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Modification of The HPM by Using Optimal Newton Interpolation Polynomial for Quadratic Riccati Differential Equation

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Abstract

In this work, an efficient modification of the homotopy analysis method by using optimal Newton interpolation polynomials is given for the approximate solutions of the Riccati differential equations. This presented method can be applied to linear and nonlinear models. Examples show that the method is effective.

Keywords: quadratic Riccati differential equation, modification of the HPM, Newton interpolation.

1. Introduction

Riccati differential equations are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science [1]. The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The applications of this equation may be found not only in random processes, optimal control, and diffusion problems [2] but also in stochastic realization theory, optimal control, robust stabilization, network synthesis and financial mathematics. Solitary wave solutions of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [3]. Therefore, one has to go for numerical techniques or approximate approaches for getting its solution. Recently various iterative methods are employed for the numerical and analytical solution of functional equations such as Adomian's decomposition method (ADM) [4, 5], homotopy analysis method (HAM) [6], homotopy perturbation method (HPM) [7], variational iteration method (VIM) [8], and differential transform method (DTM) [9]. In [10], Liao has shown that HPM equations are equivalent to HAM equations when $\hbar= -1$, and too, matrix differential transform method for solving of Riccati types matrix differential equations [11]. In this work, we introduce a new modification the HPM using optimal Newton interpolation polynomials. The schemes are tested for some examples.

This study is organized as follows: In section 2, we present the standard HAM. In section 3, we present the modification technique of HAM. In section 4, the method is applied to a variety of examples to show efficiency and simplicity of the method.

2. Basic ideas of HAM

For the convenience of the reader, we will first present a brief account of HAM [12]. Let us consider the following differential equation

$$N[u(x,t)] = 0, \tag{1}$$

where N is a nonlinear operator, u(x, t) is an unknown function, and x and t denote spatial and temporal independent variables. By means of homotopy analysis method, we first construct the so-called zeroth-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = \hbar p N[u(x,t)], \tag{2}$$

where p ϵ [0; 1] is the embedding parameter, ħ is a non-zero auxiliary parameter, L is an auxiliary linear operator, $u_0(x; t)$ is an initial guess of u(x, t) and φ(x, t; p) is a unknown function. Obviously, when p = 0 and p = 1, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t), \tag{3}$$

Thus, as p increases from 0 to 1, the solution φ(x, t; p) varies from the initial guess $u_0(x; t)$ to the exact solution u(x, t). Expanding φ(x, t; p) in Taylor series with respect to p, we have

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m, \tag{4}$$

Where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\,\Big|_{p=0}\,. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter ħ, and the auxiliary function are properly chosen, series 4 converges at p = 1, then we have

$$u(x,t) = u_0 + \sum_{m=1}^{\infty} u_m(x,t), \tag{6}$$

which must be one of solutions of original nonlinear equations. According to definition (6), the governing equation can be deduced from the zero-order deformation (4). Define the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \cdots, u_n(x,t)\}. \tag{7}$$

Differentiating (4) m-times with respect to the embedding parameter p and then setting p = 0 and finally dividing them by m!, we have the so-called mth-order deformation equation

$$L[U_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}), \tag{8}$$

where

$$R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N(x,t;p)}{\partial p^{m-1}}\,\Big|_{p=0}\,, \tag{9}$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \quad 10$$

3. Modified homotopy analysis method

For MHAM consider the following general nonlinear differential equation;

$$N[u(x,t)] = A(u) - f(x), \quad (11)$$

where A is a general differential operator, f(x) is a known analytic function. In this study, alternatively, f(x) is approximated by optimal Newton interpolation, $P_f(x)$ over the interval [a, b],

$$f(x) \approx P_f(x) = f(x_0) + \sum_{k=1}^{\vartheta} f[x_0, \cdots, x_k] \prod_{j=0}^{j=k-1} (x - x_j),$$

where x_ks are the roots of the (ϑ + 1)st Chebyshev polynomial of the first kind $T_{\vartheta+1}(x)$ in [-1, 1] translated onto the interval [a, b] given by

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2k+1}{2\vartheta+2}\pi\right), \quad k = 0, 1, \cdots, \vartheta.$$

And

$$f[x_0, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

$$f[x_j] = f(x_j), \quad \forall j = 0, \cdots, k.$$

4. Numerical Examples

In this section, we demonstrate the effectiveness of the proposed modification of the HAM by applying it to two nonlinear and one linear problems. For each example, the M-term approximation of u(x) and the exact solution is presented. Moreover, all numerical results obtained by the modification of the HAM using optimal Newton interpolation polynomial, u_M, are compared with the exact solutions. The algorithm are performed by Matlab version 7.6.0.324 (R2008a) with 10 digits precision.

Example 1. Consider the following quadratic Riccati differential equation taken from [1]

$$u'(x) - 2e^{2x}u(x) + e^{x}u(x)^2 = e^{x} - e^{3x}, \quad 0 \leq x \leq 1 \quad (12)$$

subject to the initial condition u(0) = 1 and the exact solution u(x) = e^t. To solve (12) by HAM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial u}{\partial x} - \frac{\partial u_0}{\partial x}\right) = p\hbar\left(\frac{\partial u}{\partial x} - 2e^{2x}u(x) + e^{x}u(x)^2 - f(x)\right), \quad (13)$$

Where

$$f(x) = e^{x} - e^{3x}. \quad (14)$$

Assume the solution of (12) to be in form:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \tag{15}$$

Substituting (15) into (13) and collecting terms of the same power of p gives

$$\begin{aligned}
p^0:&\quad \frac{\partial u_0}{\partial x} - \frac{\partial u_0}{\partial x} = 0,\\
p^1:&\quad \frac{\partial u_1}{\partial x} = \hbar\left(-2e^{2x}u_0 + e^x u_0^2 - f(x)\right),\\
p^2:&\quad \frac{\partial u_2}{\partial x} = \hbar\left(\frac{\partial u_1}{\partial x} - 2e^{2x}u_1 + 2e^x u_0 u_1\right) + \frac{\partial u_1}{\partial x},\\
p^3:&\quad \frac{\partial u_3}{\partial x} = \hbar\left(\frac{\partial u_2}{\partial x} - 2e^{2x}u_2 + e^x u_1^2 + 2e^x u_0 u_2\right) + \frac{\partial u_2}{\partial x},\\
p^4:&\quad \frac{\partial u_4}{\partial x} = \hbar\left(\frac{\partial u_3}{\partial x} - 2e^{2x}u_3 + 2e^x u_0 u_3 + 2e^x u_1 u_2\right) + \frac{\partial u_3}{\partial x},\\
&\vdots
\end{aligned} \tag{16}$$

The given initial value admits the use of

$$u_0(x) = 1.$$

To solve the above equations, we use the optimal Newton interpolation polynomial of f. For this reason, by setting ϑ = 10, the optimal Newton interpolation polynomial of f becomes,

$$f(x) \approx P_f(x) = f(x_0) + \sum_{k=1}^{\vartheta} f[x_0, \cdots, x_k] \prod_{j=0}^{j=k-1} (x - x_j),$$

where

$$x_k = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2k+1}{22}\pi\right), \quad k = 0, 1, \cdots, 10.$$

So, we have

$$\begin{aligned}
f(x) \approx{} & -.0000000087 - 1.9999978801x - 4.0000843980x^2 - 4.3320282556x^3\\
& - 3.3436322922x^4 - 1.9695528102x^5 - 1.1442974648x^6\\
& - .1951490620x^7 - .4310833367x^8 + .1249908871x^9\\
& - .0764204618x^{10}.
\end{aligned} \tag{17}$$

By substituting (17) into (16), we obtain $u_i(x)$, for $i \geq 1$, as

$$\begin{aligned}
u_1(x) ={} & .4999999956x + .2500005299x^2 - .0833473996x^3 - .2290035319x^4\\
& - .2551965625x^5 - .1203794009x^6 - .0691363266x^7 - .0090346142x^8\\
& - .0232449845x^9 + .0063904999x^{10},
\end{aligned}$$

$$u_2(x) = 2499999978x + .1250002649x^2 - .1249929653x^3 - .1354983648x^4 - .0723990634x^5 - .0184992586x^6 - .0860715200x^7 - .0844052679x^8 - .0804320238x^9 + .0406467414x^{10} - .0258653744x^{11} - .0114237435x^{12} - .0047451763x^{13} - .0018075012x^{14} - .0006454048x^{15} - .0002143958x^{16} - .0000650433x^{17} - .0000177719x^{18} - .0000043025x^{19} - .0000009680x^{20} - .0000001454x^{21} - .0000000441x^{22},$$

...

Here, for a given arbitrary natural number M,

$$u_M = \sum_{i=0}^{M} u_i(x),$$

It denotes the M-term approximation of the exact solution u(x) which is obtained using optimal Newton interpolation of f. In Table 1, we give the numerical results of the exact solutions and the approximate solutions obtained by the present method with ħ=-0.5, M = 10 and ϑ=10 in the interval $0 \le x \le 1$. The absolute errors for this solution are shown in Table 2. The graphs of approximated trajectory and exact trajectory are plotted in Figure 1.

Table 1: Numerical results of the modified HAM with ħ=-0.5 for Example 1

x_i	exact solution $u(x_i)$	Modified HAM M = 10
0.2	1.2214027581	1.2213091060
0.4	1.4918246976	1.4926294519
0.6	1.8221188003	1.8263387073
0.8	2.2255409284	2.2300568140

Table 2: Absolute errors of the modified HAM with ħ=-0.5 for Example 1

x_i	M = 10
0.2	9.36521e-005
0.4	8.047543e-004
0.6	4.2199070e-003
0.8	4.5158856e-003

Example 2. Consider the following quadratic Riccati differential equation taken from [13]

$$u'(x) - u(x) + u(x)^2 - \frac{1}{1+x} = 0, \quad 0 \le x \le 1 \tag{18}$$

subject to the initial condition u(0) = 1 and the exact solution u(x) =1/(1+x). To solve (18) by HAM, we construct a homotopy:

$$(1-p)\left(\frac{\partial u}{\partial x} - \frac{\partial u_0}{\partial x}\right) = p\hbar\left(\frac{\partial u}{\partial x} - u(x) + u(x)^2 + f(x)\right), \tag{19}$$

Where

$$f(x) = -\frac{1}{1+x}. \tag{20}$$

Assume the solution of (18) to be in form:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \tag{21}$$

using the optimal Newton interpolation polynomial of f, we have

$$\begin{aligned} f(x) \approx {} & -.9999999924 + .9999981568\mathrm{x} - .9999247324\mathrm{x}^2 + .9987793121\mathrm{x}^3 \\ & - .9896159497\mathrm{x}^4 + .9468535921\mathrm{x}^5 - .8224540061\mathrm{x}^6 \\ & + .5900591751\mathrm{x}^7 - .3111853779\mathrm{x}^8 + .1033970585\mathrm{x}^9 \\ & - .0159072397\mathrm{x}^{10}. \end{aligned} \tag{22}$$

By substituting (22) into (19), we obtain $u_i(x)$, for $i \geq 1$, as

$$\begin{aligned} u_1(x) = {} & -.9999999924\mathrm{x} + .4999990784\mathrm{x}^2 - .3333082441\mathrm{x}^3 + .2496948280\mathrm{x}^4 \\ & - .1979231899\mathrm{x}^5 + .1578089320\mathrm{x}^6 - .1174934294\mathrm{x}^7 + .0737573968\mathrm{x}^8 \\ & - .0345761531\mathrm{x}^9 + .0103397058\mathrm{x}^{10} - .0014461127\mathrm{x}^{11}, \end{aligned}$$

$$\begin{aligned} u_2(x) = {} & 0.4999999962\mathrm{x}^2 - .1666663594\mathrm{x}^3 + .0833270610\mathrm{x}^4 - .0499389656\mathrm{x}^5 \\ & + .0329871983\mathrm{x}^6 - .0225441331\mathrm{x}^7 + .0146866786\mathrm{x}^8 - .0081952663\mathrm{x}^9 \\ & + .0034576153\mathrm{x}^{10} - .0009399732\mathrm{x}^{11} + .0001205093\mathrm{x}^{12}, \end{aligned}$$

...

Here, for a given arbitrary natural number M,

$$u_M = \sum_{i=0}^{M} u_i(x),$$

It denotes the M-term approximation of the exact solution u(x) which is obtained using optimal Newton interpolation of f.

In Table 3, we give the numerical results of the exact solutions and the approximate solutions obtained by the present method with $\hbar = -1$, M = 10 and $\vartheta = 10$ in the interval $0 \leq x \leq 1$. The absolute errors for this solution are shown in Table 4. The graphs of approximated trajectory and exact trajectory are plotted in Figure 2.

Table 3: Numerical results of the modified HAM with $\hbar$=-1 for Example 2

x_i	exact solution $u(x_i)$	Modified HAM M = 10
0.2	0. 8333333333	0. 8333333345
0.4	0. 7142857142	0. 7142877171
0.6	0. 6250000000	0. 6251263624
0.8	0. 5555555555	0. 5578072670

Table 4: Absolute errors of the modified HAM with ħ=-1 for Example 2

x_i	M = 10
0.2	1. 2e-009
0.4	2. 0029e-006
0.6	1. 263624e-004
0.8	2. 2517115e-003

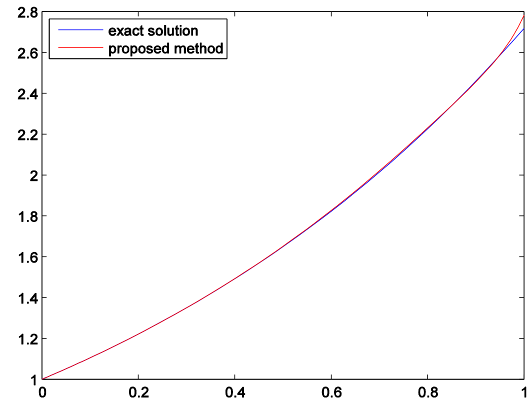

Figure 1: comparison exact trajectory approximated trajectory with presented method for Example 1, with ħ=-0.5 and M = 10.

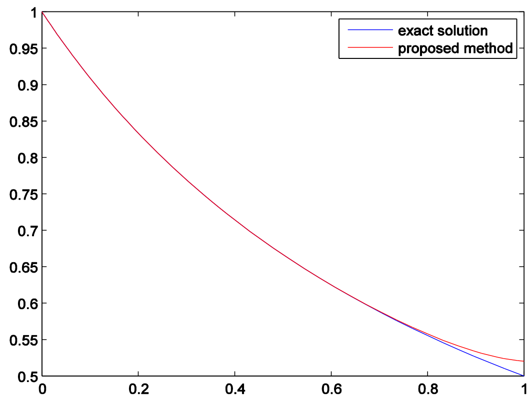

Figure 2: comparison exact trajectory approximated trajectory with presented method for Example 2, with ħ=-1 and M = 10.

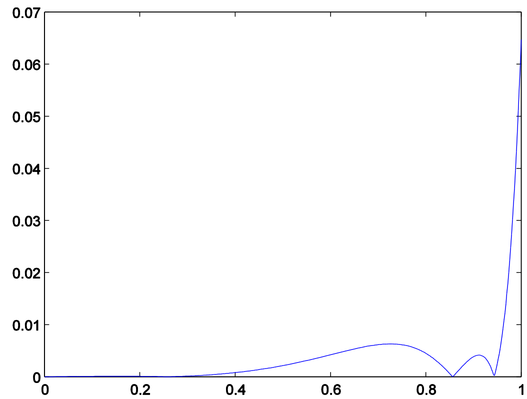

Figure 3: The error graph of the modified HAM with ħ=-0.5 and M = 10 for Example 1.

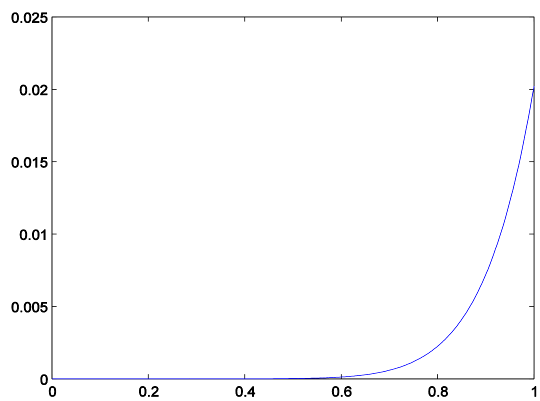

Figure 4: The error graph of the modified HAM with ħ=-1 and M = 10 for Example 2.

5. Conclusion

A simple and effective algorithm based on optimal Newton interpolation polynomial is presented which is stated for solving quadratic Riccati differential equation. The method is computationally attractive, and some examples are solved.

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