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On approximation process by certain modified Dunkl generalization of Szász-Beta type operators

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Abstract

In this paper, we give a generalization of the Szász-Beta type operators generated by Dunkl generalization of exponential function and obtain convergence properties of these operators by using Korovkin's theorem and weighted Korovkin-type theorem. We also establish the order of convergence by using the modulus of smoothness and the weighted modulus of continuity.

Keywords: Dunkl type generalization, Genuine-Szász beta operators, modulus of smoothness, Lipschitz functions.

2010 MSC: 41A10, 41A25, 41A36.

1. Introduction

Approximation theory is concerned with approximating functions of a given class using functions from another, usually more elementary, class. The theory of approximation of function is now an extremely extensive branch of mathematical analysis and this theory has very important applications in other branches. As it is known, linear positive operators play an important role in the study of approximation of functions. One of the best known of these operators is the Szász operator introduced by Szász [13] and it is as below:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!},\ x \geqslant 0,\ n \in \mathbb{N}.$$

This operator is a generalization of Bernstein polynomials to the infinite interval. Szász operators and their generalizations have been studied by many authors (see [1, 4–10, 12–15]). One of the generalizations of the Szász operator including parameters a_n and b_n was given by İspir and Atakut [7] as follows:

$$S_n^{a_n,b_n}(f;x) := S_n(f;a_n,b_n;x) = e^{-a_n x}\sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right)\frac{(a_n x)^k}{k!},\ x \geqslant 0,\ n \in \mathbb{N}, \tag{1.1}$$

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where $\{a_n\}, \{b_n\}$ are given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \ \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right). \tag{1.2}$$

Note that the parameters a_n and b_n have an important effect for a better approximation of the operator $S_n^{a_n,b_n}$. An example of this situation will be illustrated in following Figure 1.

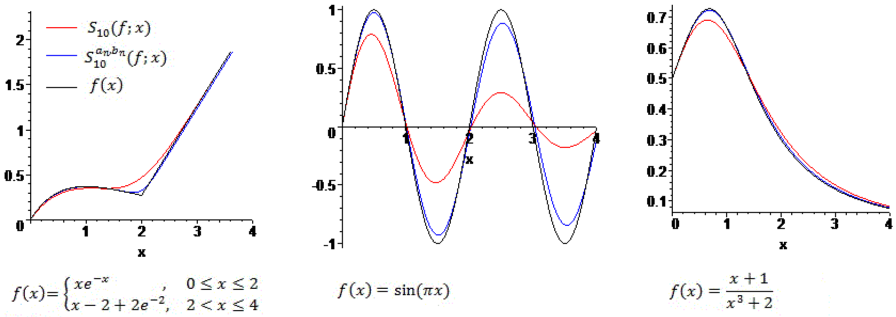

Figure 1: The effects of the a_n and b_n parameters on the approximation $(a_n = n^2, b_n = n^2 + 1)$.

For $\nu, x \in [0, \infty)$ and $f \in C[0, \infty)$, in [12], Sucu introduced the following generalization of the Szász operators (later, it was called as Dunkl analogue of Szász operators) by using the generalization of the exponential function e_ν given in [11]:

$$S_n^*(f; x) = \frac{1}{e_\nu(nx)} \sum_{k=0}^{\infty} f\left(\frac{k + 2\nu\theta_k}{n}\right) \frac{(nx)^k}{\gamma_\nu(k)},$$

where

$$e_\nu(x) = \sum_{r=0}^{\infty} \frac{x^r}{\gamma_\nu(r)}, \tag{1.3}$$

and for $r \in \mathbb{N}_0$ and $\nu > -\frac{1}{2}$, the coefficients γ_ν are given by:

$$\gamma_\nu(2r) = \frac{2^{2r} r! \Gamma\left(r + \nu + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)}, \quad \gamma_\nu(2r+1) = \frac{2^{2r+1} r! \Gamma\left(r + \nu + \frac{3}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)}, \tag{1.4}$$

where the function Γ is well-known gamma function and also for $p \in \mathbb{N}$,

$$\theta_r = \begin{cases} 0, & \text{if r=2p}, \\ 1, & \text{if r=2p+1}, \end{cases}$$

the recursion relation

$$\gamma_\nu(r+1) = (2\nu\theta_{r+1} + r + 1)\gamma_\nu(r) \tag{1.5}$$

holds. Also, it is easily seen that if we select $\nu = 0$, then the operator S_n^* turns into the operator S_n.

Very recently, in [1], Çekim et al. have given the Dunkl analogue of Szász-Beta type operators defined by

$$L_n(f; x) = \frac{(n-1)}{e_\nu(nx)} \sum_{r=1}^{\infty} \binom{n+r-2}{r-1} \frac{(nx)^r}{\gamma_\nu(r)} \int_0^\infty s^{r-1}(1+s)^{-n-r+1} f(s)\, ds + \frac{f(0)}{e_\nu(nx)}, \tag{1.6}$$

where $e_\nu(x)$ and γ_ν are as given in (1.3) and (1.4), respectively. In this paper, inspired by the operators (1.1) and (1.6), for $\nu, x \in [0,\infty)$ and $f \in C[0,\infty)$ we define a modified Dunkl analogue of the Szász-Beta type operators as follows:

$$L_n(f; a_n, b_n, x) := L_n^*(f; x)$$
$$= \frac{(n-1)}{e_\nu(a_n x)} \sum_{r=1}^{\infty} \binom{n+r-2}{r-1} \frac{(a_n x)^r}{\gamma_\nu(r)} \int_0^\infty s^{r-1}(1+s)^{-n-r+1} f\left(\frac{s}{b_n}\right) ds + \frac{f(0)}{e_\nu(a_n x)}. \tag{1.7}$$

Here the sequences $\{a_n\}$ and $\{b_n\}$ have the properties given in (1.2). Note that the well-known beta function is defined by

$$B(r, n-1) = \int_0^\infty s^{r-1}(1+s)^{-n-r+1} ds. \tag{1.8}$$

For $m \in \mathbb{N}$, the beta integral (1.8) can be written as

$$B(r+m, n-m-1) = \int_0^\infty s^{m+r-1}(1+s)^{-n-r+1} ds. \tag{1.9}$$

2. Main Results

In this section, we will give some important results for the operator L_n^*. For the proofs of the next theorems the following simple results are needed.

Lemma 2.1. *Let $f(t) = t^i$, $i = 0,1,2,3,4$. For the operator L_n^* defined by (1.7), the following statements hold:*

$$L_n^*(1; x) = 1, \tag{2.1}$$

$$|L_n^*(t; x) - x| \leqslant \left(\frac{1}{n-2}\frac{a_n}{b_n} - 1\right)x + \frac{2\nu}{n-2}, \tag{2.2}$$

$$\left|L_n^*(t^2; x) - x^2\right| \leqslant \left(\frac{1}{(n-2)(n-3)}\frac{a_n^2}{b_n^2} - 1\right)x^2 + \frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}x + \frac{4\nu^2+6\nu}{b_n^2(n-2)(n-3)}, \tag{2.3}$$

$$\begin{aligned} \left|L_n^*(t^3; x) - x^3\right| \leqslant \frac{1}{(n-2)(n-3)(n-4)} &\left\{\left(\frac{a_n^3}{b_n^3} - (n-2)(n-3)(n-4)\right)x^3\right. \\ &\left. + 6(\nu+1)\frac{a_n^2}{b_n^3}x^2 + \frac{(12\nu^2+20\nu+6)a_n}{b_n^3}x + \frac{(12\nu^3+6\nu^2+20\nu)}{b_n^3}\right\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left|L_n^*(t^4; x) - x^4\right| \leqslant 1&(n-2)(n-3)(n-4)(n-5) \\ &\times\left\{\left(\frac{a_n^4}{b_n^4} - (n-2)(n-3)(n-4)(n-5)\right)x^4\right. \\ &+ \frac{(8\nu+12)a_n^3}{b_n^4}x^3 + \frac{(24\nu^2+64\nu+36)a_n^2}{b_n^4}x^2 \\ &\left. + \frac{(32\nu^3+112\upsilon^2+96\nu+64)a_n}{b_n^4}x + \frac{(16\nu^4+64\nu^3+84\nu^2+84\nu)}{b_n^4}\right\}. \end{aligned} \tag{2.5}$$

Proof. Firstly, we consider $f(t)=1$. Using (1.9) and definition of e_ν, we have

$$L_n^*(1;x)=\frac{(n-1)}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\binom{n+r-2}{r-1}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty s^{r-1}(1+s)^{-n-r+1}ds+\frac{1}{e_\nu(a_nx)}$$
$$=\frac{1}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}+\frac{1}{e_\nu(a_nx)}=\frac{1}{e_\nu(a_nx)}\sum_{r=0}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}=1.$$

Now, consider $f(t)=t$. For each $n>2$, by using (1.5), (1.9), and $e_\nu(x)$, respectively, we obtain

$$L_n^*(t;x)=\frac{(n-1)}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty\frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\frac{s}{b_n}ds+\frac{f(0)}{e_\nu(a_nx)}$$
$$=\frac{1}{b_ne_\nu(a_nx)}\frac{1}{n-2}\sum_{r=1}^{\infty}r\frac{(a_nx)^r}{\gamma_\nu(r)}+\frac{f(0)}{e_\nu(a_nx)}$$
$$=\frac{1}{b_ne_\nu(a_nx)}\frac{1}{n-2}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}(r+2\nu\theta_r)-\frac{1}{b_ne_\nu(a_nx)}\frac{2\nu}{n-2}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\theta_r$$
$$=\frac{1}{n-2}\left[\left(\frac{a_n}{b_n}x-n+2\right)-\frac{2\nu}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\theta_r\right].$$

Thus,

$$|L_n^*(t;x)-x|\leqslant\frac{1}{n-2}\left[\left(\frac{a_n}{b_n}x-(n-2)x\right)-\frac{2\nu}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\theta_r\right]\leqslant\left(\frac{1}{(n-2)}\frac{a_n}{b_n}-1\right)x+\frac{2\nu}{(n-2)}$$

for all $n>2$, which shows that (2.2) holds. For $n\geqslant3$ and $f(t)=t^2$ using (1.9), we get

$$L_n^*\left(t^2;x\right)=\frac{(n-1)}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\binom{n+r-2}{r-1}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty s^{r-1}(1+s)^{-n-r+1}\frac{s^2}{b_n^2}ds+\frac{f(0)}{e_\nu(a_nx)}$$
$$=\frac{1}{b_n^2(n-2)(n-3)}\frac{1}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}r(r+1)+\frac{f(0)}{e_\nu(a_nx)}.$$

Then by using

$$r(r+1)=-(r-1)2\nu\theta_r+(r+2\nu\theta_r)(r-1)+2r$$

and (1.5),

$$L_n^*\left(t^2;x\right)$$
$$=\frac{1}{b_n^2(n-2)(n-3)}\frac{1}{e_\nu(a_nx)}\left\{\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}(r+2\nu\theta_r)(r-1)-(r-1)2\nu\theta_r+2r)\right\}$$
$$=\frac{1}{b_n^2(n-2)(n-3)}\frac{1}{e_\nu(a_nx)}\left\{\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}(r+2\nu\theta_r)(r-1)-2\nu\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}(r-1)\theta_r+\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}2r\right\}$$
$$=\frac{1}{b_n^2(n-2)(n-3)}\frac{1}{e_\nu(a_nx)}\left\{\sum_{r=0}^{\infty}\frac{(a_nx)^{r+1}}{\gamma_\nu(r+1)}\frac{\gamma_\nu(r+1)}{\gamma_\nu(r)}r-2\nu\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}(r-1)\theta_r+\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}2r\right\}.$$

Therefore, since $\theta_r\leqslant1$, we obtain

$$\left|L_n^*\left(t^2;x\right)-x^2\right|\leqslant\frac{1}{(n-2)(n-3)}\left\{\left(\frac{a_n^2}{b_n^2}x^2-(n-2)(n-3)x^2\right)+2\nu a_nx+2\nu a_nx+4\nu^2+2\nu+2a_nx+4\nu\right\}$$

$$= \frac{1}{(n-2)(n-3)}\left\{\left(\frac{a_n^2}{b_n^2}-(n-2)(n-3)\right)x^2+\frac{(4\nu a_n+2a_n)}{b_n^2}x+\frac{4\nu^2+6\nu}{b_n^2}\right\}$$
$$= \left(\frac{1}{(n-2)(n-3)}\frac{a_n^2}{b_n^2}-1\right)x^2+\frac{4\nu^2+6\nu}{b_n^2(n-2)(n-3)}.$$

Similarly, it can be shown that the inequalities (2.4) and (2.5) hold. □

The moments for the operator L_n^* are stated in next lemma.

Lemma 2.2. *The operators L_n^* satisfy the following inequalities:*

$$\Delta_1 := L_n^*(t-x;x) \leqslant \left(\frac{1}{n-2}\frac{a_n}{b_n}-1\right)x+\frac{2\nu}{n-2},$$

$$\begin{aligned}\Delta_2 := L_n^*\left((t-x)^2;x\right) &\leqslant \left(\frac{a_n^2}{(n-2)(n-3)b_n^2}-\frac{2a_n}{(n-2)b_n}+1\right)x^2\\ &+\left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}-\frac{4\nu}{n-2}\right)x+\frac{4\nu^3+6\nu}{b_n^2(n-2)(n-3)},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\Delta_3 &:= L_n^*\left((t-x)^4;x\right)\\ &\leqslant \left(\frac{a_n^4}{(n-2)(n-3)(n-4)(n-5)b_n^4}+\frac{6a_n^2}{(n-2)(n-3)b_n^2}\right.\\ &\quad\left.-\frac{4a_n^3}{(n-2)(n-3)(n-4)b_n^3}-\frac{4a_n}{(n-2)b_n}+1\right)x^4+\left(\frac{8\nu}{n-2}-\frac{24(\nu+1)a_n^2}{b_n^3(n-2)(n-3)(n-4)}\right.\\ &\quad\left.+\frac{(8\nu+12)a_n^3}{b_n^4(n-2)(n-3)(n-4)(n-5)}+\frac{12a_n(1+2\nu)}{b_n^2(n-2)(n-3)}\right)x^3\\ &\quad+\left(\frac{(24\nu^2+64\nu+36)a_n^2}{b_n^4(n-2)(n-3)(n-4)(n-5)}+\frac{6(4\nu^2+6\nu)}{b_n^2(n-2)(n-3)}\right.\\ &\quad\left.-\frac{4(12\nu^2+20\nu+6)a_n}{b_n^3(n-2)(n-3)(n-4)}\right)x^2+\left(-\frac{4(12\nu^3+6\nu^2+20\nu)}{b_n^3(n-2)(n-3)(n-4)}\right.\\ &\quad\left.+\frac{(32\nu^3+112\nu^2+96\nu+64)a_n}{b_n^4(n-2)(n-3)(n-4)(n-5)}\right)x+\frac{16\nu^4+64\nu^3+84\nu^2+84\nu}{b_n^4(n-2)(n-3)(n-4)(n-5)}.\end{aligned} \tag{2.7}$$

By using Lemma 2.1 and applying the well-known Korovkin theorem, we have the following useful result.

Theorem 2.3. *Let L_n^* be given by (1.7). Then for any $f \in C[0,\infty)\cap E$, we have*

$$L_n^*(f;x) \rightrightarrows f(x) \text{ as } n\to\infty$$

on each compact subset K of $[0,\infty)$. Here

$$E = \left\{f\in C[0,\infty): f \text{ satisfies the condition } \lim_{x\to\infty}\frac{f(x)}{1+x^2}<\infty\right\}.$$

The next theorem gives approximation of the operator L_n^* in the weighted space. Firstly, the concepts of weighted spaces are introduced. Let $\mathbb{R}^+=[0,\infty)$ and $\rho(x)=1+x^2$. The weighted spaces of the functions and the norm of $B_\rho(\mathbb{R}^+)$ are defined by,

$$B_\rho(\mathbb{R}^+)=\{f:\ |f(x)|\leqslant m_f\rho(x)\},$$

$$C_{\rho}(\mathbb{R}^{+}) = \left\{f \in B_{\rho}(\mathbb{R}^{+}) : f \text{ is continuous on } \mathbb{R}^{+}\right\},$$
$$C_{\rho}^{*}(\mathbb{R}^{+}) = \left\{f \in C_{\rho}(\mathbb{R}^{+}) : \lim_{x\to\infty} \frac{f(x)}{\rho(x)} \text{ is finite}\right\},$$
$$\|f\|_{\rho} = \sup_{x\in[0,\infty)} \frac{f(x)}{\rho(x)}, \ f \in B_{\rho}(\mathbb{R}^{+}).$$

Note that a weighted Korovkin-type theorem is given by Gadjiev [2, 3]. The next theorem presents the approximation of the operators L_n^* in the weighted space.

Theorem 2.4. *For the operators* L_n^* *given in* (1.7) *and each function* $f \in C_{\rho}^{*}(\mathbb{R}^{+})$, *one has*

$$\lim_{n\to\infty} \|L_n^*(f;x) - f(x)\|_{\rho} = 0.$$

Proof. From (2.1), we can write $\lim_{n\to\infty} \|L_n^*(1;x) - 1\|_{\rho} = 0$. By (2.2) and the following calculation

$$\sup_{x\in[0,\infty)} \frac{|L_n^*(t;x) - x|}{1+x^2} \leqslant \left(\frac{1}{n-2}\frac{a_n}{b_n} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{2\nu}{n-2} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \leqslant \left(\frac{1}{n-2}\frac{a_n}{b_n} - 1\right) + \frac{2\nu}{n-2},$$

we get

$$\lim_{n\to\infty} \|L_n^*(t;x) - x\|_{\rho} = 0.$$

Finally, by (2.3) and the following calculation

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\left|L_n^*(t^2;x) - x^2\right|}{1+x^2} &\leqslant \left(\frac{1}{(n-2)(n-3)}\frac{a_n^2}{b_n^2} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\ &\quad + \frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{4\nu^2+6\nu}{b_n^2(n-2)(n-3)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leqslant \left(\frac{1}{(n-2)(n-3)}\frac{a_n^2}{b_n^2} - 1\right) + \frac{2a_n(1+2\gamma) + 4\nu^2 + 6\nu}{b_n^2(n-2)(n-3)},\end{aligned}$$

we have

$$\lim_{n\to\infty} \left\|L_n^*(t^2;x) - x^2\right\|_{\rho} = 0.$$

Thus, we get $\lim_{n\to\infty} \|L_n^*(f;x) - f(x)\|_{\rho} = 0$ for each $f \in C_{\rho}^{*}(\mathbb{R}^{+})$ according to weighted Korovkin-type theorem. □

The simplest method of estimating the degree of approximation by positive linear operators is with the aid of the first and second order modulus of continuity given by:

$$\omega(f;\delta) = \sup_{|x-y|\leqslant\delta} \{|f(x) - f(y)| : x, y \in [0,\infty)\}$$

and

$$\omega_2(f;\delta) = \sup_{0<h\leqslant\delta} \{|f(x+h) - 2f(x) + f(x-h)| : x \in [0,\infty)\},$$

respectively.

In the next theorems, we will give the degree of approximation using the operator L_n^* by considering the first and second order modulus of continuity, in terms of the moments for L_n^* obtained in Lemma 2.2.

Theorem 2.5. *Let $f \in \widetilde{C}[0,\infty) \cap E$. Then the operators given in (1.7) satisfy the following inequality*

$$|L_n^*(f;x) - f(x)| \leqslant 2\omega(f;\delta_{n,x}),$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$, ω is the first order modulus of continuity of f, and

$$\delta_{n,x} = \sqrt{\left(\frac{a_n^2}{(n-2)(n-3)b_n^2} - \frac{2a_n}{(n-2)b_n} + 1\right)x^2 + \left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)} - \frac{4\nu}{(n-2)}\right)x + \frac{4\nu^3+6\nu}{b_n^2(n-2)(n-3)}}.$$

Furthermore, if $f \in Lip_M(\alpha)$, the following inequality

$$|L_n^*(f;x) - f(x)| \leqslant K(\tau_n(x))^{\alpha/2}$$

holds, where $\tau_n(x) = \Delta_2$ given by (2.6).

Proof. In view of (2.4), one gets

$$|L_n^*(f;x) - f(x)| \leqslant L_n^*(|f(t)-f(x)|;x) \leqslant \left(1 + \frac{1}{\delta}L_n^*(|t-x|;x)\right)\omega(f;\delta) \leqslant \left(1 + \frac{1}{\delta}\sqrt{\Delta_2}\right)\omega(f;\delta)$$

using Cauchy-Schwarz inequality. If we select $\delta = \delta_{n,x} = \sqrt{\Delta_2}$, we obtain the desired result. Thus, the proof of first part of the theorem is finished.

Now, let $f \in Lip_M(\alpha)$, then we have

$$|L_n^*(f;x) - f(x)| \leqslant L_n^*(|f(t)-f(x)|;x) \leqslant K L_n^*(|t-x|^{\alpha};x).$$

From Lemma 2.2 and using Hölder's inequality, one gets

$$|L_n^*(f;x) - f(x)| \leqslant K(\Delta_2)^{\alpha/2}.$$

Then choosing $\tau_n(x) = \Delta_2$, the proof of the theorem is now completed. □

Lemma 2.6. *Let $C_B[0,\infty)$ be the space of continuous and bounded functions on $[0,\infty)$. For $f \in C_B^2[0,\infty) = \left\{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\right\}$, we have*

$$|L_n^*(f;x) - f(x)| \leqslant \chi_n(x)\|f\|_{C_B^2[0,\infty)},$$

where

$$\chi_n(x) = \Delta_1 + \Delta_2.$$

Proof. For $f \in C_B^2[0,\infty)$, using the Taylor's formula of the function f, we write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(\sigma), \sigma \in (x,t),$$

by linearity of L_n^*, we get

$$L_n^*(f;x) - f(x) = f'(x)L_n^*(t-x;x) + \frac{f''(\sigma)}{2!}L_n^*\left((t-x)^2;x\right) = f'(x)\Delta_1 + \frac{f''(\sigma)}{2!}\Delta_2.$$

Then using Lemma 2.2, one obtains

$$|L_n(f;x) - f(x)| \leqslant \left[\left(\frac{1}{n-2}\frac{a_n}{b_n} - 1\right)x + \frac{2\nu}{n-2}\right]\left\|f'\right\|_{C_B^2[0,\infty)}$$

$$+\frac{1}{2}\left\{\left[\frac{a_n^2}{(n-2)(n-3)b_n^2}-\frac{2a_n}{(n-2)b_n}+1\right]x^2\right.$$
$$\left.+\left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}-\frac{4\nu}{n-2}\right)x+\frac{4\nu^3+6\nu}{b_n^2(n-2)(n-3)}\right\}\left\|f''\right\|_{C_B^2[0,\infty)}$$
$$\leqslant(\Delta_1+\Delta_2)\|f\|_{C_B^2[0,\infty)}.$$

Choosing $\chi_n(x)=\Delta_1+\Delta_2$ finishes the proof. □

Theorem 2.7. *The operators L_n^* satisfy the following inequality*

$$|L_n^*(f;x)-f(x)|\leqslant 2M\left\{\min\left(1,\frac{\chi_n(x)}{2}\right)\|f\|_{C_B[0,\infty)}+\omega_2\left(f;\sqrt{\frac{\chi_n(x)}{2}}\right)\right\},$$

where $f\in C_B[0,\infty)$, $x\in[0,\infty)$, M is a positive constant independent of n and $\chi_n(x)$.

Proof. For any $g\in C_B^2[0,\infty)$, we use Lemma 2.6 and the triangle inequality to get the following inequality

$$\begin{aligned}|L_n^*(f;x)-f(x)|&\leqslant|L_n^*(f-g;x)|+|L_n^*(g;x)-g(x)|+|f(x)-g(x)|\\&\leqslant 2\|f-g\|_{C_B[0,\infty)}+\chi_n(x)\|g\|_{C_B^2[0,\infty)}\\&=2\left\{\|f-g\|_{C_B[0,\infty)}+\frac{\chi_n(x)}{2}\|g\|_{C_B^2[0,\infty)}\right\}.\end{aligned}$$

With the help of the Peetre's K functional, and using the well-known relation between K_2 and ω_2 one has

$$|L_n^*(f;x)-f(x)|\leqslant 2M\left\{\min\left(1,\frac{\chi_n(x)}{2}\right)\|f\|_{C_B[0,\infty)}+\omega_2\left(f;\sqrt{\frac{\chi_n(x)}{2}}\right)\right\}$$

and the proof of the theorem is completed. □

In the last theorem, we will give rate of approximation of the operators L_n^* in the weighted space by using the weighted modulus of continuity. The details about the weighted modulus of continuity are included below. For $f\in C_\rho^*(\mathbb{R}^+)$, the weighted modulus of continuity is defined by

$$\Omega(f;\delta)=\sup_{\substack{x\in[0,\infty)\\|h|\leqslant\delta}}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}$$

and has the following properties:

$$\lim_{\delta\to 0}\Omega(f;\delta)=0,$$

and

$$|f(s)-f(x)|\leqslant 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)(1+x^2)\left(1+(t-x)^2\right)\Omega(f;\delta), \tag{2.8}$$

where $t,x\in[0,\infty)$. For further properties of the weighted modulus of continuity see [7].

Theorem 2.8. *Let $f\in C_\rho^*(\mathbb{R}^+)$. Then the following inequality holds:*

$$\sup_{x\in[0,\infty)}\frac{|L_n^*(f;x)-f(x)|}{(1+x^2)^3}\leqslant M_\nu\left(1+\frac{1}{n}\right)\Omega\left(f;\frac{1}{\sqrt{n}}\right),$$

where M_ν is a constant independent of n.

Proof. By Lemma 2.2 and (2.8), we get

$$\begin{aligned}
|L_n^*(f;x)-f(x)| &\leqslant \frac{(n-1)}{e_\nu(a_nx)}\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\\
&\quad\times\left|f\left(\frac{s}{b_n}\right)-f(x)\right|ds+\frac{f(0)}{e_\nu(a_nx)}\\
&\leqslant 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\frac{(n-1)}{e_\nu(a_nx)}\\
&\quad\times\sum_{r=1}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\left(1+\frac{\left|\frac{s}{b_n}-x\right|}{\delta}\right)\left(1+\left(\frac{s}{b_n}-x\right)^2\right)ds\\
&\quad+\frac{f(0)}{e_\nu(a_nx)}\\
&= 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\frac{(n-1)}{e_\nu(a_nx)}\\
&\quad\times\left\{\sum_{r=0}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}ds\right.\\
&\quad+\frac{1}{\delta}\sum_{r=0}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\left|\frac{s}{b_n}-x\right|ds\\
&\quad+\sum_{r=0}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\left(\frac{s}{b_n}-x\right)^2ds\\
&\quad\left.+\frac{1}{\delta}\sum_{r=0}^{\infty}\frac{(a_nx)^r}{\gamma_\nu(r)}\int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}}\binom{n+r-2}{r-1}\left(\frac{s}{b_n}-x\right)^3ds\right\}.
\end{aligned}$$

We use the Cauchy-Schwarz inequality to get

$$|L_n(f;x)-f(x)|\leqslant 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\Delta_2}+\Delta_2+\frac{1}{\delta}\sqrt{\Delta_2\Delta_3}\right).$$

With the help of (2.6) and (2.7), we have

$$\begin{aligned}
&|L_n^*(f;x)-f(x)|\\
&\leqslant 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\left\{1+\left(\frac{a_n^2}{(n-2)(n-3)b_n^2}-\frac{2a_n}{(n-2)b_n}+1\right)x^2\right.\\
&+\left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}-\frac{4\nu}{n-2}\right)x\\
&+\frac{1}{\delta}\sqrt{\left(\frac{a_n^2}{(n-2)(n-3)b_n^2}-\frac{2a_n}{(n-2)b_n}+1\right)x^2+\left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}-\frac{4\nu}{n-2}\right)x}\\
&\left.+\frac{1}{\delta}\sqrt{\left(\left(\frac{a_n^2}{(n-2)(n-3)b_n^2}-\frac{2a_n}{(n-2)b_n}+1\right)x^2+\left(\frac{2a_n(1+2\nu)}{b_n^2(n-2)(n-3)}-\frac{4\nu}{n-2}\right)x\right)\Delta_3}\right\}
\end{aligned}$$

and finally, by choosing $\delta=\frac{1}{\sqrt{n}}$ we complete the proof. □

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