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Oscillation criteria for a class of third order neutral distributed delay differential equations with damping

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Abstract

In this paper, the oscillation criteria of a class of third order neutral distributed delay differential equations with damping are investigated. This work is the continuation of the study by Saker [S. H. Saker, Math. Slovaca, **56** (2006), 433–450] and the extension of the work by Zhang [Q. X. Zhang, L. Gao, Y. H. Yu, Appl. Math. Lett., **25** (2012), 1514–1519] on oscillation properties of nonlinear third order delay differential equation. By choosing the appropriate functions and using a generalized Riccati transformation, some new oscillation criteria are presented to insure that every solution of this equation oscillates or converges to zero. The presented results correct and improve the earlier ones in existing literature. Finally, several illustrative examples are included.

Keywords: Oscillation criteria, third order, distributed delay, damping, Riccati transformation.

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1. Introduction

In this article, we consider third order neutral distributed delay differential equations with damping of the form

$$\begin{aligned}\left(r_1(t)\left(r_2(t)\left(x(t)+\int_a^b p(t,\mu)x(\sigma(t,\mu))d\mu\right)'\right)'\right)'&\\ +m(t)\left(r_2(t)\left(x(t)+\int_a^b p(t,\mu)x(\sigma(t,\mu))d\mu\right)'\right)'+\int_c^d q(t,\zeta)f(x(\tau(t,\zeta)))d\zeta=0.&\end{aligned} \tag{1.1}$$

Throughout the whole paper, we assume that the following hypotheses hold:

(C_1) $r_1(t)\in C([t_0,\infty),(0,\infty))$, $r_2(t)\in C([t_0,\infty),(0,\infty))$, $m(t)\in C([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty}\frac{1}{r_2(t)}dt=\infty$, $\int_{t_0}^{\infty}\frac{1}{r_1(t)}\exp(-\int_{t_0}^{t}\frac{m(s)}{r_1(s)}ds)dt=\infty$;

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(C_2) $p(t,\mu) \in C([t_0,\infty) \times [a,b], (0,\infty)), \quad 0 \leqslant p(t) \equiv \int_a^b p(t,\mu)d\mu \leqslant p < 1$;

(C_3) $\sigma(t,\mu) \in C([t_0,\infty) \times [a,b], (0,\infty))$ and $\tau(t,\zeta) \in C([t_0,\infty) \times [c,d], (0,\infty))$ are not the decreasing functions with respect to μ and ζ, respectively, and satisfy $\sigma(t,\mu) \leqslant t, \tau(t,\zeta) \leqslant t$, $\lim_{t\to\infty} \inf_{\mu\in[a,b]} \sigma(t,\mu) = \infty$ and $\lim_{t\to\infty} \inf_{\zeta\in[c,d]} \tau(t,\zeta) = \infty$;

(C_4) $q(t,\zeta) \in C([t_0,\infty) \times [c,d], (0,\infty))$;

(C_5) $f(\nu) \in C((0,\infty), (0,\infty)), \quad \frac{f(\nu)}{\nu} \geqslant \alpha > 0, \quad \nu \neq 0.$

Define the function by

$$z(t) = x(t) + \int_a^b p(t,\mu)x(\sigma(t,\mu))d\mu. \tag{1.2}$$

By a solution of (1.1) this means a function $x(t) \in C([T_x,\infty))$ which has the property $x(t), r_2(t)z'(t)$, $r_1(t)(r_2(t)z'(t))' \in C^1[T_x,\infty)$ and satisfies (1.1) on $[T_x,\infty)$. Our attention is restricted to those solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t_1 \leqslant t \leqslant \infty\} > 0$ for all $t_1 \geqslant t_0$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called nonoscillatory.

Motivated by [4, 8, 10, 18, 21, 24], the study focuses on the oscillation behavior of neutral distributed delay differential equations with damping. The oscillation of functional differential equations have received a great deal of interest in recent years. But we notice that most of the investigations are concerned with oscillation of first or second order differential equations, while relatively less attention has been paid to oscillation of third order differential equations, see [1–3, 5–7, 9, 11–16, 18, 20, 22, 24], especially the distributed delay equations with damping. On the basis of the studies [18] and [24], by using a generalized Riccati transformation and choosing appropriate functions, the aim of this paper is to establish some new sufficient conditions which insure that any solution of this equation oscillates or converges to zero. In fact, by choosing appropriate functions, we shall present several easily verifiable oscillation criteria. The methods and arguments used in the present paper are different from those used in [4, 8, 10, 21], and the results correct, extend and improve a number of existing results, especially the results in [18] and [24].

The paper is organized as follows. We first need to state and prove some lemmas in Section 2, which will be used in the proof of our main results. We will establish some new criteria of oscillatory behavior for (1.1) by a generalized Riccati transformation technique in Section 3, and then present some applications for our results in Section 4.

2. Some preliminary lemmas

Lemma 2.1. *Let* $x(t)$ *be a positive solution of* (1.1). *Then* $z(t)$ *has only one of the following two properties:*

(I) $z(t) > 0,\ z'(t) > 0,\ (r_2(t)z'(t))' > 0$;

(II) $z(t) > 0,\ z'(t) < 0,\ (r_2(t)z'(t))' > 0$,

where $t \geqslant t_1$ *for sufficiently large* t_1.

Proof. Let $x(t)$ be a positive solution of (1.1) on $[t_0,\infty)$. Then we have $z(t) > x(t) > 0$ from (C_2) for $t \geqslant t_1$. Based on (1.2), it follows from (C_3) and (C_4) that

$$\begin{aligned}\left(r_1(t)\left(r_2(t)z'(t)\right)'\right)' + m(t)\left(r_2(t)z'(t)\right)' &= -\int_c^d q(t,\zeta)f(x(\tau(t,\zeta)))d\zeta \\ &\leqslant -\alpha\int_c^d q(t,\zeta)x(\tau(t,\zeta))d\zeta \\ &< 0.\end{aligned} \tag{2.1}$$

Then we have

$$\frac{d}{dt}\left[\exp(\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)r_1(t)\left(r_2(t)z'(t)\right)'\right] < 0.$$

Thus, $\exp(\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)r_1(t)\left(r_2(t)z'(t)\right)'$ is a decreasing function and eventually of one sign, and then from (C_1) we know that

$$\left(r_2(t)z'(t)\right)' < 0 \quad \text{or} \quad \left(r_2(t)z'(t)\right)' > 0,$$

for $t \geqslant t_2 \geqslant t_1$. We assert that $(r_2(t)z'(t))' > 0$. Suppose $(r_2(t)z'(t))' < 0$, then there exists a constant $M > 0$ such that

$$\exp(\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)r_1(t)\left(r_2(t)z'(t)\right)' \leqslant -M.$$

Integrate the above inequality on $[t_2, t]$ to get

$$r_2(t)z'(t) \leqslant r_2(t_2)z'(t_2) - M(\int_{t_2}^{t}\frac{1}{r_1(s)}\exp(-\int_{t_1}^{s}\frac{m(\eta)}{r_1(\eta)}d\eta)ds \to -\infty \ (t \to \infty),$$

by the condition (C_1). Thus, there are an integer $t_3 \geqslant t_2$ and $C > 0$ such that $r_2(t)z'(t) \leqslant -C$ for $t \geqslant t_3$. Integrating on $[t_3, t]$ to get

$$z(t) \leqslant z(t_3) - C\int_{t_3}^{t}\frac{1}{r_2(s)}ds \to -\infty \ (t \to \infty),$$

from the condition (C_1), and then $z(t) < 0$ which contradicts $z(t) > 0$. Thus we have $(r_2(t)z'(t))' > 0$. Therefore, $z(t)$ has only one of the two properties (I) and (II). This completes the proof. □

Lemma 2.2. *Let* $x(t)$ *be a positive solution of* (1.1)*, and* $z(t)$ *has the property* (I)*. Then*

$$z'(\tau(t)) \geqslant \frac{r_1(t)\left(r_2(t)z'(t)\right)' R(\tau(t))}{r_2(\tau(t))},$$

where $\tau(t) = \tau(t, c), \ R(t) = \int_{t_0}^{t}\frac{1}{r_1(s)}ds, \ t \geqslant t_0.$

Proof. Let $x(t)$ be a positive solution of (1.1). Since $z(t)$ has the property (I), we know

$$\left(r_1(t)\left(r_2(t)z'(t)\right)'\right)' < 0,$$

from (2.1). Then we have

$$\begin{aligned} r_2(t)z'(t) &= r_2(t_0)z'(t_0) + \int_{t_0}^{t}\left(r_2(s)z'(s)\right)' ds \\ &\geqslant \int_{t_0}^{t}\frac{r_1(s)\left(r_2(s)z'(s)\right)'}{r_1(s)}ds \\ &\geqslant r_1(t)\left(r_2(t)z'(t)\right)' R(t). \end{aligned}$$

Thus

$$\begin{aligned} r_2(\tau(t))z'(\tau(t)) &\geqslant r_1(\tau(t))\left(r_2(\tau(t))z'(\tau(t))\right)' R(\tau(t)) \\ &\geqslant r_1(t)\left(r_2(t)z'(t)\right) R(\tau(t)). \end{aligned}$$

Hence, we obtain

$$z'(\tau(t)) \geqslant \frac{r_1(t)\left(r_2(t)z'(t)\right)' R(\tau(t))}{r_2(\tau(t))},$$

which completes the proof. □

Lemma 2.3. *Let* $x(t)$ *be a positive solution of* (1.1), *and* $z(t)$ *has the property* (II). *If*

$$\int_{t_0}^{\infty}\frac{1}{r_2(\nu)}\int_{\nu}^{\infty}\frac{1}{r_1(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\zeta)d\zeta ds du d\nu=\infty, \tag{2.2}$$

then $\lim\limits_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be a positive solution of (1.1). According to the property (II), we have $z(t)>0,\ z'(t)<0$, then $\lim\limits_{t\to\infty}z(t)=l\geqslant 0$. We assert that $l=0$. Suppose that $l>0$, then we get $l+\varepsilon>z(t)>l$ for $\varepsilon>0$ and $t\geqslant t_1\geqslant t_0$. By choosing $\varepsilon<\dfrac{l(1-p)}{p}$, from (C_2) and property (II) we have

$$\begin{aligned}x(t)&=z(t)-\int_a^b p(t,\mu)x(\sigma(t,\mu))d\mu\\&\geqslant z(t)-z(\sigma(t,a))\int_a^b p(t,\mu)d\mu\\&\geqslant l-p(t)z(\sigma(t,a))\\&\geqslant l-p(l+\varepsilon)\\&>kz(t),\end{aligned}$$

where $k=\dfrac{l(1-p)-p\varepsilon}{l+\varepsilon}>0$. Then from (C_5), we get

$$\begin{aligned}\left(r_1(t)\left(r_2(t)z'(t)\right)'\right)'+m(t)\left(r_2(t)z'(t)\right)'&\leqslant-\alpha\int_c^d q(t,\zeta)x(\tau(t,\zeta))d\zeta\\&\leqslant-k\alpha z(\tau(t,d))\int_c^d q(t,\zeta)d\zeta\\&=-q_1(t)z(\tau_1(t)),\end{aligned}$$

where $q_1(t)=k\alpha\int_c^d q(t,\zeta)d\zeta,\ \tau_1(t)=\tau(t,d)$. Thus

$$\left(\exp(\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)r_1(t)\left(r_2(t)z'(t)\right)'\right)'\leqslant-\exp(\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)q_1(t)z(\tau_1(t)).$$

Integrating the above inequality from t to ∞, we obtain

$$r_1(t)\left(r_2(t)z'(t)\right)'\geqslant\exp(-\int_{t_1}^{t}\frac{m(s)}{r_1(s)}ds)\int_t^{\infty}\exp(\int_{t_1}^{\sigma}\frac{m(s)}{r_1(s)}ds)q_1(\sigma)z(\tau_1(\sigma))d\sigma.$$

From $z(\tau_1(t))>l$ and $\dfrac{d}{dt}\exp(\int_{t_1}^{t}\dfrac{m(s)}{r_1(s)}ds)>0$, we have

$$\left(r_2(t)z'(t)\right)'>\frac{l}{r_1(t)}\int_t^{\infty}q_1(s)ds.$$

Integrate the above inequality on $[t,\infty)$ to get

$$-r_2(t)z'(t)>l\int_t^{\infty}\frac{1}{r_1(u)}\int_u^{\infty}q_1(s)dsdu.$$

Further integrating on $[t_1,\infty)$ leads to

$$\int_{t_1}^{\infty}\frac{1}{r_2(\nu)}\int_{\nu}^{\infty}\frac{1}{r_1(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\zeta)d\zeta ds du d\nu<\frac{z(t_1)}{k\alpha l},$$

which contradicts (2.2), and then we have $l=0$. According to $z(t)>x(t)>0$, we obtain $\lim\limits_{t\to\infty}x(t)=0$. □

3. Main results

In this section, we obtain new oscillatory criteria for (1.1) by using the generalized Riccati transformation. Let

$$D=\{(t,s):t_0\leqslant s\leqslant t<\infty\},\quad D_0=\{(t,s):t_0\leqslant s<t<\infty\}.$$

A function $H\in C^1(D,R)$ is said to belong to X class $(H\in X)$ if it satisfies

(i) $H(t,t)=0,\ t\geqslant t_0,\quad H(t,s)>0,\ (t,s)\in D_0$;

(ii) $\dfrac{\partial H(t,s)}{\partial s}\leqslant 0,(t,s)\in D$, and there exists $h(t,s)\in C(D_0,R)$ such that

$$\frac{\partial H(t,s)}{\partial s}=-h(t,s)\sqrt{H(t,s)}.$$

Then we present the following main results of this article.

Theorem 3.1. *Assume that* (2.2) *holds. If there exists* $\rho(t)\in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(C(s)-\frac{B^2(s)}{4A(s)}\right)ds=\infty,\tag{3.1}$$

where

$$A(t)=\frac{\tau'(t)R(\tau(t))}{\rho(t)r_2(\tau(t))},\quad B(t)=\frac{\rho'(t)}{\rho(t)}-\frac{m(t)}{r_1(t)},\quad C(t)=\alpha(1-p)\rho(t)q(t),\tag{3.2}$$

and $q(t)=\int_c^d q(t,\zeta)d\zeta$, *then every solution* $x(t)$ *of* (1.1) *either oscillates or converges to zero.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assert $x(t)>0$ on $[t_1,\infty)$, and then $x(\sigma(t,\mu))>0,\ (t,\mu)\in[t_1,\infty)\times[a,b],\ x(\tau(t,\zeta))>0,\ (t,\zeta)\in[t_1,\infty)\times[c,d]$ for sufficiently large t_1. By Lemma 2.1, we know that $z(t)$ has the property (I) or the property (II).

When $z(t)$ has property (I), it follows from (C_2) and (C_3), we have

$$\begin{aligned}x(t)&=z(t)-\int_a^b p(t,\mu)x(\sigma(t,\mu))d\mu\\&\geqslant z(t)-\int_a^b p(t,\mu)z(\sigma(t,\mu))d\mu\\&\geqslant(1-p(t))z(t)\\&\geqslant(1-p)z(t).\end{aligned}$$

Then

$$\begin{aligned}\left(r_1(t)\left(r_2(t)z'(t)\right)'\right)'+m(t)\left(r_2(t)z'(t)\right)'&\leqslant-\alpha(1-p)\int_c^d q(t,\zeta)z(\tau(t,\zeta))d\zeta\\&\leqslant-\alpha(1-p)z(\tau(t))q(t).\end{aligned}$$

Let

$$w(t)=\rho(t)\frac{r_1(t)\left(r_2(t)z'(t)\right)'}{z(\tau(t))},\ t\geqslant t_1.$$

Then from Lemma 2.2, we obtain

$$\begin{aligned}
w'(t) &= \rho'(t)\frac{r_1(t)\left(r_2(t)z'(t)\right)'}{z'(\tau(t))} + \rho(t)\frac{\left(r_1(t)\left(r_2(t)z'(t)\right)'\right)'}{z(\tau(t))} - \rho(t)\frac{z'(\tau(t))\tau'(t)r_1(t)\left(r_2(t)z'(t)\right)'}{z^2(\tau(t))}\\
&\leqslant \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[\frac{m(t)\left(r_2(t)z'(t)\right)'}{z(\tau(t))} + \frac{\alpha(1-p)z(\tau(t))q(t)}{z(\tau(t))}\right]\\
&\quad - \rho(t)\frac{\tau'(t)R(\tau(t))(r_1(t)\left(r_2(t)z'(t)\right)')^2}{r_2(\tau(t))z^2(\tau(t))}\\
&\leqslant -C(t) + B(t)w(t) - A(t)w^2(t)\\
&\leqslant -C(t) + \frac{B^2(t)}{4A(t)}.
\end{aligned}$$

Integrate the above inequality from t_2 to t, then we have

$$\int_{t_1}^{t}\left(C(s) - \frac{B^2(s)}{4A(s)}\right)ds \leqslant w(t_2),$$

from $w(t) > 0$, which contradicts (3.1), and then the solution $x(t)$ of (1.1) is oscillatory.

When $z(t)$ has property (II), from (2.2) we know $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.3. The proof is complete. □

Remark 3.2. The proof of Theorem 3.1 is based on $Bu - Au^2 \leqslant \frac{B^2}{4A}$ for $A > 0, u \in R$.

Theorem 3.3. *Assume that* (2.2) *holds. If there exist* $H \in X$ *and* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)C(s) - \frac{h_1^2(t,s)}{4A(s)}\right)ds = \infty, \tag{3.3}$$

where

$$h_1(t,s) = h(t,s) - B(s)\sqrt{H(t,s)}, \tag{3.4}$$

and $A(t), B(t), C(t)$ *are defined in* (3.2), *then every solution* $x(t)$ *of* (1.1) *either oscillates or converges to zero.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assert $x(t) > 0, t \geqslant t_1 \geqslant t_0,\ x(\sigma(t,\mu)) > 0,\ (t,\mu) \in [t_1,\infty)\times[a,b],\ x(\tau(t,\zeta)) > 0,\ (t,\zeta) \in [t_1,\infty)\times[c,d]$. By Lemma 2.1, we know that $z(t)$ has property (I) or property (II).

When $z(t)$ has the property (I), we proceed as in the proof of Theorem 3.1 and have

$$w'(t) \leqslant -C(t) + B(t)w(t) - A(t)w^2(t).$$

Multiplying the above inequality by $H(t,s)$ and integrating the inequality from t_1 to t, we obtain

$$\begin{aligned}
\int_{t_1}^{t} H(t,s)C(s)ds &\leqslant w(t_1)H(t,t_1) - \int_{t_1}^{t}\left(h_1(t,s)\sqrt{H(t,s)}w(s) + H(t,s)A(s)w^2(s)\right)ds\\
&= w(t_1)H(t,t_1) - \int_{t_1}^{t}\left(\sqrt{H(t,s)A(s)}w(s) + \frac{h_1(t,s)}{2\sqrt{A(s)}}\right)^2 ds \qquad (3.5)\\
&\quad + \int_{t_1}^{t}\frac{h_1^2(t,s)}{4A(s)}ds,
\end{aligned}$$

where $h_1(t,s)$ is defined as (3.4). Then

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)C(s)-\frac{h_1^2(t,s)}{4A(s)}\right)ds \leqslant w(t_1),$$

which contradicts (3.3). Thus the solution $x(t)$ of (1.1) is oscillatory.

When $z(t)$ has property (II), from (2.2) we know $\lim_{t\to\infty} x(t)=0$ by Lemma 2.3. The proof is complete. □

Theorem 3.4. *Assume that* (2.2) *holds. If there exist* $H\in X$, $\rho(t)\in C^1([t_0,\infty),(0,\infty))$ *and* $\varphi\in C([t_0,\infty),R)$ *such that*

$$0<\inf_{s\geqslant T}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)}\right]\leqslant\infty, \tag{3.6}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\frac{h_1^2(t,s)}{A(s)}ds<\infty,$$

$$\int_{t_0}^{\infty}A(t)\varphi_+^2(t)dt=\infty,$$

and

$$\varphi(T)\leqslant\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)C(s)-\frac{h_1^2(t,s)}{4A(s)}\right)ds, \tag{3.7}$$

for $t\geqslant T\geqslant t_0$, *where*

$$\varphi_+(t)=\max\{\varphi(t),0\}, \tag{3.8}$$

$A(t),B(t),C(t)$ *and* $h_1(t,s)$ *are defined in* (3.2) *and* (3.4), *respectively, then every solution* $x(t)$ *of* (1.1) *either oscillates or converges to zero.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Proceeding as in the proof of Theorem 3.3, when $z(t)$ has property (I), from (3.5) we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)C(s)-\frac{h_1^2(t,s)}{4A(s)}\right)ds\\ &\quad\leqslant w(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(\sqrt{H(t,s)A(s)}w(s)+\frac{h_1(t,s)}{2\sqrt{A(s)}}\right)^2ds.\end{aligned}$$

It follows from (3.7) that

$$\varphi(T)\leqslant w(T),$$

and then

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(\sqrt{H(t,s)A(s)}w(s)+\frac{h_1(t,s)}{2\sqrt{A(s)}}\right)^2ds\leqslant w(T)-\varphi(T)<\infty.$$

Thus we know

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)A(s)w^2(s)+\sqrt{H(t,s)}h_1(t,s)w(s)\right)ds<\infty.$$

Here we assert

$$\int_{t_1}^{\infty} A(s)w^2(s)ds < \infty.$$

Suppose $\int_{t_1}^{\infty} A(s)w^2(s)ds = \infty$. From (3.6), we have

$$\inf_{s \geqslant T}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)}\right] > \mu,$$

for $\mu > 0$, then $\frac{H(t,t_2)}{H(t,T)} > \mu$ for $t \geqslant t_2 \geqslant t_1$. There exists $M_1 > 0$ such that

$$\int_{t_1}^{t} A(s)w^2(s)ds \geqslant \frac{M_1}{\mu}.$$

Thus for $t \geqslant t_2$,

$$\begin{aligned}\frac{1}{H(t,T)}\int_{t_2}^{t} H(t,s)A(s)w^2(s)ds &= \frac{1}{H(t,T)}\int_{t_2}^{t} -\frac{\partial H(t,s)}{\partial s}\int_{t_2}^{s} A(\eta)w^2(\eta)d\eta ds\\ &\geqslant \frac{1}{H(t,T)}\frac{M_1}{\mu}\int_{t_2}^{t} -\frac{\partial H(t,s)}{\partial s}ds\\ &= \frac{M_1}{\mu}\frac{H(t,t_2)}{H(t,T)}\\ &\geqslant M_1.\end{aligned}$$

Hence, we have

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t} H(t,s)A(s)w^2(s)ds = \infty.$$

The reminder of the proof is similar to that of similar theorems in [17, 19, 23] and hence is omitted. Then we get that the solution $x(t)$ of (1.1) is oscillatory.

When $z(t)$ has property (II), from (2.2) we find $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.3. The proof is complete. □

4. Examples

In this section, we will give several examples to illustrate our main results.

Example 4.1. We consider the equation

$$\begin{aligned}&\left(\frac{1}{(t+1)^2}\left(x(t)+\int_{-1}^{0}\left(\frac{1}{2}+\frac{2}{3}e^{-2t}+\mu\right)x(t+\frac{\mu}{2})d\mu\right)'\right)''\\ &+\frac{1}{t}\left(\frac{1}{(t+1)^2}\left(x(t)+\int_{-1}^{0}\left(\frac{1}{2}+\frac{2}{3}e^{-2t}+\mu\right)x(t+\frac{\mu}{2})d\mu\right)'\right)''\\ &+\int_{-1}^{0} e^{t+\zeta}(3+sinx(t+\zeta))x(t+\zeta)d\zeta = 0, \quad t > 1.\end{aligned} \tag{4.1}$$

Based on (4.1), we find that $r_1(t) = 1$, $r_2(t) = \frac{1}{(t+1)^2}$, $p(t,\mu) = \frac{1}{2}+\frac{2}{3}e^{-2t}+\mu$, $p = \frac{2}{3}$, $a = c = -1$, $b = d = 0$, $t_0 = 1$, $\sigma(t,\mu) = t+\frac{\mu}{2} \leqslant t, \mu \in [-1,0]$, $m(t) = \frac{1}{t}$, $\tau(t,\zeta) = t+\zeta \leqslant t, \zeta \in [-1,0]$, $q(t,\zeta) =$

$e^{t+\zeta}$, $f(v) = (3+\sin v)v$, $\alpha = 2$. It is clear that the conditions (C_1)–(C_5) are satisfied. By choosing $H(t,s) = (t-s)^2$, $\rho(t) = 1$, from Theorem 3.3 we have $h(t,s) = 2$, $q(t) = (1-e^{-1})e^t$, $R(t) = t-1$, $\tau(t) = t-1$, $A(t) = t^2(t-2)$, $B(t) = -\frac{1}{t}$, $C(t) = \frac{2(1-e^{-1})}{3}e^t$, $h_1(t,s) = 1+\frac{t}{s}$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)C(s) - \frac{h_1^2(t,s)}{4A(s)} \right) ds = \infty.$$

Hence by Theorem 3.3, every solution $x(t)$ of (1.1) is oscillatory or $x(t) \to 0$ as $t \to \infty$.

Example 4.2. Consider the equation

$$\begin{aligned} &\left(e^{-t} \left(x(t) + \int_1^2 \frac{\mu}{3t} x(\frac{1}{2}t\mu) d\mu \right)' \right)'' + \frac{1}{t} \left(e^{-t} \left(x(t) + \int_1^2 \frac{\mu}{3t} x(\frac{1}{2}t\mu) d\mu \right)' \right)'' \\ &\quad + \int_0^1 t\zeta x(t-\zeta) d\zeta = 0, \quad t > 1. \end{aligned} \tag{4.2}$$

Here from (4.2), we note that $r_1(t) = 1$, $r_2(t) = e^{-t}$, $p(t,\mu) = \frac{\mu}{3t}$, $p = \frac{1}{2}$, $a = 1$, $b = 2$, $c = 0$, $d = 1$, $t_0 = 1$, $\sigma(t,\mu) = \frac{1}{2}t\mu \leqslant t, \mu \in [1,2]$, $m(t) = \frac{1}{t}$, $\tau(t,\zeta) = t-\zeta \leqslant t, \zeta \in [0,1]$, $q(t,\zeta) = t\zeta$, $f(v) = v$, $\alpha = 1$. Consequently the conditions (C_1)–(C_5) are satisfied. Choose $H(t,s) = (t-s)^2$, $\rho(t) = e^t$, $\varphi(t) = t$, then from Theorem 3.4 we have $h(t,s) = 2$, $q(t) = \frac{t}{2}$, $R(t) = t-1$, $\tau(t) = t$, $A(t) = t-1$, $B(t) = 1-\frac{1}{t}$, $C(t) = \frac{1}{4}te^t$, $h_1(t,s) = 1+\frac{t}{s}-t+s$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)C(s) - \frac{h_1^2(t,s)}{4A(s)} \right) ds \geqslant T = \varphi(T).$$

Clearly, it is easy to check that the other conditions of Theorem 3.4 are also satisfied. Thus, we can conclude from Theorem 3.4 that every solution $x(t)$ of (1.1) is oscillatory or $x(t) \to 0$ as $t \to \infty$.

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