functions for oids. We discover, new compact semigroups by using a certain subspace of functions $\mathcal{U}^{\infty}(T)$ of $C(T)$ for an oid T for which f^{β} is continuous on $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$,where$(T \cup T^{\infty} \cup TT^{\infty})$ is a suitable subspace of βT for a wide range.

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1. Introduction

Let S be a semigroup and topological space. S is called a topological semigroup if the multiplication $(s,t) \to st: S \times S \to S$ is jointly continuous. Civin and Yood [4] shows that βS the Stone-Čech compactification of a discrete semigroup S could be given a semigroup structure, which need not be commutative on S and is continuous in the left-hand variable; (that is for fixed $\nu \in \beta S$, the map $\mu \to \mu\nu: \beta S \to \beta S$ is continuous).Indeed the operation on S extends uniquely to βS, so that S contained in it's topological center [5]. Pym [7] introduced the concept of an oid (see Section 2 for precise definition). Oids are important because nearly all semigroups contain them and all oids are oid-isomorphic [6].We shall present our theory in a fairly concrete setting, so that our methods and results will be more readily accessible. Through out this paper we will let T be a commutative standard oid with a discrete topology. Then the compact space βT produces a compact right topological semigroup, "at infinity”T^{∞},so that its topological center is empty and it is not commutative(we refer the reader to [2],for these facts). Our aim of the present paper is to introduce a new compact topological semigroup for an oid T,using a certain space of functions on T which have jointly continuous extensions on subspace $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$ of $T^{\infty} \times \beta T$ where $(T \cup T^{\infty} \cup TT^{\infty})$ is a suitable subspace of βT which is as large as possible. $C(T)$ is the C^*-algebra of all bounded continuous complex valued functions defined on the discrete space T and $C(T)^*$ is the dual space of $C(T)$; we indicate the supremum norm on $C(T)$ by$\|.\|$.We define a subset $\mathcal{U}^{\infty}(T)$ containing all $f \in C(T)$ such that f^{β} is jointly continuous on $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$ where f^{β} is a unique continuous extension f to βT. Then $\mathcal{U}^{\infty}(T)$ is a C^*-subalgebra of $C(T)$(Lemma 3.3), so that $\mathcal{U}^{\infty}(T) \subseteq WAP^{\infty}(T)$(see[1],for definition). Indeed, $WAP^{\infty}(T)$need not be a subset of $\mathcal{U}^{\infty}(T)$(Example4.27). From the functions space $\mathcal{U}^{\infty}(T)$ we shall able to define an equivalence relation $\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ on T^{∞} by $\mu \ \mathcal{R}_{\mathcal{U}^{\infty}(T)} \ \nu$ if and only if $f^{\beta}(\mu) = f^{\beta}(\nu)$ for all $f \in \mathcal{U}^{\infty}(T)$. This does determine a closed congruence relation on T^{∞} Which makes the quotient $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$a compact Hausdorff commutative topological semigroup which is a new semigroup to consider. Also, we conclude by establishing some properties of $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$, for example $(T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)})^2$ is not dense in $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$*(Proposition 4.14), it contains* 2^c *idempotents (Theorem 5.4) and* $K(T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)})$*, the minimal ideal of* $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ *contains a free abelian group on* 2^c *generators (Theorem6.2).*

2. Definitions and preliminaries

Let $x = (x(n))_{n\in\mathbb{N}}$be any sequence consisting of$1's$and $\infty's$.Write $1.1 = 1, 1.\infty = \infty. 1 = 1$.We define

$$supp(x(n))_{n\in\mathbb{N}} = \{n \in \mathbb{N}: x(n) = \infty\},$$

and write

$$T = \{\big(x(n)\big)_{n\in\mathbb{N}}: supp\big(x(n)\big)_{n\in\mathbb{N}}\ is\ finite\ and\ non-empty\}.$$

A commutative standard oid is the set T together with the product xy defined in T if and only if$(supp\ x) \cap (supp\ y) = \emptyset$ to be $(x(n)y(n))$. Thus the product $x(n)y(n)$ is required to be defined if and only if either $x(n)$or$y(n)$ is 1. Obviously, the product in T is associative where defined and $supp(xy) = (supp\ x) \cup (supp\ y)$ whenever xy is defined in T (oids are discussed in [7]). Any commutative standard oid T can be considered as$\bigoplus_{n=1}^{\infty}\{1,\infty\}\backslash \{(1,1,\dots,1)\}$. We use epithet “standard”to indicate that the index set is $\mathbb{N}$(in [7],oids could have any index set). For $x, y \in T$,$supp\ x < supp\ y$ means that $n < m$ if $n \in supp\ x$ and $m \in supp\ y$, and $supp\ x_{\alpha} \to \infty$ for some net (x_{α}) in T will mean that for arbitrary $k \in \mathbb{N}$ eventually $\min(supp\ x_{\alpha}) > k$. Then for a fixed $t \in T$, eventually $supp\ t < supp\ x_\alpha$ and so eventually tx_α is defined in T.Write $u_n = (1,1,\dots,\infty,1,1,\dots)$ (with ∞ in the *n*th place). Put $= \{u_n: n \in \mathbb{N}\}$. Then U is countable subset of T. Moreover, any $x \in T$ can be written uniquely as a finite product $x = u_{i_1}u_{i_2}\dots u_{i_k}$ with $i_1 < i_2 < \cdots < i_k$,$supp\ x = \{i_1,\dots,i_k\}$.The compact space βT is the Stone-Čech compactification of the discrete space T and if f maps T to some compact space,f^β is the unique continuous extension of f to βT.We define

$$T^\infty = \{\mu \in \beta T:\ \mu = lim_\alpha x_\alpha\ with\ supp\ x_\alpha \to \infty\}.$$

Obviously, $T \cap T^\infty = \emptyset$. For $\mu \in \beta T, \nu \in T^\infty$ the product $\mu\nu \in C(T)^*$ is defined by $\mu\nu = \mu \circ L_\nu$, where $L_\nu f(t) = \lim_\beta f(ty_\beta)$, if $t \in T$,$f \in C(T)$ and $y_\beta \to \nu$ with $supp\ y_\beta \to \infty$. Then $L_\nu f \in C(T)$,$L_\nu f(t) = (L_t f)^\beta(\nu)$. Further,$L_\nu$ is a bounded linear operator on $C(T)$. Of course $\mu \in \beta T$ is a bounded linear functional on $C(T)$,with$\|\mu\| \le 1$,if$\mu(f) = f^\beta(\mu)$. In fact, the product $(\mu,\nu) \to \mu\nu: \beta T \times T^\infty \to T^\infty$ is defined and is right continuous,and left continuity holds when $\mu = t \in T$[1].Also $\mu\nu = lim_\alpha lim_\beta x_\alpha y_\beta$ where (x_α) is a net in T with $x_\alpha \to \mu$. If $\mu \in T^\infty$, then $L_{\mu\nu} = L_\mu \circ L_\nu$, so that $(\mu,\nu) \to \mu\nu: T^\infty \times T^\infty \to T^\infty$ is a binary operation on T^∞ relative to which that T^∞ is a compact right topological semigroup. If $\subseteq T$, then 1_A denotes the indicator function of A,that is,the function whose value is 1 on A and 0 on $T\backslash A$.

Remark2.1.*For $\nu \in T^\infty$ and $\mu \in \beta T$, $\nu\mu$ can not always be defined in a standard oid T, if we require that multiplication is right continuous. This is true even if $\mu \in T$. If $z_n = u_1u_2\dots u_n$, $n \in \mathbb{N}$, $u_n \in U$ and $z_{n_i} \to \lambda \in \beta T$ for some subnet $(z_{n_i})$of(z_n), then for any $t \in T$, $\lim_i tz_{n_i}$ is not defined .But we can define $\nu\mu$ for standard oids only on a subset of $T^\infty \times \beta T$. This subset includes $T^\infty \times (T \cup T^\infty)$. Now let $x_\alpha \to \mu$ in βT with $supp\ x_\alpha \to \infty$ and let $\lambda = t\lambda'$ where $t \in T$,$\lambda' \in T^\infty$such that $y_\beta \to \lambda'$ with $supp\ y_\beta \to \infty$. Then eventually $supp\ t < supp\ x_\alpha$ and for such α,eventually $supp\ x_\alpha < supp\ y_\beta$, so that eventually $tx_\alpha y_\beta$ is defined in T and hence $lim_\alpha lim_\beta(tx_\alpha y_\beta) = t\mu\lambda'\ (= \mu t\lambda')$ (see*[1]*,Definition* 3.5*). Therefore ,we can defined $\mu\lambda$ on $T^\infty \times (T \cup T^\infty \cup TT^\infty)$, whenever $(T \cup T^\infty \cup TT^\infty)$ is a suitable subspace of βT for a wide range.*

Definition2.2.(*i*) The **cardinal function** is the map $c: T \to \mathbb{N}$ given by $c(x) = card(supp\ x)$ (that is, the number of elements of the support of x).Then c extends to a unique continuous extension c^β from βT into the one-point compactification $\mathbb{N} \cup \{\infty\}$. If $(supp\ x) \cap (supp\ y) = \emptyset$ so that xy is defined in ,$(xy) = c(x) + c(y)$, and so for$\mu \in \beta T, \nu \in T^\infty$ then $c^\beta(\mu\nu) = c^\beta(\mu) + c^\beta(\nu)$. Thus c^β is a homomorphism on T^∞. We denote $1/c(x)$ by $h(x)$ for $x \in T$.

(*ii*) The **length function** is the map $l: T \to \mathbb{N}$ by letting $l(x)$ (The length Of support of x) be the integer $i_k - i_1 + 1$ where $supp\ x = \{i_1,\dots,i_k\}$.

Then l extends to a unique continuous extension l^β from βT into the one-point compactification $\mathbb{N} \cup \{\infty\}$. We denote $1/l(x)$ by $r(x)$ for $x \in T$.
(*iii*) The ***z*-function** is the map $z: T \to \mathbb{Z}^+$ by letting $z(x)$ be the largest number of consecutive $1's$ between $\min(supp\ x)$ and $\max(supp\ x)$. Then z extends to a unique continuous extension z^β from βT into the one-point compactification $\mathbb{Z}^+ \cup \{\infty\}$. We denote $1/z(x)+1$ by $k(x)$ for $\in T$.

We next have some useful results which we will need later.

Proposition2.3.*For $\mu \in T^\infty, \nu \in (T \cup T^\infty \cup TT^\infty)$ then $l^\beta(\mu\nu) = \infty$.*
Proof. Let $\nu = t \in T$,and let $x_\alpha \to \mu$ for some net (x_α) in T with $supp\ x_\alpha \to \infty$. Then eventually $supp\ t < supp\ x_\alpha$, so that eventually $l(tx_\alpha) = \infty$. Since $tx_\alpha \to t\mu$ in βT and l^β is continuous onβT,from which it follows that $l^\beta(\mu t) = l^\beta(t\mu) = \infty$. If$\nu \in T^\infty$, and $y_\beta \to \nu$ for some net (y_β)in T with $supp\ y_\beta \to \infty$,then $l^\beta(\mu\nu) = lim_\alpha lim_\beta l(x_\alpha y_\beta) = \infty$, by a similar reason. Suppose that $\nu = t\lambda, \lambda \in T^\infty$. Then $\mu\lambda \in T^\infty$,sinceT^∞ is a semigroup,
Hence $l^\beta(\mu\nu) = l^\beta(\mu t\lambda) = l^\beta(t\mu\lambda) = \infty$ and the result follows. □

The next result is an immediate consequence of Definition2.2(*ii*), Proposition2.3.

Corollary2.4. *For $\mu \in T^\infty, \nu \in (T \cup T^\infty \cup TT^\infty)$. Then $r^\beta(\mu\nu) = 0$.*

Proposition2.5.*Let $\mu \in T^\infty, \nu \in (T \cup T^\infty \cup TT^\infty)$. Then $z^\beta(\mu\nu) = \infty$.*

Proof. This uses Definition 2.2(*iii*),the proof is parallel to that of Proposition 2.3. □

Corollary2.6. *Let $\mu \in T^\infty, \nu \in (T \cup T^\infty \cup TT^\infty)$. Then $k^\beta(\mu\nu) = 0$.*
Proof is straightforward. □

3. Space of jointly continuous functions

Our aim of the present section is to introduce a new kind of C^*-subalgebra of the C^*-algebra $C(T)$. In this section we try to find an analogue of almost periodic functions for oids.
Definition3.1. Let T be a commutative standard oid. We use $\mathcal{U}^\infty(T)$ to denote the set of all bounded complex valued functions on T for which $(\mu,\nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is jointly continuous. Clearly $\mathcal{U}^\infty(T)$ is conjugate closed and contains all constant functions.

Example3.2.(*i*) *Let $h = 1/c$ be as in Definition 2.2*(*i*). *Then by a routine* *argument we see that for* $\mu \in T^{\infty}$,$\nu \in (T \cup T^{\infty} \cup TT^{\infty})$, $c^{\beta}(\mu\nu) = c^{\beta}(\mu) + c^{\beta}(\nu)$, *and so* $(\mu,\nu) \to h^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \to \mathbb{C}$ *is jointly continuous. Therefore* $h \in \mathcal{U}^{\infty}(T)$.

(ii)Let $r = 1/l$ *be as in Definition 2.2(ii). Then by Corollary 2.4,*$r^{\beta}(\mu\nu) = 0$ *for* $\mu \in T^{\infty}$ *and* $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$, *and so* $(\mu,\nu) \to r^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \to \mathbb{C}$ *is jointly continuous . Thus* $r \in \mathcal{U}^{\infty}(T)$.

(iii) Let $k = 1/z + 1$ *be as in Definition 2.2(iii).Then by Corollary 2.6,* $k^{\beta}(\mu\nu) = 0$ *for* $\mu \in T^{\infty}$, $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$, *and so*$(\mu,\nu) \to k^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \to \mathbb{C}$ *is jointly continuous ,hence* $k \in \mathcal{U}^{\infty}(T)$.

Lemma3.3.$\mathcal{U}^{\infty}(T)$ *is a* C^**-sub algebra of the* C^**-algebra* $C(T)$.

Proof. It is easily seen that $\mathcal{U}^{\infty}(T)$ is a subalgebra of the algebra $C(T)$. To prove that $\mathcal{U}^{\infty}(T)$ is a C^*-subalgebra it is enough to prove that $\mathcal{U}^{\infty}(T)$ is a closed subalgebra of $C(T)$ because the other conditions are satisfied easily. For this purpose, let $(f_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{U}^{\infty}(T)$, $f \in C(T)$ with $\|f_n - f\| \to 0$, as $n \to \infty$.Suppose that $\mu_\alpha \to \mu$ in T^{∞}, $\nu_\alpha \to \nu$ in $(T \cup T^{\infty} \cup TT^{\infty})$.Then given $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $\|f_n - f\| < \varepsilon/3$ for all $n \geq k$. Fix $n_0 > k$. Then choose α_0 such that $\alpha > \alpha_0$,
$\left|f_{n_0}^{\beta}(\mu_\alpha\nu_\alpha) - f_{n_0}^{\beta}(\mu\nu)\right| < \varepsilon/3$. For such α, then

$$\begin{aligned}\left|f^{\beta}(\mu_\alpha\nu_\alpha) - f^{\beta}(\mu\nu)\right| &\leq \left|f^{\beta}(\mu_\alpha\nu_\alpha) - f_{n_0}^{\beta}(\mu_\alpha\nu_\alpha)\right| + \left|f_{n_0}^{\beta}(\mu_\alpha\nu_\alpha) - f_{n_0}^{\beta}(\mu\nu)\right| \\ &\quad + \left|f_{n_0}^{\beta}(\mu\nu) - f^{\beta}(\mu\nu)\right| \\ &\leq \left\|f^{\beta} - f_{n_0}^{\beta}\right\| + \left|f_{n_0}^{\beta}(\mu_\alpha\nu_\alpha) - f_{n_0}^{\beta}(\mu\nu)\right| + \left\|f_{n_0}^{\beta} - f^{\beta}\right\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Hence $\lim_\alpha f^{\beta}(\mu_\alpha\nu_\alpha) = f^{\beta}(\mu\nu)$ and so $(\mu,\nu) \to f^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$ is jointly continuous, as desired. □

Our next result will be useful in later.

Theorem3.4.*Let*$f \in \mathcal{U}^{\infty}(T)$,$\eta \in T^{\infty}$. *Then* $L_\eta f \in \mathcal{U}^{\infty}(T)$.

Proof. It is easy to check that $\nu\eta \in (T \cup T^{\infty} \cup TT^{\infty})$ whenever $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$. From this and that the product $(\mu,\nu) \to \mu\nu: \beta T \times T^{\infty} \to \beta T$ is right continuous, it follows that the map $\nu \to \nu\eta$ is continuous of$(T \cup T^{\infty} \cup TT^{\infty})$ into itself. Therefore the composite map $(\mu,\nu) \to (\mu,\nu\eta) \to f^{\beta}(\mu\nu\eta)$ is continuous from$T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$ to $\mathbb{C}$ for each $f \in \mathcal{U}^{\infty}(T)$.Thus

$$f^{\beta}(\mu\nu\eta) = \mu\nu\eta(f) = \mu\nu\circ L_\eta(f) = \mu\nu\big(L_\eta f\big) = \big(L_\eta f\big)^{\beta}(\mu\nu).$$

It follows that,$(\mu,\nu) \to \big(L_\eta f\big)^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \to \mathbb{C}$ is continuous, and therefore by Definition 3.1,$L_\eta f \in \mathcal{U}^{\infty}(T)$,as desired. □

Definition3.5. For $f \in \mathcal{U}^{\infty}(T)$ and $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$, we define $L_\nu f^{\beta}(\mu) = f^{\beta}(\mu\nu)$, where $\mu \in T^{\infty}$.

Remark3.6.*It is easy to check that* $L_\nu f^{\beta}$ *is continuous on compact space*

T^∞ *, since* $(\mu,\nu) \to \mu\nu: \beta T \times T^\infty \to \beta T$ *is right continuous and left continuity holds when* $\mu = t \in T$*.Moreover,* L_ν *is a bounded linear operator with* $\|L_\nu\| \leq 1$.

Theorem3.7.*Let* $f \in C(T)$*. Then* $(\mu,\nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ *is jointly continuous if and only if* $\nu \to L_\nu f^\beta: (T \cup T^\infty \cup TT^\infty) \to C(T^\infty)$ *is norm continuous.*

Proof. Define $\psi: T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ by $\psi(\mu,\nu) = f^\beta(\mu\nu)$. Then ψ Is a bounded function, sincef^β is continuous on βT.It follows readily that $\mu \to \psi(\mu,\nu): T^\infty \to \mathbb{C}$ is a continuous function for each $\nu \in (T \cup T^\infty \cup TT^\infty)$. Let $C(T^\infty)$ have the uniform norm. Since $T^\infty$is a compact space and $(T \cup T^\infty \cup TT^\infty)$ is a subspace of βT,ψ is jointly continuous if and only if the mapping $\nu \to \psi(0,\nu): (T \cup T^\infty \cup TT^\infty) \to C(T^\infty)$ is continuous (see[10] , Chapter 1, Lemma 1.8(a)). □

Lemma3.8.*If* $\nu \to L_\nu f^\beta: T^\infty \to C(T^\infty)$ *is norm-continuous, then* $\{L_\nu f^\beta:\ \nu \in T^\infty\}$ *is relatively norm compact in*$C(T^\infty)$

Proof is straightforward. □

The next result is an immediate consequence of Definition 3.1,Theorem 3.7 and Lemma3.8.

Corollary3.9. *Let* $f \in \mathcal{U}^\infty(T)$*.Then* $\{L_\nu f^\beta:\ \nu \in T^\infty\}$ *is a norm relatively compact in* $C(T^\infty)$*.*

4. Compact topological semigroups

In this section by starting with $\mathcal{U}^\infty(T)$we will produce a new compact Commutative topological semigroup,and make an investigation of it's properties. Assume that τ is the topology induced on a compact right topological Semigroup $T^\infty$by βT and $\tau_{\mathcal{U}^\infty(T)}$ is the weak topology induced on $T^\infty$by the family$\{f^\beta:\ f \in \mathcal{U}^\infty(T)\}$.Then the identity map from (T^∞,τ)onto $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$ is continuous, thus $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$is compact [9].

Definition4.1. For$\mu,\nu \in T^\infty$,define $\mu\ \mathcal{R}_{\mathcal{U}^\infty(T)}\nu$ if and only if$f^\beta(\mu) = f^\beta(\nu)$ For all $f \in \mathcal{U}^\infty(T)$. Clearly $\mathcal{R}_{\mathcal{U}^\infty(T)}$is a closed relation on $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$.

Remark4.2.*It should be noted from Definition 3.1,It follows that if* $f \in \mathcal{U}^\infty(T)$*,* $(\mu,\nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty) \to \mathbb{C}$ *is separately continuous, so that* $f^\beta(\mu\nu) = f^\beta(\nu\mu)$ *for all* $\mu,\nu \in T^\infty$*. Therefore* $f \in W_1^\infty(T)$*(see [1], for details).*

Proposition4.3. $\mathcal{R}_{\mathcal{U}^\infty(T)}$*Is a congruence relation on* T^∞.

Proof. To prove that $\mathcal{R}_{\mathcal{U}^\infty(T)}$is congruence, we use Remark 4.2 and Theorem 3.4. Let $\mu\ \mathcal{R}_{\mathcal{U}^\infty(T)}\mu'$and $\nu\ \mathcal{R}_{\mathcal{U}^\infty(T)}\nu'$where $\mu,\nu,\mu',\nu' \in T^\infty$.

Pick $f \in \mathcal{U}^\infty(T)$. Then

$$f^\beta(\mu\nu) = (L_\nu f)^\beta(\mu) = (L_\nu f)^\beta(\mu') = f^\beta(\mu'\nu) = f^\beta(\nu\mu') = \left(L_{\mu'} f\right)^\beta(\nu)$$
$$= \left(L_{\mu'} f\right)^\beta(\nu') = f^\beta(\nu'\mu') = f^\beta(\mu'\nu')$$

Thus $\mu\nu\ \mathcal{R}_{\mathcal{U}^\infty(T)}\mu'\nu'$, as claimed. □

Proposition4.4.$(T^\infty,\tau_{\mathcal{U}^\infty(T)})$ *Is a compact topological semigroup.*

Proof. We know that $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$ is a compact space. Now let (μ_α)be a net in T^∞ converging to μ in$(T^\infty,\tau_{\mathcal{U}^\infty(T)})$. Then $\mu_{\alpha_\beta} \to \mu_0$in (T^∞,τ) for some subnet (μ_{α_β}) of (μ_α). Since identity map from (T^∞,τ) onto $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$ is continuous, it follows that $\mu_{\alpha_\beta} \to \mu_0$ in $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$.Hence for each $f \in \mathcal{U}^\infty(T)$, $f^\beta(\mu_0) = \lim_\beta f^\beta(\mu_{\alpha_\beta}) = f^\beta(\mu)$. So $\mu_0\ \mathcal{R}_{\mathcal{U}^\infty(T)}\mu$. Similarly if $\nu_\alpha \to \nu$ and $\nu_{\alpha_\beta} \to \nu_0$ in $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$ then $\nu_0\ \mathcal{R}_{\mathcal{U}^\infty(T)}\nu$. Hence, as $\mathcal{R}_{\mathcal{U}^\infty(T)}$is a congruence relation on T^∞ by Proposition 4.3, then $\mu_0\nu_0\ \mathcal{R}_{\mathcal{U}^\infty(T)}\mu\nu$, so that $f^\beta(\mu_0\nu_0) = f^\beta(\mu\nu)$ for all $f \in \mathcal{U}^\infty(T)$. Now let $\mu_\alpha \to \mu, \nu_\alpha \to \nu$in $(T^\infty,\tau_{\mathcal{U}^\infty(T)})$, let $(\mu_{\alpha_\beta}\nu_{\alpha_\beta})$ be a subnet of $(\mu_\alpha\nu_\alpha)$. Using compactness of (T^∞,τ) we find subnets$(\mu_{\alpha_{\beta_\gamma}})$, $(\nu_{\alpha_{\beta_\gamma}})$ with $\mu_{\alpha_{\beta_\gamma}} \to \mu_0$, $\nu_{\alpha_{\beta_\gamma}} \to \nu_0in(T^\infty,\tau)$. Then for each $f \in \mathcal{U}^\infty(T)$, we have $\lim_\gamma f^\beta(\mu_{\alpha_{\beta_\gamma}}\ \nu_{\alpha_{\beta_\gamma}}) = f^\beta(\mu_0\nu_0) = f^\beta(\mu\nu)$, since $(\mu,\nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is jointly continuous, hence $\mu_\alpha\nu_\alpha \to \mu\,\nu$, as required. □

Corollary4.5.*Let the quotient semigroup* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$*have the quotient topology. Then* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$*is compact Hausdorff topological semigroup.*

Proof. Use Definition 4.1 and Proposition 4.3. □

Corollary4.6. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *is a commutative semigroup.*

Proof. Take $\mu,\nu \in T^\infty$and $f \in \mathcal{U}^\infty(T)$. Then $f^\beta(\mu\nu) = f^\beta(\nu\mu)$(Remark 4.2),and thus $\mu\nu\ \mathcal{R}_{\mathcal{U}^\infty(T)}\nu\mu$, which implies the assertion. □

We conclude this section with some results (both algebraic, topological) on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$.

Theorem 4.7.$K(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})$*the minimal ideal of* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *Is compact topological group.*

Proof. This follows from Corollaries 4.5,4.6 and Corollary 1.5.3[3]. □

Remark4.8. *For each $n \in \mathbb{N}$, write $H_n = \{\mu \in T^\infty : c^\beta(\mu) = n\}$ and $H_\infty = \{\mu \in T^\infty : c^\beta(\mu) = \infty\}$. Then $T^\infty = H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots \cup H_\infty$. Hence H_n is clopen and each $\mu \in H_n$ is a limit of a net (x_α) in T with $c(x_\alpha) = n$ for each α. Further, $H_n H_m \subseteq H_{n+m}$ for all $m, n \in \mathbb{N}$, so $H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots$ is a sub semigroup of T^∞. Recall that by Definition 2.2(ii), l is the length function and $r(x) = 1/l(x)$, $x \in T$.*

Lemma4.9. *Let $\xi \in H_n$ for some $n \in \mathbb{N}$ with $l^\beta(\xi) < \infty$, let $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ be the quotient map. Then $\pi(\xi)$ is not a product.*

Proof. Let $g_l \pi = l^\beta$. Then g_l is a continuous functions on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$, since $r = 1/l$ and $r \in \mathcal{U}^\infty(T)$ (Example 3.2 (ii)) and that $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ have the quotient topology (see [9], Chapter 3, Theorem 9). If $\pi(\xi) = \pi(\xi_1)\pi(\xi_2)$ for some $\xi_1, \xi_2 \in T^\infty$, then $l^\beta(\xi) = l^\beta(\xi_1\xi_2) = \infty$(Proposition2.3), which contradicts $l^\beta(\xi) < \infty$. □

Theorem4.10. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *has no identity.*

Proof. If $\pi(e)$ is an identity element for $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$, where $e \in T^\infty$, then $\pi(\xi) = \pi(e)\pi(\xi) = \pi(\xi)\pi(e)$ for all $\xi \in T^\infty$, which is impossible by Lemma 4.9.

Remark 4.11. *It is easy to verify that, $l^\beta(e) = \infty$, whenever $\pi(e)$ is an idempotent in $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$. We denote the set of all idempotents in $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ by $E(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})$. Thus we obtain that $E(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}) \subseteq \{\pi(\xi): \xi \in T^\infty, l^\beta(\xi) = \infty\}$.*

Proposition4.12. *Let $\xi \in H_n$ for some $n \in \mathbb{N}$ with $l^\beta(\xi) < \infty$. Then $\pi(\xi)$ Is not a left zero.*

Proof. Left zeros are idempotents and we saw above that $l^\beta(\xi) = \infty$ if ξ is an idempotent. □

We next have the following theorem.

Theorem4.13. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *is not a left zero semigroup.*

Proposition4.14. $(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})^2$ *is not dense in* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$.

Proof. Let $g_r \pi = r^\beta$, where $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is the quotient map. Then g_r is continuous on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$, since $r \in \mathcal{U}^\infty(T)$(Example 3.2 (ii)), and that $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ have the quotient topology. By Corollary 2.4,

$$(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})^2 \cap g_r^{-1}(0,1) = \emptyset$$

But $g_r^{-1}(0,1)$ is a non-empty open set in $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$, as claimed. □

Remark4.15. Let $\mu_0 \in T^\infty$ be the cluster point of the sequence $(u_n u_{n+3})_{n=1}^{\infty}$ in βT, where $u_n \in U$ for all $n \in \mathbb{N}$, let $g_r \pi = r^\beta$ (Proposition 4.14). Then $r^\beta(\mu_0) = 1/4$ and since by Corollary 2.4, $r^\beta((T^\infty)^2) = 0$ which implies that $\pi(\mu_0)$ is not the limit of a net of elements $(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})^2$. Thus we obtain an alternative proof of the Proposition 4.14 which will be required in the next result.

Theorem4.16. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *is not a left (resp, right) simple semigroup.*

Proof. Indeed, $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}\pi(\mu_0) \subseteq (T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})^2$, and $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}\pi(\mu_0)$ is closed in $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$. Thus $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}\pi(\mu_0) \neq T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ by Proposition 4.14, so $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is not a left simple semigroup. Thus $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is not a group (see [3], for more details). □

From Theorem 4.16 and Definition 1.5.6[3],we get the following result.

Corollary4.17. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *is not topologically left (resp, right) simple.*

Corollary4.18. $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *is not cancellative (and hence is not group).*
Proof. Use Theorem 4.16 and Corollaries 3.13,3.14[3]. □

Remark4.19. *If for each* $k \in \mathbb{N}$, *let* $x_m^{(k)} = u_m u_{m+1} \dots u_{m+k-1}$, $m \in \mathbb{N}$ *and* $y_n = u_n u_{n+1} \dots u_{n^2}$. $n \in \mathbb{N}$, *where* $u_n \in U$ *for all* n, *then* $supp\ x_m^{(k)} \to \infty$, $supp\ y_n \to \infty$. *Let* $\mu^{(k)}, \nu \in T^\infty$ *be the cluster points of* $(x_m^{(k)})_{m=1}^{\infty}, (y_n)_{n=1}^{\infty}$ *In* βT *respectively. Then* $l^\beta(\mu^{(k)}) = k$, $l^\beta(\nu) = \infty$. *Now, suppose that* $g_l \pi = l^\beta$ *(Lemma 4.9). Then* $g_l \pi(\mu^{(k)}) = l^\beta(\mu^{(k)}) = k$, $g_l \pi(\nu) = l^\beta(\nu) = \infty$, *which implies that* g_l *and* l^β *map* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *and* T^∞ *onto the one-point compactification* $\mathbb{N} \cup \{\infty\}$ *respectively.*

Next we shall prove the following result.

Proposition 4.20. $(g_l^{-1}(\infty))^2$ *is not dense in* $g_l^{-1}(\infty)$ *(and hence isnot dense in* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$*).*

Proof. Let k be as in Definition 2.2 (iii). Define $g_k \pi = k^\beta$, where $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is the quotient map. Then g_k is a continuous function on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$, since $k \in \mathcal{U}^\infty(T)$ (Example 3.2 (iii)). Suppose now that, $(y_n)_{n=1}^{\infty}, \nu \in T^\infty$ be as in Remark 4.19. Then $\pi(\nu) \notin g_l^{-1}(\infty)$, since $l^\beta(\nu) = \infty$ and $g_l \pi = l^\beta$. On the other hand, $k^\beta(\nu) = 1$, since $z(y_n) = 0$ (see Definition 2.2 (iii)). But $(g_l^{-1}(\infty))^2 \subseteq (T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})^2$, an application of Corollary 2.6 then shows that $g_k(g_l^{-1}(\infty))^2 =$

0.Hence, $\pi(\nu) \notin cl(g_l^{-1}(\infty))^2$, which implies the desired conclusion.

Recall that by Definition 2.2(*i*),$c^\beta$is a continuous homomorphism on T^∞, and $h = 1/c$, $h \in \mathcal{U}^\infty(T)$(Example 3.2 (i)). Let $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$be the quotient map, and let $g_c\pi = c^\beta$. Then $g_c$is a continuous homomorphism on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$. Let $\mu^{(k)}, \nu$be as in Remark 4.19. Then $c^\beta(\mu^{(k)}) = k$, $c^\beta(\nu) = \infty$, and so we obtain that g_c maps $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ onto the one-point compactification $\mathbb{N} \cup \{\infty\}$. □

The proof of the following proposition is essentially the same as that of Proposition4.20.

Proposition 4.21.$(g_c^{-1}(\infty))^2$*is not dense in* $g_c^{-1}(\infty)$ *(and hence is not dense in* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$*).*

Theorem 4.22.*Let* $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ *be the quotient map. Then the set* $\{\pi(\xi):\ \xi \in T^\infty, l^\beta(\xi) < \infty\}$ *is not dense in* $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$*.*

Proof. Let $(y_n)_{n=1}^\infty$, $\nu \in T^\infty$ be as in Remark 4.19. Put $Y = \{y_n:\ n \in \mathbb{N}\}$and let 1_Y be the indicator function of Y. Then $1_Y^\beta(\nu) = 1$.We complete the proof by showing that $1_Y \in \mathcal{U}^\infty(T)$. To see this, suppose that $\mu \in T^\infty$, $\eta \in (T \cup T^\infty \cup TT^\infty)$ and let $x_\alpha \to \mu$ for some net (x_α) in T with $supp\ x_\alpha \to \infty$. If $\eta = t \in T$, then eventually $supp\ t < supp\ x_\alpha$ and for such α, eventually $supp\ x_\alpha < supp\ y_n$,so that eventually $tx_\alpha \notin Y$.Hence$1_Y^\beta(t\mu) = 0$.If $\eta \in T^\infty$and $y_\beta \to \eta$ for some net(y_β)in$T$with $supp y_\beta \to \infty$, then $1_Y^\beta(\mu\nu) = lim_\alpha lim_\beta 1_Y(x_\alpha y_\beta) = 0$ by a similar reason. Finally, if $\eta = t\lambda$, $\lambda \in T^\infty$. Then $\mu\lambda \in T^\infty$,sinceT^∞ is a semigroup and hence $1_Y^\beta(\mu\eta) = 1_Y^\beta(t\mu\lambda) = 0$.Consequently, $1_Y \in \mathcal{U}^\infty(T)$. Let$g_{1_Y}\pi = 1_Y^\beta$.Then$g_{1_Y}$is continuous on $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$. Take $\xi \in T^\infty$ with$l^\beta(\xi) = k$, $k \in \mathbb{N}$. There exists a net(z_γ) in T such that$z_\gamma \to \xi$ with $supp\ z_\gamma \to \infty$. It follows that,eventually $l(z_\gamma) = k$,hence eventually$z_\gamma \notin Y$. Therefore $1_Y^\beta(\xi) = 0$, so$g_{1_Y}\pi(\xi) = 0$. On the other hand,$g_{1_Y}\pi(\nu) = 1_Y^\beta(\nu) = 1$ and hence $\pi(\nu) \notin cl\{\pi(\xi):\ \xi \in T^\infty, l^\beta(\xi) < \infty\}$, which implies the desired conclusion. □

For remainder of this section we consider the more general results of deseribing about inclusion between $\mathcal{U}^{\infty}(T)$and$WAP^{\infty}(T)$(see[1],for more details).

Theorem 4.23.*For $f \in C(T)$, if $(\mu,\nu) \to f^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \to \mathbb{C}$ is separately continuous,then$(\mu,\nu,\eta) \to f^{\beta}(\mu\nu\eta): T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty}) \times T^{\infty} \to \mathbb{C}$ is also separately continuous.*

Proof. Let(μ_{α}) is a net inT^{∞}converging to$\mu \in T^{\infty}$ and let $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$, $\eta \in T^{\infty}$.

(*i*) If $\nu \in T^{\infty}$,then as is readily verified that$f^{\beta}(\mu_{\alpha}\nu\eta) \to f^{\beta}(\mu\nu\eta)$,since $\nu\eta \in T^{\infty}$and$(\mu,\nu) \to \mu\nu: \beta T \times T^{\infty} \to \beta T$ is right continuous.

(*ii*) If$\nu = t\lambda$, $t \in T$and$\lambda \in T^{\infty}$.Then$\lambda\eta \in T^{\infty}$since$T^{\infty}$is semigroup and $t\mu_{\alpha} \to t\mu$in$T$.Using Definition 3.5[1] and that $(\mu,\nu) \to \mu\nu: \beta T \times T^{\infty} \to \beta T$ is right continuous, it follows that $f^{\beta}(\mu_{\alpha}\nu\eta) \to f^{\beta}(\mu\nu\eta)$.

(*iii*) If $\nu = t \in T$,then by a similar argument,$f^{\beta}(\mu_{\alpha}\nu\eta) \to f^{\beta}(\mu\nu\eta)$.

Let(η_{α})isanetinT^{∞} converging to $\eta \in T^{\infty}$. Take $\mu \in T^{\infty}$, $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$.

(*i*) Let$\nu \in T^{\infty}$. Then $\mu\nu \in T^{\infty}$.In fact $\eta_{\alpha} \to \eta$ in $(T \cup T^{\infty} \cup TT^{\infty})$ and by hypothesis, $(\mu,\nu) \to f^{\beta}(\mu\nu)$ is separately continuous on $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$,from which it follows that $f^{\beta}(\mu\nu\eta_{\alpha}) \to f^{\beta}(\mu\nu\eta)$.

(*ii*) Let $\nu = t\lambda$, $t \in T$and$\lambda \in T^{\infty}$.Then $\mu\lambda \in T^{\infty}$. Indeed, $t\eta_{\alpha} \to t\eta$ in $(T \cup T^{\infty} \cup TT^{\infty}) \subseteq \beta T$ and $(\mu,\nu) \to f^{\beta}(\mu\nu)$ is separately continuous on $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$, hence$f^{\beta}(\mu\nu\eta_{\alpha}) \to f^{\beta}(\mu\nu\eta)$.

(*iii*) Let$\nu = t \in T$.The proof is similar.

Finally, suppose that (ν_{α}) is a net in$(T \cup T^{\infty} \cup TT^{\infty})$ converging to $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$,and let $\mu, \eta \in T^{\infty}$. Then $\nu_{\alpha}\eta \to \nu\eta$in$(T \cup T^{\infty} \cup TT^{\infty})$.But $(\mu,\nu) \to f^{\beta}(\mu\nu)$ is separately continuous on $T^{\infty} \times (T \cup T^{\infty} \cup TT^{\infty})$,hence $f^{\beta}(\mu\nu_{\alpha}\eta) \to f^{\beta}(\mu\nu\eta)$. This proves our assertion. □

Theorem4.24.*For$f \in C(T)$, let both$(\mu,\nu) \to f^{\beta}(\mu\nu): T^{\infty} \times (T \cup T^{\infty}) \to \mathbb{C}$ and $(\mu,\nu,\eta) \to f^{\beta}(\mu\nu\eta): T^{\infty} \times (T \cup T^{\infty}) \times T^{\infty} \to \mathbb{C}$ are separately continuous. Then $f \in W_2^{\infty}(T)$.*

Proof. We use Theorem 3.11[1]. Suppose $\mu,\nu,\eta \in T^{\infty}$,and (x_{α})isanet in$T$converging to μ with $supp\, x_{\alpha} \to \infty$. Then $f^{\beta}(x_{\alpha}\eta\nu) \to f^{\beta}(\mu\eta\nu)$,since $(\mu,\nu) \to \mu\nu: \beta T \times T^{\infty} \to \beta T$ is right continuous, and $\eta\nu \in T^{\infty}$. By hypothesis, $(\mu,\nu) \to f^{\beta}(\mu\nu)$ is separately continuous on$T^{\infty} \times (T \cup T^{\infty})$,hence from Remark 4.2,and that T^{∞} is a semigroup we see that

$$f^{\beta}(\mu\eta\nu) = f^{\beta}(\nu\mu\eta) = f^{\beta}(\eta\nu\mu) \quad , \quad f^{\beta}(\eta\mu\nu) = f^{\beta}(\nu\eta\mu) = f^{\beta}(\mu\nu\eta).$$

On the other hand, $\lim_{\alpha} f^{\beta}(x_{\alpha}\eta\nu) = \lim_{\alpha} f^{\beta}(\eta x_{\alpha}\nu) = f^{\beta}(\eta\mu\nu)$, since $(\mu,\nu,\eta) \to f^{\beta}(\mu\nu\eta)$ is separately continuous on $T^{\infty} \times (T \cup T^{\infty}) \times T^{\infty}$ and $x_{\alpha} \to \mu$ in $(T \cup T^{\infty})$. Thus $f^{\beta}(\mu\eta\nu) = f^{\beta}(\eta\mu\nu)$, and we have that $f \in W_2^{\infty}(T)$, as desired. □

Definition 3.1and the preceding theorems now imply that the following result.

Corollary 4.25. *Let T be a commutative standard oid. Then $\mathcal{U}^{\infty}(T) \subseteq W_2^{\infty}(T)$.*

Corollary 4.26. *Let T be a commutative standard oid. Then $\mathcal{U}^{\infty}(T) \subseteq WAP^{\infty}(T)$.*

The proof now follow by the preceding Corollary,Remark4.2 and Definition 4.1[1]. □

Example 4.27. *We have already seen that $\mathcal{U}^{\infty}(T) \subseteq WAP^{\infty}(T)$. Now we show that the converse is not true. For this purpose, consider two sets $E = \{(x(n))_{n\in\mathbb{N}} \in T\colon\ x(1) = 1\}$ and $F = \{(x(n))_{n\in\mathbb{N}} \in T\colon\ x(1) = \infty\}$. Then $T = E \cup F$. Now let $(u_{2m})_{m\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ be two sequences in T such that $u_{2m}, u_{2n+1} \in U$ for all m,n. Take $u_1 \in U$. Put $D = \{u_1 u_{2m} u_{2n+1}\colon m > n\} \subseteq F$. Define $f\colon T \to \mathbb{C}$ by*

$$f(x) = \begin{cases} 1/c(x) & , \quad x \in E \\ 1 & , \quad x \in D \\ 0 & , \quad otherwise \end{cases} \qquad (x \in T)$$

Then $f \in W_1^{\infty}(T)$ (Example 3.7[1]). We show that $f \in W_2^{\infty}(T)$ (Definition 3.9[1]). Assume that $(x_{\alpha}), (y_{\beta})$ and (z_{γ}) be nets in T with supp $x_{\alpha} \to \infty$, supp $y_{\beta} \to \infty$ and supp $z_{\gamma} \to \infty$. Then (for all sufficiently large α,β,γ) $x_{\alpha}(1) = 1$, $y_{\beta}(1)$ and $z_{\gamma}(1) = 1$,so that $x_{\alpha}, y_{\beta}, z_{\gamma} \in E$. It is therefore easy To verify that $f \in W_2^{\infty}(T)$, since $h = 1/c \in W_2^{\infty}(T)$ (Example 3.10[1]) hence $f \in WAP^{\infty}(T)$. To prove that $f \notin \mathcal{U}^{\infty}(T)$, we remind the reader that for any $f \in \mathcal{U}^{\infty}(T)$, $f^{\beta}(t\mu\nu) = f^{\beta}(t\nu\mu)$ where $t \in T$ and $\mu,\nu \in T^{\infty}$. We assume that μ,ν be the cluster points of the sequences $(u_{2m})_{m\in\mathbb{N}}$, $(u_{2n+1})_{n\in\mathbb{N}}$ in βT

respectively. Then $\mu, \nu \in T^{\infty}$*,since* $supp\ u_{2m} \to \infty$*and*$supp\ u_{2n+1} \to \infty$*.*

Hence, $lim_m lim_n f(u_1 u_{2m} u_{2n+1}) = 0$*,* $lim_n lim_m f(u_1 u_{2m} u_{2n+1}) = 1$*, so that Both iterated limits of*$f(u_1 u_{2m} u_{2n+1})$*exist. But* $lim_m lim_n f(u_1 u_{2m} u_{2n+1}) = f^{\beta}(u_1 \mu\nu)$*,and* $lim_n lim_m f(u_1 u_{2m} u_{2n+1}) = f^{\beta}(u_1 \nu\mu)$*, which is clearly impossible. Thus* $f \notin \mathcal{U}^{\infty}(T)$*,asclaimed.* □

5. Idempotents

Recall that T^{∞}*is a compact right topological semigroup and*$\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ *is a relation on* T^{∞}*(Definition 4.1). Then* $\mathcal{R}_{\mathcal{U}^{\infty}(T)}$*is congruence and under certain topology on* T^{∞}*, is also a closed relation, so that the quotient space* $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ *is a compact topological semigroup (Corollary 4.5), which is commutative (Corollary 4.6). In this section we are concerned with obtaining of idempotents* $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$*. In connection with the present section special suboids of an oid* T*play an important role. Each special suboid corresponds to a strictly increasing sub sequence of* $\mathbb{N}$*.A more details analysis of special suboids can be found in*[1].

For an infinite subset$A \subseteq \mathbb{N}$*,the special suboid of an oid* T *corresponding to*

The strictly increasing sequence of A *is denoted by* $S(A)$*.Then* $S(A)$ *produces a compact right topological semigroup* $S^{\infty}(A)$ *which is a sub semigroup of* T^{∞}*.*

Indeed, $\pi(S^{\infty}(A))$ *is a compact commutative sub semigroup of* $T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$*, where* $\pi: T^{\infty} \to T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ *is the quotient map.*

Proposition5.1. *Let*$A \subseteq \mathbb{N}$ *be an in finite set and let* $1_{S(A)}$ *be the indicator function of* $S(A)$*. Then*$1^{\beta}_{S(A)}(\mu\nu) = 1^{\beta}_{S(A)}(\mu)1^{\beta}_{S(A)}(\nu)$ *for all* $\mu \in T^{\infty}$*and* $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$*.*

Proof. It is a straightforward argument that$1_{S(A)}(xy) = 1_{S(A)}(x)1_{S(A)}(y)$ For $x, y \in T$such that $(supp\ x) \cap (supp\ y) = \emptyset$. Suppose that $t \in T$,$\mu \in T^{\infty}$ such that$x_{\alpha} \to \mu$for some net (x_{α}) in T with $supp\ x_{\alpha} \to \infty$. Then eventually $supp\ t < supp\ x_{\alpha}$,so that eventually $1_{S(A)}(tx_{\alpha}) = 1_{S(A)}(t)1_{S(A)}(x_{\alpha})$.

Since$\{0,1\}$ is a compact commutative semigroup with the usual multiplication, and $t\mu = \lim_{\alpha} tx_{\alpha}$, it follows that $1^{\beta}_{S(A)}(t\mu) = 1_{S(A)}(t)1^{\beta}_{S(A)}(\mu)$. Now let $\nu \in T^{\infty}$ with $y_{\beta} \to \nu$ for some net (y_{β})in$T$such that $supp\ y_{\beta} \to \infty$.

Then by a similar reason, we see that

$$1^{\beta}_{S(A)}(\mu\nu) = lim_{\alpha} lim_{\beta} 1_{S(A)}(x_{\alpha}y_{\beta}) = lim_{\alpha} 1_{S(A)}(x_{\alpha})1^{\beta}_{S(A)}(\nu) = 1^{\beta}_{S(A)}(\mu)1^{\beta}_{S(A)}(\nu).$$

Finally if $\nu = t\lambda$, $t \in T$, $\lambda \in T^\infty$, then $\mu\lambda \in T^\infty$, and $\mu\nu = \mu t\lambda = t\mu\lambda$. Thus, by above paragraph, we obtain that $1^\beta_{S(A)}(\mu\nu) = 1^\beta_{S(A)}(\mu)1^\beta_{S(A)}(\nu)$, as desired. □

Corollary5.2. *For an infinite set $A \subseteq \mathbb{N}$, the indicator function $1_{S(A)}$ of $S(A)$ is in $\mathcal{U}^\infty(T)$.*

Proof. To show that $1_{S(A)} \in \mathcal{U}^\infty(T)$, it is adequate by Theorem 3.7, to show that $\nu \to L_\nu f^\beta : (T \cup T^\infty \cup TT^\infty) \to C(T^\infty)$ is norm-continuous. Assume that (ν_α) be any net in $(T \cup T^\infty \cup TT^\infty)$ such that $\nu_\alpha \to \nu$ in $(T \cup T^\infty \cup TT^\infty)$. Then $1^\beta_{S(A)}(\nu_\alpha) \to 1^\beta_{S(A)}(\nu)$. Thus for $\varepsilon > 0$, there exists α_0 such that for $\alpha \geq \alpha_0$, $\left|1^\beta_{S(A)}(\nu_\alpha) - 1^\beta_{S(A)}(\nu)\right| < \varepsilon/2$. However, by definition of $1_{S(A)}$, we have $1^\beta_{S(A)}(\mu) \in \{0,1\}$ for all $\mu \in T^\infty$, which is clearly that from Proposition 5.1, and for $\alpha \geq \alpha_0$,
$\left|1^\beta_{S(A)}(\mu)1^\beta_{S(A)}(\nu_\alpha) - 1^\beta_{S(A)}(\mu)1^\beta_{S(A)}(\nu)\right| = \left|1^\beta_{S(A)}(\mu)\right|\left|1^\beta_{S(A)}(\nu_\alpha) - 1^\beta_{S(A)}(\nu)\right| < \varepsilon/2$ $(\mu \in T^\infty)$. Hence, for $\alpha \geq \alpha_0$, $\left\|L_{\mu_\alpha} 1^\beta_{S(A)} - L_\mu 1^\beta_{S(A)}\right\| < \varepsilon$ and the proof of the corollary is complete. □

Proposition5.3. *If A, B be two infinite subsets of $\mathbb{N}$ with $A \cap B$ is finite. Then $\pi(S^\infty(A)) \cap \pi(S^\infty(B)) = \emptyset$, where $\pi : T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is quotient map.*

Proof. This uses Corollary 5.2, the proof is essentially the same as that of Proposition7.1[1]. □

We finish the present section by giving the following main result. In connection with this result we will use non-principal ultrafilters on $\mathbb{N}$. We remind the reader that, if $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$ and $A \in \mathcal{F}$, Then A is an infinite set. Moreover, the number of non-principal ultrafilters on $\mathbb{N}$ is 2^c [8].

Here is the main result of this section.

Theorem5.4. *$T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ contains at least 2^c idempotents.*

Proof. This uses Proposition5.3, the proof is parallel to that of Theorem 7.2[1]. □

6. Free abelian groups

In connection with Theorem 4.7, of the Section 4, it should be mentioned that, it is possible for a compact commutative topological semigroup $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ have a minimal idempotent with a unique minimal left ideal and a unique minimal right ideal, so that $K(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})$, the minimal ideal of $T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)}$ is a maximal group(see[3]*, for more details). The aim of this Section is the search for existence in $K(T^\infty/\mathcal{R}_{\mathcal{U}^\infty(T)})$ a free abelian group on 2^c generator. Let us first give the following result.*

Lemma6.1. *Let ψ is an arbitrary function from $\mathbb{N}$ to $\mathbb{R}$, let $f: T \to \mathbb{C}$ be defined by $f(t) = expi\sum\{\psi(n):\ n \in supp\ t\}$. Then $f \in \mathcal{U}^{\infty}(T)$.*

Proof. It is easily seen that $f^{\beta}(t\mu) = f(t)f^{\beta}(\mu)$ for $t \in T, \mu \in T^{\infty}$, since f is an oid-map (that is, $f(st) = f(s)f(t)$ whenever $s, t \in T$ with $(supp\ s) \cap (supp\ t) = \emptyset$). Moreover, f^{β} is a homomorphism of T^{∞} to the circle group $\mathbb{T}$. Thus $f^{\beta}(\mu\nu) = f^{\beta}(\mu)f^{\beta}(\nu)$ for $\mu \in T^{\infty}$, $\nu \in (T \cup T^{\infty} \cup TT^{\infty})$. To prove that $f \in \mathcal{U}^{\infty}(T)$, we show that $\nu \to L_{\nu}f^{\beta}: (T \cup T^{\infty} \cup TT^{\infty}) \to C(T^{\infty})$ is norm-continuous(Theorem 3.7). Now suppose that (ν_{α}) be any net in $(T \cup T^{\infty} \cup TT^{\infty})$ with $\nu_{\alpha} \to \nu$ in $(T \cup T^{\infty} \cup TT^{\infty})$.

Then $f^{\beta}(\nu_{\alpha}) \to f^{\beta}(\nu)$, hence for $\varepsilon > 0$, there exists α_0 such that for $\alpha \geq \alpha_0$, $|f^{\beta}(\nu_{\alpha}) - f^{\beta}(\nu)| < \varepsilon/2$. But since, $|f^{\beta}(\mu)| = 1$ for all $\mu \in \beta T$, it follows directly that for $\alpha \geq \alpha_0$ and for all $\mu \in T^{\infty}$,

$$|f^{\beta}(\mu)f^{\beta}(\nu_{\alpha}) - f^{\beta}(\mu)f^{\beta}(\nu)| = |f^{\beta}(\mu)||f^{\beta}(\nu_{\alpha}) - f^{\beta}(\nu)| < \varepsilon/2.$$

Thus, for $\alpha \geq \alpha_0$, $\|L_{\nu_{\alpha}}f^{\beta} - L_{\nu}f^{\beta}\| < \varepsilon$, that is $f \in \mathcal{U}^{\infty}(T)$. □

We remind the reader that, if $M = \{\mu \in T^{\infty}:\ c^{\beta}(\mu) = 1\}$, then it is easy to verify that $M = \{\mu \in T^{\infty}:\ \mu \in cl\{u_m:\ m \in \mathbb{N}\}\}$, and $card(M) = 2^{c}$ (see [2], Remark5.8). Recall that, $\pi: T^{\infty} \to T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)}$ is the quotient map which is a Continuous epimorphism. We now give the following result which is the last major result of this section.

Theorem6.2. *$K(T^{\infty}/\mathcal{R}_{\mathcal{U}^{\infty}(T)})$ contains a free abelian group on 2^{c} generators.*

Proof. This use Lemma6.1 and Corollary 4.26, the proof is similar to that of Theorem 8.1[1]. □

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