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On a subclass of bi-univalent functions associated with the q-derivative operator

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Abstract

In this paper, we consider a new subclass of analytic and bi-univalent functions associated with q-Ruscheweyh differential operator in the open unit disk $\mathbb{U}$. For functions belonging to the class $\Sigma_q(\lambda, \phi)$, we obtain estimates on the first two Taylor-Maclaurin coefficients. Further, we derive another subclass of analytic and bi-univalent functions as a special consequences of the results.

Keywords: Bi-univalent functions, q-derivative, coefficient estimates, q-starlike functions.

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1. Introduction

Denote by $\mathcal{A}$ the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For two analytic functions f and g in $\mathbb{U}$, the subordination between them is written as $f \prec g$. The function $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function w with $w(0) = 0, |w(z)| < 1$, for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f \prec g \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The well-known Koebe one-quarter theorem [12] ensures that the image of $\mathbb{U}$ under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Hence, every univalent function f has an inverse f^{-1} satisfying

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$f^{-1}(f(z)) = z,\ (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, \quad (|w| < r_0(f), r_0(f) \geqslant \frac{1}{4}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi univalent functions in $\mathbb{U}$ given by (1.1). Some functions in the class Σ are as below (see Srivastava et al. [24]):

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

In 1986, Brannan and Taha [7] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses of starlike and convex functions of order α. In 2012, Ali et al. [5] widen the result of Brannan and Taha by using subordination. Since then, various subclasses of the bi-univalent function class Σ were introduced. The estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the TaylorMaclaurin series expansion (1.1) were found in several recent studies (see [8, 9, 16]) and still an interest to many researchers.

In [17, 18], Jackson defined the q-derivative operator D_q of a function as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0, q \neq 0),$$

and $D_q f(z) = f'(0)$. In case $f(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1}.$$

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^k \to k.$$

Quite a number of great mathematicians studied the concepts of q-derivative, for example by Gasper and Rahman [15], Aral et al. [6] and many others (see [1–3, 7–14]).

Making use of the q-derivative, we define the subclass $\mathcal{S}_q^*(\alpha)$ of the class $\mathcal{A}$ for $0 \leqslant \alpha < 1$ by

$$\mathcal{S}_q^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zD_q(f(z))}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\}.$$

This class is introduced and studied by Seoudy and Aouf [23] and also by Aldweby and Darus [4].

Noting that

$$\lim_{q\to 1} \mathcal{S}_q^*(\alpha) = \left\{ f \in \mathcal{A} : \lim_{q\to 1} \operatorname{Re}\left(\frac{zD_q(f(z))}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\} = \mathcal{S}^*(\alpha),$$

where $\mathcal{S}^*(\alpha)$ is the class of starlike of order α ([19, 21]).

Next, we state the q-analogue of Ruscheweyh operator given by Aldweby and Darus [3], that will be used throughout.

Definition 1.1. Let $f \in \mathcal{A}$. Denote by $\mathcal{R}_q^\lambda$ the q-analogue of Ruscheweyh operator defined by

$$\mathcal{R}_q^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q! [k-1]_q!} a_k z^k,$$

where $[k]_q!$ given by

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \cdots [1]_q, & k = 1, 2, \cdots, \\ 1, & k = 0. \end{cases} \tag{1.2}$$

From the definition we observe that if $q \to 1$, we have

$$\lim_{q\to 1} \mathcal{R}_q^\lambda f(z) = z + \lim_{q\to 1}\left[\sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k\right] = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{(\lambda)!(k-1)!} a_k z^k = \mathcal{R}^\lambda f(z),$$

where $\mathcal{R}_q^\lambda$ is Ruscheweyh differential operator defined in [22].

Let φ be an analytic function with positive real part in $\mathbb{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.3}$$

With this brief introduction, we define the following class of bi-univalent functions and finding the coefficient estimates with the help of q-derivative.

Definition 1.2. Let $\lambda > -1$. A function $f \in \Sigma$ is said to be in the class $\Sigma_q(\lambda, \varphi)$, if each of the following subordination condition holds true:

$$\frac{z\, D_q(\mathcal{R}_q^\lambda(f(z)))}{\mathcal{R}_q^\lambda(f(z))} \prec \varphi(z), \quad z \in \mathbb{U},$$

and

$$\frac{w\, D_q(\mathcal{R}_q^\lambda(g(w)))}{\mathcal{R}_q^\lambda(g(w))} \prec \varphi(w), \quad w \in \mathbb{U},$$

where $g(w) = f^{-1}(w)$.

In order to derive our main results, we have to recall here the following lemma.

Lemma 1.3 ([20]). *Let the function $p \in \mathcal{P}$ be given by the following series:*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad (z \in \mathbb{U}).$$

The sharp estimate given by

$$|p_n| \leqslant 2, \quad (n \in \mathbb{N}),$$

holds true.

2. A set of main results

For functions f in the class $\Sigma_q(\lambda, \varphi)$, the following result is obtained.

Theorem 2.1. *Let $f \in \Sigma_q(\lambda, \varphi)$ be of the form (1.2). Then*

$$|a_2| \leqslant \frac{B_1^{\frac{3}{2}}}{\sqrt{\left|q[\lambda+1]_q\left[q^\lambda B_1^2 + q[\lambda+1]_q(B_1 - B_2)\right]\right|}}, \tag{2.1}$$

and

$$|a_3| \leqslant \frac{B_1}{q[\lambda]_q + q^{\lambda+1}}\left(\frac{B_1}{[\lambda+1]_q} + \frac{1}{[\lambda+2]_q}\right),$$

where the coefficients B_1 and B_2 are given as in (1.3).

Proof. Let $f \in \Sigma_q(\lambda, \varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, satisfying the following conditions:

$$\frac{z\,D_q(\mathcal{R}_q^{\lambda}(f(z)))}{\mathcal{R}_q^{\lambda}(f(z))} = \varphi(u(z)), \quad z \in \mathbb{U}, \tag{2.2}$$

and

$$\frac{w\,D_q(\mathcal{R}_q^{\lambda}(g(w)))}{\mathcal{R}_q^{\lambda}(g(w))} = \varphi(v(w)), \quad w \in \mathbb{U}. \tag{2.3}$$

Define the functions p and q by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots,$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots.$$

Then p and q are analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$.

Since $u, v : \mathbb{U} \to \mathbb{U}$, each of the functions p and q has a positive real part in $\mathbb{U}$. Therefore, in view of the above lemma, we have

$$|p_n| \leqslant 2 \quad \text{and} \quad |q_n| \leqslant 2, \quad (n \in \mathbb{N}).$$

Solving for $u(z)$ and $v(z)$, we get

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2\right] + \cdots, \quad (z \in \mathbb{U}), \tag{2.4}$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2\right] + \cdots, \quad (z \in \mathbb{U}). \tag{2.5}$$

Clearly, upon substituting from (2.4) and (2.5) into (2.2) and (2.3), respectively, if we make use of (1.3), we obtain

$$\frac{z\,D_q(\mathcal{R}_q^{\lambda}(f(z)))}{\mathcal{R}_q^{\lambda}(f(z))} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + B_1 p_1 z + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]]z^2 + \cdots,$$

and

$$\frac{w\,D_q(\mathcal{R}_q^{\lambda}(g(w)))}{\mathcal{R}_q^{\lambda}(g(w))} = \varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + B_1 q_1 w + \left[\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right]]w^2 + \cdots.$$

Also

$$\frac{z\,D_q(\mathcal{R}_q^{\lambda}(f(z)))}{\mathcal{R}_q^{\lambda}(f(z))} = 1 + q[\lambda+1]_q a_2 z + \left\{q[\lambda+1]_q[\lambda+2]_q a_3 - q[\lambda+1]_q^2 a_2^2\right\} z^2 + \cdots,$$

and

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$$\frac{w\,D_q(\mathcal{R}_q^{\lambda}(g(w)))}{\mathcal{R}_q^{\lambda}(g(w))} = 1 - q[\lambda+1]_q a_2 w + \left\{-q[\lambda+1]_q[\lambda+2]_q a_3 + q[\lambda+1]_q\left(2[\lambda+2]_q - [\lambda+1]_q\right) a_2^2\right\} w^2 + \cdots.$$

Now equating the coefficients in (2.2) and (2.3), we find that

$$q[\lambda+1]_q a_2 = \frac{1}{2}B_1 p_1, \tag{2.6}$$

and

$$q[\lambda+1]_q[\lambda+2]_q a_3 - q[\lambda+1]_q^2 a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2. \tag{2.7}$$

Also we have

$$-q[\lambda+1]_q a_2 = \frac{1}{2}B_1 p_1, \tag{2.8}$$

and

$$-q[\lambda+1]_q[\lambda+2]_q a_3 + q[\lambda+1]_q\left(2[\lambda+2]_q - [\lambda+1]_q\right)a_2^2 = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2. \tag{2.9}$$

From (2.6) and (2.8), we get

$$p_1 = -q_1, \tag{2.10}$$

and

$$2q^2[\lambda+1]_q^2 a_2^2 = \frac{1}{4}B_1^2(p_1^2+q_1^2). \tag{2.11}$$

Now by adding equation (2.7) and equation (2.9), we get

$$2q^{\lambda+2}[\lambda+1]_q a_2^2 = \frac{1}{2}B_1\left[p_2 + q_2 - \left(\frac{p_1^2+q_1^2}{2}\right)\right] + \frac{1}{4}B_2[p_1^2+q_1^2].$$

By using (2.11), we get

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4q[\lambda+1]_q\left[q^\lambda B_1^2 + q[\lambda+1]_q(B_1-B_2)\right]}.$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leqslant \frac{B_1^{\frac{3}{2}}}{\sqrt{\left|q[\lambda+1]_q\left[q^\lambda B_1^2 + q[\lambda+1]_q(B_1-B_2)\right]\right|}}.$$

This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.9) from (2.7) and also from (2.10), we get $p_1^2 = q_1^2$, hence

$$2q[\lambda+1]_q[\lambda+2]_q a_3 - \left[2q[\lambda+1]_q[\lambda+2]_q\right]a_2^2 = \frac{1}{2}B_1(p_2-q_2).$$

Using (2.11) and applying Lemma 1.3 once again for the coefficients p_2 and q_2, we have

$$|a_3| \leqslant \frac{B_1}{q[\lambda]_q + q^{\lambda+1}}\left(\frac{B_1}{[\lambda+1]_q} + \frac{1}{[\lambda+2]_q}\right).$$

This completes the proof of Theorem 2.1. □

3. Applications of the main result

If we set

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots, \quad (z \in \mathbb{U}, 0 \leqslant \beta < 1),$$

in Definition 1.2 of the bi-univalent functions class $\Sigma_q(\lambda,\varphi)$, we obtain a new class $\Sigma_q^1(\lambda,\beta)$ given by Definition 3.1.

Definition 3.1. A function $f \in \Sigma$ is said to be in the class $\Sigma_q^1(\lambda, \beta)$, if the following conditions hold true:

$$\mathrm{Re}\left(\frac{z\, D_q(\mathcal{R}_q^\lambda(f(z)))}{\mathcal{R}_q^\lambda(f(z))}\right) > \beta, \quad (z \in \mathbb{U}),$$

and

$$\mathrm{Re}\left(\frac{w\, D_q(\mathcal{R}_q^\lambda(g(w)))}{\mathcal{R}_q^\lambda(g(w))}\right) > \beta, \quad (w \in \mathbb{U}),$$

where $g(w) = f^{-1}(w)$.

Using the parameter setting of Definition 3.1 in the Theorem 2.1, we get the following corollary.

Corollary 3.2. *Let the function* $f \in \Sigma_q^1(\lambda, \beta)$ *be of the form* (1.1). *Then*

$$|a_2| \leqslant \sqrt{\frac{2(1-\beta)}{q^{\lambda+1}[\lambda]_q + q^{2\lambda+1}}},$$

and

$$|a_3| \leqslant \frac{2(1-\beta)}{q[\lambda]_q + q^{\lambda+1}} \left(\frac{[2(1-\beta)([\lambda]_q + q^\lambda[2]_q) + [\lambda]_q + q^\lambda]}{[\lambda+1]_q[\lambda+2]_q}\right).$$

Remark 3.3. For special case, when $\lambda = 0$, Corollary 3.2 simplifies to the following form.

Corollary 3.4. *Let the function* f *given by* $f \in \Sigma_q^2(\beta) := \Sigma_q^1(0, \beta)$ *be of the form* (1.1). *Then*

$$|a_2| \leqslant \sqrt{\frac{2(1-\beta)}{q}},$$

and

$$|a_3| \leqslant \frac{2(1-\beta)}{q}\left(2(1-\beta) + \frac{1}{1+q}\right).$$

If we set

$$\varphi(z) = (\frac{1+z}{1-z})^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots, \quad (0 < \alpha \leqslant 1, z \in \mathbb{U}),$$

in Definition 1.2 of the bi-univalent function class $\Sigma_q(\lambda, \varphi)$, we obtain a new class $\Sigma_q^3(\lambda, \alpha)$ defined as follows.

Definition 3.5. A function $f \in \Sigma$ is said to be in the class $\Sigma_q^3(\lambda, \alpha)$, if the following conditions hold true

$$\left|\arg\left(\frac{z\, D_q(\mathcal{R}_q^\lambda(f(z)))}{\mathcal{R}_q^\lambda(f(z))}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leqslant 1; z \in \mathbb{U}),$$

and

$$\left|\arg\left(\frac{w\, D_q(\mathcal{R}_q^\lambda(g(w)))}{\mathcal{R}_q^\lambda(g(w))}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leqslant 1; w \in \mathbb{U}),$$

where $g(w) = f^{-1}(w)$.

Using the parameter setting of Definition 3.5 in the Theorem 2.1, we get the following corollary.

Corollary 3.6. *Let the function* $f \in \Sigma_q^3(\lambda, \alpha)$ *be of the form* (1.1). *Then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{|q[\lambda+1]_q[2\alpha q^\lambda + q[\lambda+1]_q(1-\alpha)]|}},$$

and

$$|a_3 \leqslant \frac{2\alpha}{q[\lambda]_q + q^{\lambda+1}}\left(\frac{2\alpha}{[\lambda+1]_q} + \frac{1}{[\lambda+2]_q}\right).$$

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