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FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract

In this paper we study the approximation of common fixed points of a finite family of nonexpansive mappings in uniformly smooth Banach spaces. Also we show that the convergence of the proposed algorithm can be proved under some types of control conditions.

Keywords: Nonexpansive mapping, Strong convergence, Common fixed point, Uniformly smooth Banach space.

1. Introduction

Let X be a real Banach space. A mapping $T : X \to X$ is said to be nonexpansive provided that

$$\| Tx - Ty \| \le \| x - y \|, \qquad x, y \in X.$$

Let T_i, $i = 1, 2, \ldots, N$ be a finite family of nonexpansive self-mappings on X, and let $Fix(T_i) = \{x \in X : T_i x = x\}$. We shall assume that $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Assume that for each natural number n, there correspond N real numbers $\alpha_{n,1}, \alpha_{n,2}, ..., \alpha_{n,N}$ in the half-open interval (0, 1]. Given N self-mappings $T_1, T_2, \ldots, T_N$ as in [4] one can define, for each n the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ in the following way:

$$U_{n,1} = \alpha_{n,1}T_1 + (1-\alpha_{n,1})I$$
$$U_{n,2} = \alpha_{n,2}T_2U_{n,1} + (1-\alpha_{n,2})I \quad .$$
$$.$$
$$.$$
$$U_{n,N-1} = \alpha_{n,N-1}T_{n-1}U_{n,N-2} + (1-\alpha_{n,N-1})I$$
$$W_n := U_{n,N} = \alpha_{n,N}T_NU_{n,N-1} + (1-\alpha_{n,N})I \qquad (1.1)$$

Nonexpansivity of T_i yields the nonexpansivity of W_n. Moreover, in [4], it is shown that $Fix(W_n) = F$.

In the case X equals a real Hilbert space H, Y. Yao in [8] introduced an iterative algorithm to approximate the common fixed points of a finite family of nonexpansive self-mappings defined on the real Hilbert space H. To write down Yao's proposed scheme, we recall that $f : H \to H$ is said to be a contraction if there exists a number $0 < \alpha < 1$ such that

$$\| f(x) - f(y) \| \le \alpha \| x - y \|, \quad x, y \in H .$$

Recall also that a bounded linear operator A on the Hilbert space H is said to be strongly positive if there exists a positive number $\bar{\gamma}$ such that

$$\langle Ax, x \rangle \ge \bar{\gamma} \| x \|^2, \quad x \in H.$$

Assuming that I denotes the identity operator on H and $\{\lambda_n\}$ is a sequence of real numbers, and $u_0 \in H$ is arbitrarily chosen, Yao introduced

$$u_{n+1} = \alpha_n \gamma f(u_n) + \beta u_n + ((1-\beta)I - \lambda_n A)W_n u_n$$

Where γ, β are two positive real numbers such that $\beta < 1$, $f : H \to H$ is a contraction with coefficient $0<\alpha<1$, A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and W_n is the self-mapping of H generated by (1.1).

Under the assumption that the sequence $\{\lambda_n\}$ satisfies the following two control conditions

$$(C_1): \lim_{n\to\infty} \lambda_n = 0,$$

$$(C_2): \sum_{n=1}^{\infty} \lambda_n = \infty,$$

Yao proved that the iterative sequence $\{u_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)u^*, u - u^* \rangle \ge 0 \qquad u \in F$$

Let X be a real Banach space. For t > 0, we define

$$\rho(\mathrm{t}) = \frac{1}{2} \sup \{ \|\mathrm{x} + \mathrm{ty}\| + \|\mathrm{x} - \mathrm{ty}\| - 2 : \|x\| = \|y\| = 1\} .$$

A Banach space X is said to be uniformly smooth if

$$\lim_{t\to 0} \frac{\rho(t)}{t} = 0 .$$

Geometrically, this means that the function $x \mapsto h(x) = \| x\|$ is uniformly Ferechet-differentiable on the unit sphere, that is,

$$\lim_{t\to 0^+} \sup_{\|x\| = \|y\| = 1} \left| \frac{\| x+ty\| - \| x\|}{t} - \langle h'(x), y \rangle \right| = 0.$$

Let X be a real Banach space. Recall the (normalized) duality mapping J from X to X^* the dual space of X, is given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \| x\|^2 = \| x^*\|^2\} \qquad (1.2)$$

In this paper we assume that X is a uniformly smooth Banach space and C is a closed convex subset of X. We further assume that $f : C \to C$ is a contraction with coefficient $0 < \alpha < 1$. By a slight modification of the sequence introduced by Yao, and replacing the strongly positive operator A by the identity operator I acting instead on a uniformly smooth Banach space X, we shall see that the sequence

$$u_{n+1} = \lambda_n f(u_n) + \beta u_n + (1-\beta-\lambda_n)W_n u_n$$

Converges strongly to the unique solution of the variational inequality

$$\langle (I-f)u^*, u-u^* \rangle \geq 0 \qquad u \in F. \qquad (1.3)$$

Here $0 < \beta < 1$, and W_n 's are the self-mappings of C generated by (1.1), moreover the sequence $\{\lambda_n\}$ satisfies the control conditions $(C_1), (C_2)$.

2. PRELIMINARIES

This section collects some lemmas which will be used in the proofs for the main results in the next section.

Lemma 2.1. [7] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\| y_{n+1} - y_n\| - \| x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \| y_n - x_n\| = 0$.

Lemma 2.2. [6] Let X be a real Banach space. Then for all $x, y \in X$

$$\| x+y \|^2 \leq \| x\|^2 + 2\langle y, J(x+y) \rangle$$

Lemma 2.3. [1] Assume $\{a_n\}$ to be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n) a_n + \delta_n \gamma_n, \quad \forall n \geq 0$$

where $\{\gamma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$

(ii) either $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0.$

Proposition2.4. [3] If X is a uniformly smooth Banach space, then the normalized duality mapping J defined by (1.2) is single-valued and uniformly continuous on each bounded subset of X from the norm topology of X to the norm topology of X^*.

Lemma 2.5. [2] Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T:C\to C$ a nonexpansive mapping with $Fix(T)\neq\emptyset$, and $f:C\to C$ be a contraction. then $\{x_t\}$ defined by

$$x_t = tf(x_t)+(1-t)Tx_t \qquad (2.1)$$

Convergence strongly to a point in $Fix(T)$.

3. Main results

We now state and prove the main result of this paper.

Theorem 3.1. Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, f be a contraction on C with coefficient $0<\alpha<1$, choose any $u_0\in C$. Define $\{u_n\}$ by

$$u_{n+1} = \lambda_n f(u_n)+\beta u_n+(1-\beta-\lambda_n)W_n u_n$$

Where β is a positive real number such that $\beta<1$ and W_n is the self-mapping of C generated by (1.1), suppose the sequence $\{\alpha_{n,i}\}$ satisfy $\lim_{n\to\infty}(\alpha_{n,i}-\alpha_{n-1,i})=0$ and the sequence $\{\lambda_n\}$ satisfying the control conditions $(C_1),(C_2)$. Then the sequence $\{u_n\}$ converges strongly to the unique solution of the variational inequality defined by (1.3).

Proof. First we observe that $\{u_n\}$ is bounded. Indeed, pick any $p\in F$ to obtain

$$\begin{aligned}\| u_{n+1}-p\| &\le \lambda_n\| f(u_n)-f(p)\| +\beta\| u_n-p\| +(1-\beta-\lambda_n)\| W_n u_n-p\| +\lambda_n\| f(p)-p\|\\ &\le (\lambda_n\alpha+1-\lambda_n)\| u_n-p\| +\lambda_n\| f(p)-p\|\\ &=[1-\lambda_n(1-\alpha)]\| u_n-p\| +\lambda_n\| f(p)-p\| \qquad (3.1)\end{aligned}$$

It follows from (1.3) by induction that

$$\| u_n-p\| \le \max\{\| u_0-p\|,\frac{\| f(p)-p\|}{1-\alpha}\}$$

Hence, $\{u_n\}$ is bounded, and so are $\{f(u_n)\}$ and $\{W_n u_n\}$.

Next, we claim that $\lim_{n\to\infty}\| u_{n+1}-u_n\| =0.$

Define

$$u_{n+1}=(1-\beta)z_n+\beta u_n$$

We shall use M to denote the possible different constants appearing in the following reasoning. Observe that from the definition of z_n, we obtain

$$z_{n+1}-z_n=\frac{u_{n+2}-\beta u_{n+1}}{1-\beta}-\frac{u_{n+1}-\beta u_n}{1-\beta}$$
$$=\frac{\lambda_{n+1}f(u_{n+1})+(1-\beta-\lambda_{n+1})W_{n+1}u_{n+1}}{1-\beta}-\frac{\lambda_n f(u_n)+(1-\beta-\lambda_n)W_n u_n}{1-\beta}$$
$$=\frac{\lambda_{n+1}}{1-\beta}f(u_{n+1})-\frac{\lambda_n}{1-\beta}f(u_n)+W_{n+1}u_{n+1}-W_n u_n+\frac{\lambda_n}{1-\beta}W_n u_n-\frac{\lambda_{n+1}}{1-\beta}W_{n+1}u_{n+1}$$
$$=\frac{\lambda_{n+1}}{1-\beta}[f(u_{n+1})-W_{n+1}u_{n+1}]+W_{n+1}u_{n+1}+\frac{\lambda_n}{1-\beta}[W_n u_n-f(u_n)]-W_n u_n\pm W_{n+1}u_n.$$

It follows that

$$\|z_{n+1}-z_n\|-\|u_{n+1}-u_n\|\le\frac{\lambda_{n+1}}{1-\beta}(\|f(u_{n+1})\|+\|W_{n+1}u_{n+1}\|)$$
$$+\frac{\lambda_n}{1-\beta}(\|f(u_n)\|+\|W_n u_n\|)+\|W_{n+1}u_{n+1}-W_{n+1}u_n\|$$
$$+\|W_{n+1}u_n-W_n u_n\|-\|u_{n+1}-u_n\|$$
$$\le\frac{\lambda_{n+1}}{1-\beta}(\|f(u_{n+1})\|+\|W_{n+1}u_{n+1}\|)$$
$$+\frac{\lambda_n}{1-\beta}(\|f(u_n)\|+\|W_n u_n\|)+\|W_{n+1}u_n-W_n u_n\| \quad (3.2)$$

From (1.1), since T_N and $U_{n,N}$ are nonexpansive,

$$\|W_{n+1}u_n-W_n u_n\|-\|u_{n+1}-u_n\|$$
$$=\|\alpha_{n+1,N}T_N U_{n+1,N-1}u_n+(1-\alpha_{n+1,N})u_n-\alpha_{n,N}T_N U_{n,N-1}u_n-(1-\alpha_{n,N})u_n\|$$
$$\le|\alpha_{n+1,N}-\alpha_{n,N}|\|u_n\|+\|\alpha_{n+1,N}(T_N U_{n+1,N-1}u_n-T_N U_{n,N-1}u_n)\|$$
$$\le 2M|\alpha_{n+1,N}-\alpha_{n,N}|+\alpha_{n+1,N-1}\|U_{n+1,N-1}u_n-U_{n,N-1}u_n\| \quad (3.3)$$

Again, from (1.1),

$$\|U_{n+1,N-1}u_n-U_{n,N-1}u_n\|=\|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}u_n+(1-\alpha_{n+1,N-1})u_n$$
$$-\alpha_{n,N-1}T_{N-1}U_{n,N-2}u_n-(1-\alpha_{n,N-1})u_n$$
$$\le|\alpha_{n+1,N-1}-\alpha_{n,N-1}|\|u_n\|+\|\alpha_{n+1,N-1}U_{n+1,N-2}u_n-\alpha_{n,N-1}T_{N-1}U_{n,N-2}u_n\|$$
$$\le|\alpha_{n+1,N-1}-\alpha_{n,N-1}|\|u_n\|+\alpha_{n+1,N-1}\|U_{n+1,N-2}u_n-U_{n,N-2}u_n\|$$
$$+|\alpha_{n+1,N-1}-\alpha_{n,N-1}|\|T_{N-1}U_{n,N-2}u_n\|$$
$$\le 2M|\alpha_{n+1,N-1}-\alpha_{n,N-1}|+\|U_{n+1,N-2}u_n-U_{n,N-2}u_n\| \quad (3.4)$$

Therefore, we have

$$\|U_{n+1,N-2}u_n-U_{n,N-2}u_n\|\le 2M|\alpha_{n+1,N-1}-\alpha_{n,N-1}|+2M|\alpha_{n+1,N-2}-\alpha_{n,N-2}|$$
$$\le 2M\sum_{i=2}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}|+\|U_{n+1,1}u_n-U_{n,1}u_n\|$$
$$=\|\alpha_{n+1,1}T_1u_n+(1-\alpha_{n+1,1})-\alpha_{n,1}T_1u_n-(1-\alpha_{n,1})u_n\|+2M\sum_{i=2}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}|$$

$$\leq |\alpha_{n+1,1}-\alpha_{n,1}| \, \| u_n \| + \| \alpha_{n+1,1} T_1 u_n - \alpha_{n,1} T_1 u_n \| + 2M \sum_{i=2}^{N-1} | \alpha_{n+1,i} - \alpha_{n,i} |$$

$$\leq 2M \sum_{i=1}^{N-1} | \alpha_{n+1,i} - \alpha_{n,i} | \qquad (3.5)$$

Substituting (3.5) into (3.3), we have

$$\begin{aligned} \| W_{n+1} u_n - W_n u_n \| &\leq 2M \, |\alpha_{n+1,N} - \alpha_{n,N}| + 2\alpha_{n+1,N} M \sum_{i=1}^{N-1} | \alpha_{n+1,i} - \alpha_{n,i} | \\ &\leq 2M \, |\alpha_{n+1,N} - \alpha_{n,N}| + 2M \sum_{i=1}^{N-1} | \alpha_{n+1,i} - \alpha_{n,i} | \\ &= 2M \sum_{i=1}^{N} | \alpha_{n+1,i} - \alpha_{n,i} | \qquad (3.6) \end{aligned}$$

Using (3.6) in (3.2), we get

$$\begin{aligned} \| z_{n+1} - z_n \| - \| u_{n+1} - u_n \| \leq & \frac{\lambda_{n+1}}{1-\beta} (\| f(u_{n+1}) \| + \| W_{n+1} u_{n+1} \|) \\ & + \frac{\lambda_n}{1-\beta} (\| f(u_n) \| + \| W_n u_n \|) + 2M \sum_{i=1}^{N} |\alpha_{n+1,i} - \alpha_{n,i}| \end{aligned}$$

Which implies that

$$\limsup_{n\to\infty} (\| z_{n+1} - z_n \| - \| u_{n+1} - u_n \|) \leq 0$$

We can obtain $\lim_{n\to\infty} \| z_n - u_n \| = 0$ easily by lemma (2.1). Consequently

$$\lim_{n\to\infty} \| u_{n+1} - u_n \| = \lim_{n\to\infty} (1-\beta) \| z_n - u_n \| = 0$$

We can write the sequence $\{u_n\}$ as follow

$$u_{n+1} = \lambda_n f(u_n) - \lambda_n W_n u_n + y_n$$

Where $y_n = W_n u_n + \beta (u_n - W_n u_n)$

Observing that $u_{n+1} - y_n = \lambda_n (f(u_n) - W_n u_n)$, we can easily get

$$\lim_{n\to\infty} \| y_n - u_{n+1} \| = 0$$

$$\| u_n - y_n \| \leq \| u_n - u_{n+1} \| + \| u_{n+1} - y_n \|$$

That is, $\lim_{n\to\infty} \| y_n - u_n \| = 0$. On the other hand, we have

$$\| W_n u_n - u_n \| \leq \| u_n - y_n \| + \| y_n - W_n u_n \| \leq \| u_n - y_n \| + \beta \| u_n - W_n u_n \|$$

Which implies $(1-\beta) \| W_n u_n - u_n \| \leq \| u_n - y_n \|$ but we have $\beta \in (0,1)$ so we obtain

$$\lim_{n\to\infty} \| W_n u_n - u_n \| = 0 \qquad (3.7)$$

Next, we claim that

$$\limsup_{n\to\infty} \langle f(p) - p, J(u_{n+1} - p) \rangle \leq 0 \qquad (3.8)$$

Let u_t be the unique fixed point of the contraction mapping given by

$$u \mapsto t\, f(u) + (1-t)\, W_n u$$

Then, u_t solves the fixed point equation $u_t = t\, f(u_t) + (1-t)\, W_n u_t \quad t \in (0,1)$. Thus

$$u_t - u_n = t\ (f(u_t) - u_n) + (1-t)\ (W_n u_t - u_n).$$

We apply Lemma 2.2 to get

$$\| u_t - u_n\|^2 \le (1-t)^2 \| W_n u_t - u_n\|^2 + 2t\langle f(u_t) - u_n, J(u_t - u_n)\rangle$$
$$\le (\| W_n u_t - W_n u_n\| + \| W_n u_n - u_n\|)^2 + 2t\ \langle f(u_t) - u_t, J(u_t - u_n)\rangle + 2t\langle u_t - u_n, J(u_t - u_n)\rangle$$
$$\le (1-t)^2 (\| u_t - u_n\|^2 + a_n(t)) + 2t\| u_t - u_n\|^2 + 2t\langle f(u_t) - u_t, J(u_t - u_n)\rangle \qquad (3.9)$$

Where

$$a_n(t) = (2\|u_t - u_n\| + \| W_n u_n - u_n\|)\|W_n u_n - u_n\| \qquad (3.10)$$

Noting (3.7) $\lim_{n\to\infty} a_n(t) = 0$

The last inequality (3.9) implies

$$\langle u_t - f(u_t), J(u_t - u_n)\rangle \le \frac{t}{2}\| u_t - u_n\|^2 + \frac{(1-t)^2}{2t} a_n(t)$$

It follows that

$$\limsup_{n\to\infty}\ \langle u_t - f(u_t), J(u_t - u_n)\rangle \le \frac{t}{2} M^2 \qquad (3.11)$$

Where $M > 0$ is a constant such that $M \ge \| u_t - u_n\|$ for all $t \in (0,1)$ and $n \ge 1$. Letting $t \to 0$ in (3.11) and noting (3.10) yields

$$\limsup_{n\to\infty}\ \langle u_t - f(u_t), J(u_t - u_n)\rangle \le 0.$$

Moreover, we have that

$$\langle p - f(p), J(p - u_n)\rangle = \langle p - f(p), J(p - u_n) - J(u_t - u_n)\rangle$$
$$+\langle p - f(p) - u_t + f(u_t), J(u_t - u_n)\rangle + \langle u_t - f(u_t), J(u_t - u_n)\rangle$$
$$= \langle p - f(p), J(p - u_n) - J(u_t - u_n)\rangle$$
$$+ \langle p - u_t, J(u_t - u_n)\rangle + \langle f(u_t) - f(p), J(u_t - u_n)\rangle$$
$$+ \langle u_t - f(u_t), J(u_t - u_n)\rangle$$

Then, we obtain

$$\limsup_{n\to\infty}\ \langle p - f(p), J(p - u_n)\rangle \le \limsup_{n\to\infty}\langle p - f(p), J(p - u_n) - J(u_t - u_n)\rangle$$
$$+\| p - u_t\| \limsup_{n\to\infty}\| u_t - u_n\| + \| f(u_t) - f(p)\| \limsup_{n\to\infty}\| u_t - u_n\|$$
$$+\limsup_{n\to\infty}\ \langle u_t - f(u_t), J(u_t - u_n)\rangle$$
$$\le \limsup_{n\to\infty}\langle p - f(p), J(p - u_n) - J(u_t - u_n)\rangle$$
$$+ (1+\alpha)\| p - u_t\| \limsup_{n\to\infty}\| u_t - u_n\|$$
$$+ \limsup_{n\to\infty}\langle u_t - f(u_t), J(u_t - u_n)\rangle$$

By Lemma 2.5, $u_t \to p \in F$ as $t \to 0$, which is the unique solution of the variational inequality (1.3). Noting proposition 2.4, we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle p - f(p), J(p - u_n) - J(u_t - u_n)\rangle = 0$$

Therefore we have

$$\limsup_{n\to\infty}\langle p-f(p),J(p-u_n)\rangle=\limsup_{t\to 0}\limsup_{n\to\infty}\langle p-f(p),J(p-u_n)\rangle$$
$$\le\limsup_{t\to 0}\limsup_{n\to\infty}\langle u_t-f(u_t),J(u_t-u_n)\rangle\le 0.$$

Finally we show that $\lim_{n\to\infty}u_n=p$.

From Lemma 2.2, we have

$$\|u_{n+1}-p\|^2=\|\lambda_n(f(u_n)-p)+\beta(u_n-p)+(1-\beta-\lambda_n)(W_nu_n-p)\|^2$$
$$\le(\beta\|u_n-p\|+(1-\beta-\lambda_n)\|u_n-p\|)^2+2\lambda_n\langle f(u_n)-p,J(u_{n+1}-p)\rangle$$
$$=(1-\lambda_n)^2\|u_n-p\|^2+2\lambda_n\langle f(u_n)-f(p),J(u_{n+1}-p)\rangle+2\lambda_n\langle f(p)-p,J(u_{n+1}-p)\rangle$$
$$\le(1-\lambda_n)^2\|u_n-p\|^2+\lambda_n\alpha[\|u_n-p\|^2+\|u_{n+1}-p\|^2]+2\lambda_n\langle f(p)-p,J(u_{n+1}-p)\rangle$$

Which implies that

$$\|u_{n+1}-p\|^2\le\frac{(1-\lambda_n)^2+\lambda_n\alpha}{1-\lambda_n\alpha}\|u_n-p\|^2+\frac{2\lambda_n}{1-\lambda_n\alpha}\langle f(p)-p,J(u_{n+1}-p)\rangle$$
$$\le[1-\frac{2\lambda_n(1-\alpha)}{1-\lambda_n\alpha}]\|u_n-p\|^2$$
$$+\frac{2\lambda_n(1-\alpha)}{1-\lambda_n\alpha}[\frac{1}{1-\alpha}\langle f(p)-p,J(u_{n+1}-p)\rangle+\frac{\lambda_n}{2(1-\alpha)}\|u_n-p\|^2]$$
$$=(1-\gamma_n)\|u_n-p\|^2+\delta_n\gamma_n \qquad (3.12)$$

Where $\gamma_n=\frac{2\lambda_n(1-\alpha)}{1-\lambda_n\alpha},\delta_n=\frac{1}{1-\alpha}\langle f(p)-p,J(u_{n+1}-p)\rangle+\frac{\lambda_n}{2(1-\alpha)}\|u_n-p\|^2$

It is easily seen that $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0.$

Finally we apply Lemma 2.3 to (3.12) to conclude that $\lim_{n\to\infty}u_n=p.$ This complete the proof.

References

1. H. K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 241 (2000)
2. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279-291.
3. K. Goebel, S. Reich, Uniform Convexity, Nonexpansive mappings and Hyperbolic Geometry, Dekker, 1984.
4. S. Atsushiba, W. Takahashi Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math. 41 (1999), 435–453.
5. Sh. Banerjee, B. S. Choudhury, Weak and strong convergence theorems of a new iterative process with errors for common fixed points of a finite families of asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. TJMCS, 11(2014), 79-85
6. S. S. Chang, Some problems and results in the study of nonlinear analysis, Nonlinear Anal. 33 (1997) 4197-4208.
7. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter

Nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305(2005) 227-239

8. Y. Yao, A general iterative method for a finite family of nonexpansive mappings, Nonlinear Anal. 66 (2007), 2676-2687