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Mönch type Leray–Schauder alternatives for maps satisfying weakly countable compactness conditions

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Abstract

In this paper we discuss weakly Mönch type maps and obtain Leray–Schauder alternatives for such maps.

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1. Introduction

Leray–Schauder type alternatives for weakly compact, weakly condensing and weakly Mönch type maps were established in the literature in a variety of settings, see for example [1, 2, 5] and the references therein. Using the notion of an essential map (originally introduced by Granas, see [3]) we present general Leray–Schauder alternative type theorems for general weakly Mönch type maps (see [4, 5]) and our results generalize those in the literature. Our theory is also motivated by recent fixed point theorems in the literature by the author [6, 7] and for completeness we state some of these fixed point result which would be useful in applying the results in Section 2.

Theorem 1.1. *Let* E *be a Banach space,* Q *a nonempty closed convex subset of* E, $x_0 \in Q$ *and* $F : Q \to K(Q)$ *(here* $K(Q)$ *denotes the family of nonempty, convex, weakly compact subsets of* Q*) has weakly sequentially closed graph. Assume either*

$$\begin{cases} A \subseteq Q,\ \ A = \overline{co}\,(\{x_0\} \cup F(A)) \text{ with } \ C \subseteq A \\ \text{countable and } \ C \subseteq \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies } \ \overline{C^w} \text{ is weakly compact,} \end{cases}$$

or

$$\begin{cases} A \subseteq Q,\ \ A = co\,(\{x_0\} \cup F(A)) \text{ with } \ C \subseteq A \\ \text{countable and } \ C \subseteq \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies } \ \overline{C^w} \text{ is weakly compact,} \end{cases}$$

holds. Then F *has a fixed point in* Q.

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Remark 1.2. In Theorem 1.1 we can replace E a Banach space with E a Hausdorff locally convex linear topological space provided certain conditions are satisfied (see [7]).

Theorem 1.3. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following conditions hold:*

$$\begin{cases} A \subseteq Q, \ \ A = \overline{co}\,(\{x_0\} \cup F(A)), \text{ for any} \\ \text{countable set } \ N \subseteq A \text{ there exists a countable set} \\ P \subseteq A \text{ with } \ \overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{P^w}, \end{cases}$$

and

$$\begin{cases} A \subseteq Q, \ \ A = \overline{co}\,(\{x_0\} \cup F(A)) \text{ with } \ C \subseteq A \\ \text{countable and } \ \overline{C^w} = \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies } \ \overline{C^w} \text{ is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Theorem 1.4. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following conditions hold:*

$$\begin{cases} A \subseteq Q, \ \ A = co\,(\{x_0\} \cup F(A)), \text{ for any} \\ \text{countable set } \ N \subseteq A \text{ there exists a countable set} \\ P \subseteq A \text{ with } \ \overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{P^w}, \end{cases}$$

and

$$\begin{cases} A \subseteq Q, \ \ A = co\,(\{x_0\} \cup F(A)) \text{ with } \ C \subseteq A \\ \text{countable and } \ \overline{C^w} = \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies } \ \overline{C^w} \text{ is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Theorem 1.5. *Let Q be a nonempty, closed, convex subset of a metrizable locally convex linear topological space E and let $x_0 \in Q$. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph and assume the following conditions hold:*

$$\begin{cases} A \subseteq Q, \ A = co\,(\{x_0\} \cup F(A)) \text{ with } \ \overline{C^w} = \overline{A^w}\,(= \overline{A}) \\ \text{and } \ C \subseteq A \text{ countable, implies } \ \overline{A^w}\,(= \overline{A}) \text{ is weakly compact,} \end{cases}$$

and

$$F \text{ maps separable sets in } Q \text{ to separable sets in } Q.$$

Then F has a fixed point in Q.

Remark 1.6. In Theorem 1.5 we can replace E metrizable with E a Šmulian space (i.e., E is such that if the weak closure of a subset Ω of E is weakly compact then

(i) Ω is weakly sequentially compact; and

(ii) if $x \in \overline{\Omega^w}$ then there exists a sequence (x_n) in Ω with $x_n \rightharpoonup x$).

2. Main results

Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of X. In this section we consider a class $\mathbf{A}$ of maps.

Definition 2.1. We say $F \in M(\overline{U^w}, X)$ if $F : \overline{U^w} \to 2^X$ and $F \in \mathbf{A}(\overline{U^w}, X)$. Here $\overline{U^w}$ denotes the weak closure of U in X.

Definition 2.2.

(i) We say $F \in M^M(\overline{U^w}, X)$, if $F \in M(\overline{U^w}, X)$ and if $D \subseteq \overline{U^w}$ and $D \subseteq \overline{co}\,(\{0\} \cup F(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup F(C))$, then $\overline{C^w}$ is weakly compact.

(ii) We say $G \in M^{MM}(\Omega, X)$ (here $\Omega \subseteq X$), if $G \in M(\Omega, X)$ and if $D \subseteq \Omega$, $D = \overline{co}\,(\{0\} \cup G(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup G(C))$ (or $\overline{C^w} = \overline{co}\,(\{0\} \cup G(C))$), then $\overline{C^w}$ is weakly compact.

Definition 2.3. We say $F \in M^M_{\partial U}(\overline{U^w}, X)$, if $F \in M^M(\overline{U^w}, X)$ and $x \notin F(x)$ for $x \in \partial U$. Here ∂U denotes the weak boundary of U in X.

Definition 2.4. Let $F \in M^M_{\partial U}(\overline{U^w}, X)$. We say $F : \overline{U^w} \to 2^X$ is essential in $M^M_{\partial U}(\overline{U^w}, X)$ if for any map $J \in M^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = F|_{\partial U}$ there exists an $x \in U$ with $x \in J(x)$.

Remark 2.5.

(i) Note if $F \in M^M_{\partial U}(\overline{U^w}, X)$ is essential in $M^M_{\partial U}(\overline{U^w}, X)$, then there exists an $x \in U$ with $x \in F(x)$ (take $J = F$ in Definition 2.4).

(ii) In Definition 2.2 (and throughout the paper) we could replace $\{0\}$ with $\{x_0\}$ where $x_0 \in X$ is fixed.

We begin with a nonlinear alternative of Leray–Schauder type (a more general result will be presented in Theorem 2.14).

Theorem 2.6. *Let X be a Hausdorff locally convex topological vector space, U a weakly open subset of X and $F \in M^M(\overline{U^w}, X)$. Assume the following conditions hold:*

$$\begin{cases} \text{the zero map (denoted by } 0\text{) is in } M(\overline{U^w}, X) \\ \text{and } \ 0 \text{ is essential in } \ M^M_{\partial U}(\overline{U^w}, X), \end{cases} \tag{2.1}$$

$$x \notin t\,F(x) \text{ for every } x \in \partial U \text{ and } t \in (0,1), \tag{2.2}$$

and

$$\begin{cases} \mu F \in M(\overline{U^w}, X) \text{ for any weakly continuous} \\ \text{map } \ \mu : \overline{U^w} \to [0,1] \text{ with } \ \mu(\partial U) = 0. \end{cases} \tag{2.3}$$

Let $\Omega = \{x \in \overline{U^w} : \ x \in t\,F(x) \text{ for some } t \in [0,1]\}$ *and we suppose*

$$\Omega \ \text{ is weakly compact.} \tag{2.4}$$

Then there exists an $x \in \overline{U^w}$ with $x \in F(x)$.

Remark 2.7. Note $0 \in M^M(\overline{U^w}, X)$, since if $D \subseteq \overline{U^w}$, $D \subseteq \overline{co}\,(\{0\} \cup 0(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup 0(C))$ then since $0(x) = \{0\}$ for $x \in C$ we have (trivially) that $\overline{C^w}$ is weakly compact.

Proof. Suppose $x \notin F(x)$ for $x \in \partial U$ (otherwise we are finished). Let Ω be as in the statement of Theorem 2.6 and note (2.1) guarantees that $\Omega \neq \emptyset$. Also note $\Omega \cap \partial U = \emptyset$ (see (2.2), $x \notin F(x)$ for $x \in \partial U$ is assumed at the beginning of the proof, and $0 \in M^M_{\partial U}(\overline{U^w}, X)$). Now $X = (X, w)$, the space X endowed with the weak topology, is completely regular. Thus there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = \mu(x)\,F(x)$ and note (2.3) guarantees that $R \in M(\overline{U^w}, X)$. We now show $R \in M^M(\overline{U^w}, X)$. To see this let $D \subseteq \overline{U^w}$ and $D \subseteq \overline{co}\,(\{0\} \cup R(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup R(C))$. Note $R(C) \subseteq co\,(\{0\} \cup F(C))$, $R(D) \subseteq co\,(\{0\} \cup F(D))$ so

$$\overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup co\,(\{0\} \cup F(D))) = \overline{co}\,(co\,(\{0\} \cup F(D))) = \overline{co}\,(\{0\} \cup F(D)),$$

and $\overline{co}\,(\{0\} \cup R(C)) \subseteq \overline{co}\,(\{0\} \cup F(C))$. Thus

$$D \subseteq \overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup F(D)),$$

and
$$C \subseteq \overline{co}\,(\{0\} \cup R(C)) \subseteq \overline{co}\,(\{0\} \cup F(C)).$$
Then since $F \in M^M(\overline{U^w}, X)$ we have that $\overline{C^w}$ is weakly compact. Thus $R \in M^M(\overline{U^w}, X)$. Next notice if $x \in \partial U$ then $R(x) = \{0\}$ (note $\mu(\partial U) = 0$) and since $0 \in U$ then $x \notin R(x)$. As a result $R \in M^M_{\partial U}(\overline{U^w}, X))$ with $R|_{\partial U} = 0|_{\partial U}$ and since 0 is essential in $M^M_{\partial U}(\overline{U^w}, X)$ then there exists a $x \in U$ with $x \in R(x) = \mu(x)\,F(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in F(x)$. □

Remark 2.8. Suppose

(i) $\mathbf{A}(\overline{U^w}, X)$ is the class of maps $F : \overline{U^w} \to K(X)$ with weakly sequentially closed graph and F takes relatively weakly compact sets into relatively weakly compact sets; and

(ii) X is a Eberlein–Šmulian space (i.e., X is a Šmulian space and X is such that the weak closure of a subset C of X is weakly compact, if and only if, C is weakly sequentially compact).

Then (2.3) and (2.4) hold.

To show (2.4) first we show Ω is weakly sequentially closed. To see this let (x_n) be sequence of Ω which converges weakly to some $x \in \overline{\Omega^w}$ (in particular $x \in \overline{U^w}$) and let (λ_n) be a sequence of $[0, 1]$ satisfying $x_n \in \lambda_n\,F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n\,z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0, 1]$ and without loss of generality assume $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges weakly to some $z \in \overline{U^w}$ with $x = \lambda\,z$. Since F has weakly sequentially closed graph then $z \in F(x)$. Hence $x \in \lambda\,F\,x$ and therefore $x \in \Omega$. Thus Ω is weakly sequentially closed. Now let $\{x_n\}_{n=1}^\infty$ be a sequence in Ω. Then there exists a sequence $\{t_n\}_{n=1}^\infty$ in $[0, 1]$ with $x_n \in t_n\,F\,x_n$ and we may assume without loss of generality that $t_n \to t \in [0, 1]$. Let $C = \{x_n\}_{n=1}^\infty$. Note C is countable and $C \subseteq co\,(\{0\} \cup F(C))$. Since $F \in M^M(\overline{U^w}, X)$ (take $D = C$) we have that $\overline{C^w}$ is weakly compact. Now since X is a Eberlein–Šmulian space then there is a subsequence N of $\{1, 2, \cdots\}$ and a $x \in \overline{C^w}$ with $x_n \rightharpoonup x$ as $n \to \infty$ in N. Now since Ω is weakly sequentially closed we have $x \in \Omega$. Consequently Ω is weakly sequentially compact, so $\overline{\Omega^w}$ is weakly compact since X is a Eberlein–Šmulian space. In fact $\overline{\Omega^w} = \Omega$. To see this let $z \in \overline{\Omega^w}$. Then there exists a sequence (z_n) in Ω with $z_n \rightharpoonup z$ (since X is a Šmulian space). Since Ω is weakly sequentially closed we have $z \in \Omega$, so $\overline{\Omega^w} = \Omega$. Thus (2.4) holds.

To show (2.3) we first note that $R = \mu\,F$ has weakly sequentially closed graph since F has weakly sequentially closed graph and μ is weakly continuous. Next suppose $A \subseteq \overline{U^w}$ is weakly compact and $y_n \in R(A)$. Then $y_n = \mu(x_n)\,z_n$ where $z_n \in F(x_n)$ and $x_n \in A$. Without loss of generality we may assume there exists $x \in A$ and $z \in F(A)$ with $x_n \rightharpoonup x$ and $z_n \rightharpoonup z$ (recall A and $\overline{F(A)}^w$ are weakly compact and in fact a standard result (see [1, P. 87] where one replaces E a Banach space with E a Šmulian space) guarantees that $F : A \to K(X)$ has weakly closed graph and from another standard result (see [1, P. 37]) we have that $F(A)$ is weakly compact). Then $z \in F(x)$ since F has weakly sequentially closed graph. Let $y = \mu(x)\,z$. Then $y_n \rightharpoonup y$ and $y \in R(A)$. Thus $R(A)$ is weakly sequentially compact so $\overline{R(A)}^w$ is weakly compact since X is a Eberlein–Šmulian space and again since X is a Šmulian space we have $\overline{R(A)}^w = R(A)$. Thus $R = \mu\,F$ takes relatively weakly compact sets into relatively weakly compact sets so $\mu\,F \in \mathbf{A}(\overline{U^w}, X)$.

We now present a result which guarantees (2.1).

Theorem 2.9. *Let X be a Hausdorff locally convex topological vector space, U a weakly open subset of X, $0 \in U$ and assume the following conditions hold:*
$$0 \in M(\overline{U^w}, X), \tag{2.5}$$

$$\begin{cases} \text{for any map } J \in M^M_{\partial U}(\overline{U^w}, X) \text{ with } J|_{\partial U} = 0|_{\partial U} \text{ and} \\ R(x) = \begin{cases} J(x), & x \in \overline{U^w}, \\ \{0\}, & x \in X \setminus \overline{U^w}, \end{cases} \\ \textit{we have that } R \in M(X, X), \end{cases} \tag{2.6}$$

$$\begin{cases} \text{for any map } J \in M^M_{\partial U}(\overline{U^w}, X) \text{ with } J|_{\partial U} = 0|_{\partial U} \text{ and for any} \\ \text{countable set } P \subseteq X \text{ with } P \cap \overline{U^w} \text{ relatively weakly compact} \\ \text{we have that the set } \overline{co}\,(\{0\} \cup J(P \cap \overline{U^w})\,) \text{ is weakly compact,} \end{cases} \tag{2.7}$$

and

$$\begin{cases} \text{for any map } H \in M^{MM}(X, X) \text{ there exists} \\ x \in X \text{ with } x \in H(x). \end{cases} \tag{2.8}$$

Then the zero map is essential in $M^M_{\partial U}(\overline{U^w}, X)$.

Remark 2.10. Note the theorems in Section 1 give conditions to guarantee (2.8) (we might have to change slightly the definition of M^M and M^{MM} depending on the theorem we use).

Remark 2.11. In the proof below we will in fact show R in (2.6) is in $M^{MM}(X, X)$ so one could replace (2.8) with "there exists $x \in X$ with $x \in R(x)$".

Proof. Let $J \in M^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = 0|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Let R be as in (2.6) and note $R \in M(X, X)$. We claim $R \in M^{MM}(X, X)$. To see this let $D \subseteq X$ and $D = \overline{co}\,(\{0\} \cup R(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup R(C))$ (or $\overline{C^w} = \overline{co}\,(\{0\} \cup R(C))$). First note $\overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w}))$ so $D = \overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w}))$ and $C \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w}))$. As a result

$$D \cap \overline{U^w} \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w})) \quad \text{and} \quad C \cap \overline{U^w} \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w})), \tag{2.9}$$

note $C \cap \overline{U^w}$ is countable. Now since $J \in M^M(\overline{U^w}, X)$ we have (see (2.9)) that $\overline{C \cap \overline{U^w}}^w$ is weakly compact. Now (2.7) guarantees that $\overline{C^w}$ is weakly compact (recall $C \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w}))$. Thus $R \in M^{MM}(X, X)$.

Now (2.8) guarantees that there exists a $x \in X$ with $x \in R(x)$. There are two cases to consider, namely $x \in U$ and $x \in X \backslash U$. If $x \in U$ then $x \in J(x)$, and we are finished. If $x \in X \backslash U$ then since $R(x) = \{0\}$ (note also $J|_{\partial U} = 0|_{\partial U}$) we have $0 \in X \backslash U$, and this contradicts $0 \in U$. □

Remark 2.12. If **A** is as in Remark 2.8, then trivially (2.5) and (2.6) hold. Also note (2.7) holds in this situation if we assume the following: if W is a weakly compact subset of X then $\overline{co}\,(W)$ is weakly compact. (This is a Krein–Šmulian type property which we know for example is true if X is a quasicomplete locally convex linear topological space). To see this we just need to note that if P and J are in (2.7) then $\overline{J(P \cap \overline{U^w})}^w$ is weakly compact and $\overline{co}\,(\{0\} \cup J(P \cap \overline{U^w})) \subseteq \overline{co}\,(\{0\} \cup \overline{J(P \cap \overline{U^w})}^w)$.

Our final two results are generalizations of Theorem 2.6.

Theorem 2.13. *Let X be a Hausdorff locally convex topological vector space, U a weakly open subset of X,* $F \in M^M(\overline{U^w}, X)$ *and* $G \in M^M_{\partial U}(\overline{U^w}, X)$ *is essential in* $M^M_{\partial U}(\overline{U^w}, X)$. *Also assume there exists a map* $H : \overline{U^w} \times [0, 1] \to 2^X$ *with* $H(\,.\,, \eta(\,.\,)) \in M^M(\overline{U^w}, X)$ *for any weakly continuous function* $\eta : \overline{U^w} \to [0, 1]$ *with* $\eta(\partial U) = 0$, $x \notin H_t(x)$ *for any* $x \in \partial U$ *and* $t \in (0, 1)$ *(here* $H_t(x) = H(x, t)$*),* $H_1 = F$, $H_0 = G$ *and* $\Omega = \{x \in \overline{U^w} : x \in H(x, t) \text{ for some } t \in [0, 1]\}$ *is weakly compact. Then there exists a* $x \in \overline{U^w}$ *with* $x \in F(x)$.

Proof. Suppose $x \notin F(x)$ for $x \in \partial U$ (otherwise we are finished). Let Ω be as in the statement of Theorem 2.13 and note $\Omega \neq \emptyset$ (note G is essential in $M^M_{\partial U}(\overline{U^w}, X)$ and $H_0 = G$). Note $X = (X, w)$ is completely regular, $\Omega \cap \partial U = \emptyset$ so there exists a weakly continuous map $\mu : \overline{U^w} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H(x, \mu(x))$. Now $R \in M^M_{\partial U}(\overline{U^w}, X)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H(x, 0) = G(x)$ and $x \notin G(x)$). Since G is essential in $M^M_{\partial U}(\overline{U^w}, X)$ there exists $x \in U$ with $x \in R(x) = H_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $x \in H_1(x) = F(x)$. □

It is also possible to generalize slightly the result in Theorem 2.13, if one modifies slightly the assumptions.

Theorem 2.14. *Let* X *be a Hausdorff locally convex topological vector space,* U *a weakly open subset of* X*,* $F \in M^M_{\partial U}(\overline{U^w}, X)$ *and* $G \in M^M_{\partial U}(\overline{U^w}, X)$ *is essential in* $M^M_{\partial U}(\overline{U^w}, X)$*. Also assume for any map* $J \in M^M_{\partial U}(\overline{U^w}, X)$ *with* $J|_{\partial U} = F|_{\partial U}$ *there exists a map* $H^J : \overline{U^w} \times [0,1] \to 2^X$ *with* $H^J(\,.\,,\eta(\,.\,)) \in M^M(\overline{U^w}, X)$ *for any weakly continuous function* $\eta : \overline{U^w} \to [0,1]$ *with* $\eta(\partial U) = 0$*,* $x \notin H^J_t(x)$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $H^J_t(x) = H^J(x,t)$*),* $H^J_1 = J$*,* $H^J_0 = G$ *and* $\Omega = \left\{x \in \overline{U^w} : \; x \in H^J(x,t) \text{ for some } \; t \in [0,1]\right\}$ *is weakly compact. Then* F *is essential in* $M^M_{\partial U}(\overline{U^w}, X)$*.*

Proof. Consider any map $J \in M^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Choose the map H^J and the set Ω as in the statement of Theorem 2.14 and note $\Omega \neq \emptyset$ (note G is essential in $M^M_{\partial U}(\overline{U^w}, X)$ and $H^J_0 = G$). Note $X = (X, w)$ is completely regular, $\Omega \cap \partial U = \emptyset$ so there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H^J(x, \mu(x))$. Now $R \in M^M_{\partial U}(\overline{U^w}, X)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x,0) = G(x)$) and since G is essential in $M^M_{\partial U}(\overline{U^w}, X)$ there exists $x \in U$ with $x \in R(x) = H^J_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $x \in H^J_1(x) = J(x)$. □

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