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The weight inequalities on Reich type theorem in b-metric spaces

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Abstract

In this note, we give a generalization of the Reich type theorem in b-metric spaces by using weight inequalities. Here, the existence of nonunique fixed points is ensured. Other known fixed point results in the literature are derived.

Keywords: Fixed point, b-metric space, weight inequalities.

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1. Introduction and preliminaries

Bakhtin [6] and Czerwik [9] introduced the notion of b-metric spaces and proved some fixed point theorems in b-metric spaces. A large number of results in fixed point theory in b-metric spaces and other generalized metric spaces has been obtained over the past ten years. For more details, see [1, 3–5, 11, 17–22, 24]. We begin with two known definitions.

Definition 1.1. Let X be a nonempty set and let $b \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leqslant b[d(x, y) + d(y, z)]$.

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A triplet (X, d, b), is called a b-metric space.

Note that a metric space is included in the class of b-metric spaces. The topological notions of a convergent sequence, a Cauchy sequence and a complete space are defined as in metric spaces.

Definition 1.2. Let (X, d, b) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(a) The sequence $\{x_n\}$ is said to be convergent in (X, d, b) to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. This fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy in (X, d, b), if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$.

(c) (X, d, b) is said to be complete, if every Cauchy sequence in X converges to some $x \in X$.

In this paper, we use the following result of Miculescu and Mihail [15, Lemma 2.2] and Suzuki [27, Lemma 6].

Lemma 1.3. *Let (X, d, b) be a b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\gamma \in [0, 1)$ satisfying $d(x_{n+1}, x_n) \leqslant \gamma d(x_n, x_{n-1})$ for any $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy.*

2. Main results

Definition 2.1. In the framework of a b-metric space (X, d, b), a mapping $T : X \to X$ is called an (r, a)-weight type contraction, if there exists $\lambda \in [0, 1)$ and such that

$$d(Tx, Ty) \leqslant \lambda M^r(T, x, y, a), \tag{2.1}$$

where $r \geqslant 0$, $a = (a_1, a_2, a_3)$, $a_i \geqslant 0, i = 1, 2, 3$ such that $a_1 + a_2 + a_3 = 1$ and

$$M^r(T, x, y, a) = \begin{cases} [a_1(d(x, y))^r + a_2(d(x, Tx))^r + a_3(d(y, Ty))^r]^{1/r}, & r > 0, \\ (d(x, y))^{a_1}(d(x, Tx))^{a_2}(d(y, Ty))^{a_3}, & r = 0, \end{cases} \tag{2.2}$$

for all $x, y \in X \backslash Fix(T)$, where $Fix(T) = \{u \in X, Tu = u\}$.

Remark 2.2. In all following cases, the $x, y \in X$ are such that $x, y \notin Fix(T)$.
1. If $r = 1, a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we obtain Reich-Rus-Ćirić type contraction,

$$d(Tx, Ty) \leqslant \frac{\lambda}{3}[d(x, y) + d(x, Tx) + d(y, Ty)],$$

where $\lambda \in [0, 1)$, see [8, 23, 25].
2. 1. If $r = 2, a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we obtain the following condition,

$$d(Tx, Ty) \leqslant \frac{\lambda}{\sqrt{3}}[d^2(x, y) + d^2(x, Tx) + d^2(y, Ty)]^{1/2},$$

where $\lambda \in [0, 1)$.
3. If $r = 1$ and $a = (a_1, a_2, a_3)$, we have a Reich type contraction,

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)],$$

where $\alpha = \lambda a_1, \beta = \lambda a_2, \gamma = \lambda a_3$, $\alpha, \beta, \gamma, \lambda \in [0, 1)$ and $\alpha + \beta + \gamma < 1$, see [23].
4. If $r = 1$ and $a = (0, \frac{1}{2}, \frac{1}{2})$, we have a Kannan type contraction,

$$d(Tx, Ty) \leqslant \frac{\lambda}{2}[d(x, Tx) + d(y, Ty)],$$

see [14].
5. If $r = 2$ and $a = (0, \frac{1}{2}, \frac{1}{2})$, we have

$$d(Tx, Ty) \leqslant \frac{\lambda}{\sqrt{2}}[d^2(x, Tx) + d^2(y, Ty)]^{1/2}.$$

6. If $r = 0$ and $a = (0, \alpha, 1-\alpha)$ with $\alpha \in (0, 1)$, we obtain an interpolative Kannan type contraction,

$$d(Tx, Ty) \leqslant \lambda(d(x, Tx))^{\alpha}(d(y, Ty))^{1-\alpha},$$

see [12].
7. If $r = 0$ and $a = (\beta, \alpha, 1-\alpha-\beta)$ with $\alpha, \beta \in (0, 1)$, we have an interpolative Reich-Rus-Ćirić type contraction,

$$d(Tx, Ty) \leqslant \lambda(d(x, y))^{\beta}(d(x, Tx))^{\alpha}(d(y, Ty))^{1-\alpha-\beta},$$

see [13].

Lemma 2.3. *If $r \leqslant s$, then we have the following weighted inequality:*

$$M^r(T, x, y, a) \leqslant M^s(T, x, y, a).$$

Proof. See for example [7]. □

Our essential main result is

Theorem 2.4. *Let (X, d, b) be a complete b-metric space and $T : X \to X$ be an (r, a)-weight type contraction mapping. Then T has a fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^* if one of the following conditions holds:*

(i) *T is continuous at such point x^*;*

(ii) $b^r a_2 < 1$;

(iii) $b^r a_3 < 1$.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geqslant 0$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. The proof is completed. From now, assume that $x_n \neq x_{n+1}$ for all $n \geqslant 0$.
1. Case $r > 0$. From condition (2.1), we have that

$$d(x_{n+1}, x_n) \leqslant \lambda[a_1(d(x_n, x_{n-1}))^r + a_2(d(x_n, x_{n+1}))^r + a_3(d(x_{n-1}, x_n))^r]^{1/r}.$$

Therefore,

$$d(x_{n+1}, x_n) \leqslant \left[\frac{\lambda^r(a_1 + a_3)}{1 - \lambda^r a_2}\right]^{1/r} d(x_n, x_{n-1}).$$

Put $\gamma = \left[\frac{\lambda^r(a_1+a_3)}{1-\lambda^r a_2}\right]^{1/r}$. We have that $\gamma \in [0, 1)$. It follows from Lemma 1.3 that $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d, b), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now, we claim that x^* is a fixed point of T. First, for any $n \in \mathbb{N}$, we have

$$d(x^*, Tx^*) \leqslant b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]. \tag{2.3}$$

(i) Suppose that T is a continuous map at the point $x^* \in X$.
Since $\lim_{n\to\infty} d(x^*, x_{n+1}) = 0$ and T is a continuous at a point x^*, we have

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = d(Tx^*, Tx^*) = 0,$$

and from (2.3), we obtain $d(x^*, Tx^*) = 0$, i.e., $Tx^* = x^*$.
(ii) Suppose that $b^r a_2 < 1$. Assume that $Tx^* \neq x^*$. We have

$$\begin{aligned} 0 < d(Tx^*, x^*) &\leqslant b[d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*)] \\ &= b[d(Tx^*, Tx_n) + d(x_{n+1}, x^*)] \\ &\leqslant b[a_1 d((x^*, x_n))^r + a_2(d(x^*, Tx^*))^r + a_3(d(x_n, x_{n+1}))^r]^{1/r} \\ &\quad + bd(x_{n+1}, x^*). \end{aligned}$$

At the limit as $n \to \infty$, we have

$$0 < d(Tx^*, x^*) \leqslant b a_2^{1/r} d(Tx^*, x^*).$$

Since $ba_2^{1/r} < 1$, we have a contradiction, that is, $Tx^* = x^*$.
(iii) Suppose that $b^r a_3 < 1$. Again, assume that $d(Tx^*, x^*) > 0$. Then

$$\begin{aligned} 0 < d(x^*, Tx^*) &\leqslant b[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leqslant bd(x^*, x_{n+1}) + b[a_1(d(x_n, x^*))^r + a_2(d(x_n, x_{n+1}))^r \\ &\quad + a_3(d(x^*, Tx^*))^r]^{1/r}. \end{aligned}$$

Taking $n \to \infty$, we have

$$0 < d(Tx^*, x^*) \leqslant b a_3^{1/r} d(Tx^*, x^*).$$

Since $ba_3^{1/r} < 1$, we get a contradiction. Thus, $Tx^* = x^*$.
2. Case $r = 0$. Here, (2.1) and (2.2) become

$$d(Tx, Ty) \leqslant \lambda (d(x, y))^{a_1} (d(x, Tx))^{a_2} (d(y, Ty))^{1-a_1-a_2},$$

for all $x, y \in X \backslash Fix(T)$, where $\lambda \in [0, 1)$ and $a_1, a_2 \in (0, 1)$. Following [13, Theorem 2.1 with its metric case], the map T has a fixed point in X. Again, following [13, Example 2.1 and Example 2.2], we have not a uniqueness of fixed points. □

Remark 2.5. We note that for $r = 0$, the proof follows from Lemma 2.3 and the case $r > 0$.

We state the following corollaries.

Corollary 2.6. *Let (X, d, b) be a complete b-metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leqslant \lambda d^{a_1}(x, y) \cdot d^{a_2}(x, Tx) \cdot d^{a_3}(y, Ty),$$

for all $x, y \in X \backslash Fix(T)$, where $\lambda \in [0, 1)$, $a_1, a_2, a_3 \geqslant 0$ and $a_1 + a_2 + a_3 = 1$. Then T has a fixed point x^* and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof. Put in Theorem 2.4, $r = 0$ and $a = (a_1, a_2, a_3)$. □

Remark 2.7. We note that from Corollary 2.6, we get [13, Theorem 2] (for metric spaces).

Corollary 2.8. *Let (X, d, b) be a complete b-metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leqslant \lambda \sqrt[3]{d(x, y) \cdot d(x, Tx) \cdot d(y, Ty)}, \tag{2.4}$$

for all $x, y \in X \backslash Fix(T)$, where $\lambda \in [0, 1)$. Then T has a fixed point x^* and for any $x_0 \in X$, the sequence $\{T^n x_0\}$

converges to x^*.

Proof. Put in Theorem 2.4, $r = 0$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. □

Corollary 2.9. *Let* (X, d, b) *be a complete* b*-metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leqslant \frac{\lambda}{3}[d(x, y) + d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X \backslash Fix(T)$*, where* $\lambda \in [0, 1)$ *, then* T *has a fixed point* x^* *and for any* $x_0 \in X$*, the sequence* $\{T^n x_0\}$ *converges to* x^* *if one of the following conditions holds:*

(i) T *is continuous at such point* $x^* \in X$*;*

(ii) $b < 3$.

Proof. Put in Theorem 2.4, $r = 1$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. □

Corollary 2.10. *Let* (X, d, b) *be a complete* b*-metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leqslant \frac{\lambda}{\sqrt{3}}[d^2(x, y) + d^2(x, Tx) + d^2(y, Ty)]^{1/2},$$

for all $x, y \in X \backslash Fix(T)$*, where* $\lambda \in [0, 1)$ *, then* T *has a fixed point* x^* *and for any* $x_0 \in X$*, the sequence* $\{T^n x_0\}$ *converges to* x^* *if one of the following conditions holds:*

(i) T *is continuous at such point* $x^* \in X$*;*

(ii) $b^2 < 3$.

Proof. Put in Theorem 2.4, $r = 2$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. □

Theorem 2.4 is illustrated by the following examples.

Example 2.11. Let $X = \{0, 1, 2, 4\}$ be a set endowed with the classical metric $d(x, y) = |x - y|$ $(b = 1)$, that is,

d(x, y)	0	1	2	4
0	0	1	2	4
1	1	0	1	3
2	2	1	0	2
4	4	3	2	0

We define a self-mapping T on X by $T : \begin{pmatrix} 0 & 1 & 2 & 4 \\ 2 & 4 & 2 & 4 \end{pmatrix}$. It is clear that T is not a Reich-Rus-Ćirić contraction. Indeed, there is no $\lambda \in [0, 1)$ such that the following inequality is fulfilled

$$d(T0, T1) \leqslant \frac{\lambda}{3} [d(0, 1) + d(0, T0) + d(1, T1)] ,$$

namely, we have,

$$2 \leqslant \frac{\lambda}{3} [1 + 2 + 3] .$$

So, we can not apply Corollary 2.9.
Also, from condition (2.4) we obtain

$$d(T0, T1) \leqslant \lambda \sqrt[3]{d(0, 1) \cdot d(0, T0) \cdot d(1, T1)},$$

i.e., $2 \leqslant \lambda\sqrt[3]{1\cdot 2\cdot 3}$, so $\lambda \geqslant \frac{2}{\sqrt[3]{6}} > 1$. Hence can not apply Corollary 2.8.

On the other hand, the conditions of Corollary 2.10 are valid. Let $x, y \in X$ be such that $x, y \in X\backslash \mathrm{Fix}(T)$. Then $x, y \in \{0, 1\}$. For $\lambda = \sqrt{\frac{6}{7}}$, we have in this case,

$$d(Tx, Ty) \leqslant \frac{\lambda}{\sqrt{3}}[d^2(x, y) + d^2(x, Tx) + d^2(y, Ty)]^{1/2},$$

for $x, y \in \{0, 1\}$ and $\lambda = \sqrt{\frac{6}{7}}$ and $\{2, 4\}$ is the set of fixed points of T.

Example 2.12. Consider the set $X = [1, 2]$. Take on X the b-metric $d(x, y) =| x - y |^2$ $(b = 2)$. Obviously, (X, d) is a complete $b-$metric space. Consider now the mapping

$$Tx = \frac{1 + x}{2}.$$

Let $x, y \in X$ be such that $x, y \in X\backslash \mathrm{Fix}(T)$. Then $x, y \in (1, 2]$. Showing that

$$d(Tx, Ty) \leqslant \frac{\lambda}{\sqrt{3}}[d^2(x, y) + d^2(x, Tx) + d^2(y, Ty)]^{1/2},$$

is equivalent to

$$3d^2(Tx, Ty) \leqslant \lambda^2[d^2(x, y) + d^2(x, Tx) + d^2(y, Ty)],$$

that is,

$$\frac{3}{16} | x - y |^4 \leqslant \lambda^2[| x - y |^4 + \frac{1}{16} | x - 1 |^4 + \frac{1}{16} | x - 1 |^4],$$

which holds when taking $\lambda \in [\frac{\sqrt{3}}{4}, 1)$. Note that T is continuous at 1. All the conditions of Corollary 2.10 are satisfied. Here, 1 is the fixed point of T.

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