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Arens Regularity of Banach Module Actions and the Strongly Irregular Property

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Abstract

Let X, Y, Z be normed spaces. We show that, if X is reflexive, then some extensions andadjointsof the bounded bilinear map $f: X \times Y \to Z$ are Arens regular. Also the left strongly irregular propertyis equivalent to the right strongly irregular property. We show that the right module action $\pi_{2_n}^*: A^{(n+1)} \times A^{(n)} \to A^*$ factors, where A is a Banach algebra.

Keywords: Arens regular, module action, derivation, topological center, factor.

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1. Introductionand Preliminaries

Arens showed in [1] that a bounded bilinear map $f: X \times Y \to Z$ on normed spaces, has two natural different extensions f^{***}, $f^{r***r}$from $X^{**} \times Y^{**}$into Z^{**}. When these extensions are equal, f is saidto be Arens regular. Throughout the article, we identify a normed space with its canonical image in thesecond dual.

Let X, Y, Z be normed spaces and$f: X \times Y \to Z$ be a bounded bilinear mapping. The natural extensions of $f$are as follows:

i) $f^*: Z^* \times X \to Y^*$, given by $\langle f^*(z^*, x), y \rangle = \langle z^*, f(x, y) \rangle$ where $x \in X$, $y \in Y$, $z^* \in Z^*$ (f^* is said the adjoint of f).

ii) $f^{**}: Y^{**} \times Z^* \to X^*$, given by $\langle f^{**}(y^{**}, z^*), x \rangle = \langle y^{**}, f^*(z^*, x) \rangle$ where $x \in X$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.

iii) $f^{***}: X^{**} \times Y^{**} \to Z^{**}$, given by $\langle f^{***}(x^{**}, y^{**}), z^* \rangle = \langle x^{**}, f^{**}(y^{**}, z^*) \rangle$ where $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.

Let $f^r: Y \times X \to Z$ be the flip of f defined by $f^r(y, x) = f(x, y)$, for every $x \in X$ and $y \in Y$. Then f^r is a bounded bilinear map and it may extends as above to $f^{r***}: Y^{**} \times X^{**} \to Z^{**}$. In general, the mapping $f^{r***r}: X^{**} \times Y^{**} \to Z^{**}$ is not equal to f^{***}. When these extensions are equal, then f is Arens regular. If the multiplication of a Banach algebra A enjoys this property, then A itself is called Arens regular. The first and the second Arens products are denoted by $\square$, $\lozenge$ respectively.

One may define similarly the mappings $f^{****}: Z^{***} \times X^{**} \to Y^{***}$ and $f^{*****}: Y^{****} \times Z^{***} \to X^{***}$ and the higher rank adjoints. Consider the nets $(x_\alpha) \subseteq X$ and $(y_\beta) \subseteq Y$ converge to $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ in the w^*–topologies, respectively, then

$$f^{***}(x^{**}, y^{**}) = w^* - \lim_\alpha w^* - \lim_\beta f(x_\alpha, y_\beta)$$

and

$$f^{r***r}(x^{**}, y^{**}) = w^* - \lim_\beta w^* - \lim_\alpha f(x_\alpha, y_\beta)$$

so Arens regularity of f is equivalent to the following

$$\lim_\alpha \lim_\beta \langle z^*, f(x_\alpha, y_\beta) \rangle = \lim_\beta \lim_\alpha \langle z^*, f(x_\alpha, y_\beta) \rangle$$

if the limits exit for each $z^* \in Z^*$. The map f^{***} is the unique extension of f such that

$x^{**} \to f^{***}(x^{**}, y^{**}): X^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $y^{**} \in Y^{**}$ and

$y^{**} \to f^{***}(x, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $x \in X$.

The left topological center of f is defined by

$$Z_l(f) = \{x^{**} \in X^{**}: y^{**} \to f^{***}(x^{**}, y^{**}): Y^{**} \to Z^{**} \text{ is } w^* - w^* \text{ continuous}\}.$$

Since $f^{r***r}: X^{**} \times Y^{**} \to Z^{**}$ is the unique extension of f such that the map $y^{**} \to f^{r***r}(x^{**}, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $x^{**} \in X^{**}$, we can set

$$Z_l(f) = \{x^{**} \in X^{**}: f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**}), (y^{**} \in Y^{**})\}.$$

The right topological center of f may therefore be defined as

$$Z_r(f) = \{y^{**} \in Y^{**}: x^{**} \to f^{r***r}(x^{**}, y^{**}): Y^{**} \to Z^{**} \text{ is } w^* - w^* \text{ continuous}\}.$$

Again since the map

$x^{**} \to f^{***}(x^{**}, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $y^{**} \in Y^{**}$, we can set

$$Z_r(f) = \{y^{**} \in Y^{**}: f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**}), (x^{**} \in X^{**})\}.$$

A bounded bilinear mapping f is Arens regular if and only if $Z_l(f) = X^{**}$, or equivalently $Z_r(f) = Y^{**}$. It is clear that $X \subseteq Z_l(f)$. If $Z_l(f) = X$ then the map f is said to be left strongly irregular. Also $Y \subseteq Z_r(f)$ and if $Z_r(f) = Y$ then the map f is said to be right strongly irregular. A bounded bilinear mapping $f: X \times Y \to Z$ is said to factor if it is onto. Let A be a Banach algebra, X be a Banach space and $\pi_1: A \times X \to X$ be a bounded bilinear map (π_1 is said the left module action of A

on X).If $\pi_1(ab,x) = \pi_1(a,\pi_1(b,x))$, for each $a,b \in A$, $x \in X$, then the pair (π_1,X) is said to be a left Banach

A−module. A right Banach A−module (X,π_2) can be defined similarly. A triple (π_1,X,π_2) is said tobe a Banach A−module if (π_1,X) and (X,π_2) are left and right Banach A−modules, respectively, and$\pi_1(a,\pi_2(x,b)) = \pi_2(\pi_1(a,x),b)$ for each $a,b \in A$, $x \in X$. Let (π_1,X,π_2) be a Banach A−module. Abounded linear mapping $D : A \to X^*$is said to be a derivation if $D(ab) = D(a).b + a.D(b)$, for each$a,b \in A$.

2. Arens regularity of bounded bilinear maps

Remark 2.1. Let f be a bounded bilinear map from $X \times Y$ into Z. f^{***n} means that the number of starsis $3n$ for every $n \in N$.

Let f be a bounded bilinear map and , $x^{**} \in X^{**}, y^{**} \in Y^{**}$. If f is Arens regular then for every $z^* \in Z^*$,

$$\langle f^{***r}(y^{**},x^{**}),z^*\rangle = \langle f^{***}(x^{**},y^{**}),z^*\rangle = \langle f^{r***r}(x^{**},y^{**}),z^*\rangle = \langle f^{r***}(y^{**},x^{**}),z^*\rangle$$

Therefore,f^r is Arens regular. Now let f^r is Arens regular, for every $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^* \in Z^*$,

$$\langle f^{r***r}(x^{**},y^{**}),z^*\rangle = \langle f^{r***}(y^{**},x^{**}),z^*\rangle = \langle f^{***r}(y^{**},x^{**}),z^*\rangle = \langle f^{***}(x^{**},y^{**}),z^*\rangle$$

Hence f is Arense regular if and only if f^r is Arens regular.

Lemma 2.2. If $f: X \times Y \to Z$ is Arens regular and X is a reflexive space, then $f^{***n}$and f^{****n} are

Arens regular for every $n \in N$.

Proof. First, we show that f^{***} is Arens regular for an arbitrary f. Then we show that $f^{******} =$

$f^{r******r}$.By $[7, Theorem\ 2.1]$, for every $x^{****} \in X^{****}$, $y^{****} \in Y^{****}$, $z^{***} \in Z^{***}$, we have

$$\begin{aligned}\langle f^{******}(x^{****},y^{****}),z^{***}\rangle &= \langle x^{****}, f^{*****}(y^{****},z^{***})\rangle\\ &= \langle y^{****}, f^{****}(z^{***},x^{****},)\rangle\\ &= \langle y^{****}, f^{r*****r}(z^{***},x^{****})\rangle\\ &= \langle f^{r******}(y^{****},x^{****}),z^{***}\rangle\\ &= \langle f^{r******r}(x^{****},y^{****}),z^{***}\rangle.\end{aligned}$$

It follows that f^{***} is Arens regular. This completes the proof of Arens regularity of f^{***n}. Now if f

is Arens regular then we show that f^{****} is Arens regula. we should show

$$(1)\quad f^{*******} = f^{****r***r}$$

Since f is Arens regular,

$$(2)\quad (f^{***})^{****} = (f^{r***r****})$$

so it is enough to show that

$$(3)\quad f^{r***r****} = f^{****r***r}$$

f^{***} is Arens regular, therefore $f^{******} = f^{***r***r}$, so from the Arens regularity of f^r, we have $f^{******} = f^{r******r}$. Therefore $f^{******r} = f^{r******}$. From the Arens regularity of f, $f^{r***r***r} = f^{r******}$. Now by $[6, Theorem\ 2.1]$, for every $x^{****} \in X^{****}$, $y^{****} \in Y^{****}$, $z^{*****} \in Z^{*****}$, we have

$$\langle f^{****r***r}(z^{*****}, x^{****}), y^{****}\rangle = \langle x^{****}, f^{*****r**}(z^{*****}, y^{****})\rangle$$
$$= \langle x^{****}, f^{r******rr**}(z^{*****}, y^{****})\rangle$$
$$= \langle z^{*****}, f^{r******}(y^{****}, x^{****})\rangle$$
$$= \langle f^{r***r****}(z^{*****}, x^{****}), y^{****}\rangle.$$

Therefore equation (3) holds and f^{****} is Arens regular. Hence f^{****n} is Arens regular, for every $n \in N$.

Lemma 2.3. Let $f: X \times Y \to Z$ be is a bounded bilinear map. If X is reflexive, then f and every adjoint and every flip map of f such that its domain contains $X, X^*, X^{**}, \ldots$ is Arens regular.

Proof. First we show that if Y is reflexive, then the result holds. $f^{****}(Z^{***}, X^{**}) \subseteq Y^{***}$ and Y^* is reflexive, therefore

$$f^{****}(Z^*, X^{**}) \subseteq f^{****}(Z^{***}, X^{**}) \subseteq Y^*.$$

Now by $[7, Theorem\ 2.1]$, f is Arens regular. Therefore f^r is Arens regular, so the result holds.

Lemma 2.4. If X is reflexive and the bounded bilinear map f^{****} factors, then f and every adjoint map and every flip map of it is Arens regular.

Proof. If X is reflexive space then by $lemma$ 2.3, f and f^{r*} are Arens regular and by $[7, Corollary\ 2.2]$ itis equivalent that $f^{****}(Z^{***}, X^{**}) \subseteq Y^*$ and f^{****} factors, therefore $Y^{***} \subseteq Y^*$ and it is equivalent that Y is reflexive. Now for every adjoint map or every flip map, X or , X^* or Y^*,is contained in a part ofits domain. Since these spaces are all reflexive, therefore by $lemma$ 2.3 the result holds.

Theorem 2.5. Let X be is reflexive and let f^{****} factors. Then f is left strongly irregular if and only if it is right strongly irregular.

Proof. By $lemma$ 2.3 f is Arens regular. From the definition, f is Arens regular if and only if $Z_l(f) = X^{**}$. X is reflexive therefore $Z_l(f) = X$, i.e. f is left strongly irregular, therefore f is Arens regular if and only f is left strongly irregular.On the other hand by $lemma$ 2.4, Y is also reflexive, therefore by definition of topological centers, f is Arens regular if and only if $Z_r(f) = Y^{**}$, since Y is reflexive so $Z_r(f) = Y$, thus f is right strongly irregular. therefore f is Arens regular if and only f is right strongly irregular. It follows that f is left strongly irregular if and only if right strongly irregular.

3. Module action

In [5] Eshaghi Gordji and Filali show that left module action of a Banach algebra A on $A^{(n)}$ factors. Now let π_{2_n} be the right module action of A on $A^{(n)}$. Thus π_{2_n} maps $A^{(n)} \times A$ into $A^{(n)}$ and $\pi_{2_n}^*$ maps $A^{(n+1)} \times A^{(n)}$ into A^*, for every $n \geq 1$. Also $\pi_{1_n} = \pi_{2_{n-1}}^{r*r}$ and $\pi_{2_n} = \pi_{1_{n-1}}^*$ such that $A^{(0)} = A$, $\pi = \pi_{1_0} = \pi_{2_0}$. In the next theorem we show that the right module action factors.

Theorem 3.1. Let A be a Banach algebra.

I) If A has a left bounded approximate identity, then $\pi_{2_n}^*$ factors for every positive even integer n.

II) If A has a right bounded approximate identity, then $\pi_{2_n}^*$ factors for odd positive even integer n.

Proof. I) We use the induction on n. Let $n = 2$ and (e_β) be a left bounded approximate identity in A with a cluster point $e^{**} \in A^{**}$. Therefore for every $a^{***} \in A^{***}$ we have $\pi_{2_2}^*(a^{***}, e^{**}) = a^{***}$.

Let (a_α^*) be a net in A^* with a cluster point $e^{***} \in A^{***}$, so for every $a \in A$,

$$\begin{aligned}
\langle \pi_{2_2}^*(a^{***}, e^{**}), a\rangle &= \langle a^{***}, \pi_{2_2}(e^{**}, a)\rangle = \langle a^{***}, \pi_{1_1}^*(e^{**}, a)\rangle \\
&= \lim_\alpha \langle e^{**}, \pi_{1_1}(a, a_\alpha^*)\rangle = \lim_\alpha \langle e^{**}, \pi_{2_0}^{r*r}(a, a_\alpha^*)\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi_{2_0}^{r*}(a_\alpha^*, a), e_\beta\rangle = \lim_\alpha \lim_\beta \langle a_\alpha^*, \pi_{2_0}(e_\beta, a)\rangle \\
&= \lim_\alpha \langle a_\alpha^*, a\rangle = \langle a^{***}, a\rangle.
\end{aligned}$$

Therefor for $n = 2$, π_{2_n} factors. Now suppose that the result holds for $n = 2k - 2$. So,

$$\begin{aligned}
\langle \pi_{2_{(2k)}}^*(a^*, e^{**}), a\rangle &= \langle a^*, \pi_{2_{(2k)}}(e^{**}, a)\rangle = \langle a^*, \pi_{1_{(2k-1)}}^*(e^{**}, a)\rangle \\
&= \langle e^{**}, \pi_{1_{(2k-1)}}(a, a^*)\rangle = \langle e^{**}, \pi_{2_{(2k-2)}}^{r*r}(a, a^*)\rangle \\
&= \langle e^{**}, \pi_{2_{(2k-2)}}^{r*}(a^*, a)\rangle = \langle a^*, \pi_{2_{(2k-2)}}^{r}(a, e^{**})\rangle \\
&= \langle a^*, \pi_{2_{(2k-2)}}(e^{**}, a)\rangle = \langle \pi_{2_{(2k-2)}}^*(a^*, e^{**}), a\rangle
\end{aligned}$$

thus $\pi_{2_{(2k)}}^*$ factors.

II) Again by induction. Let (e_α) be a right bounded approximate identity in A with a cluster point $e^{**} \in A^{**}$. for $n = 1$ it is enough to show that $\pi_{2_1}^*(e^{**}, a^*) = a^*$ for every $a^* \in A^*$.

$$\langle \pi_{2_1}^* (e^{**}, a^*), a\rangle = \langle e^{**}, \pi_{2_1} (a^*, a)\rangle = \langle e^{**}, \pi_{1_0}^*(a^*, a)\rangle = \lim_{\alpha}\langle a^*, \pi_{1_0} (a, e_\alpha)\rangle = \langle a^*, a\rangle.$$

Now suppose that is true for $n = 2k - 1$, then

$$\langle \pi_{2_{(2k+1)}}^*(e^{**}, a^*), a\rangle = \langle e^{**}, \pi_{2_{(2k+1)}}(a^*, a)\rangle = \langle e^{**}, \pi_{1_{(2k)}}^*(a^*, a)\rangle$$
$$= \langle a *, \pi_{1_{(2k)}}(a, e^{**})\rangle = \langle a^*, \pi_{2_{(2k-1)}}^{r*r}(a, e^{**})\rangle$$
$$= \langle a^*, \pi_{2_{(2k-1)}}^{r*}(e^{**}, a)\rangle = \langle e^{**}, \pi_{2_{(2k-1)}}^{r}(a, a^*)\rangle$$
$$= \langle e^{**}, \pi_{2_{(2k-1)}}(a^*, a)\rangle = \langle \pi_{2_{(2k-1)}}^*(e^{**}, a^*), a\rangle.$$

so the result holds.

Here is a new proof for the theorem [4.7.1]

Theorem 3.2. If X is a reflexive space and $D : A \to X^*$ is a derivation, then D^{**} is also a derivation.

$Proof.$ As X is reflexive, by $lemma$ 2.3 the following module actions are Arens regular,

$$\pi_1 : A \times X \to X \qquad , \qquad \pi_2 : X \times A \to X$$
$$\pi_1^* : X^* \times A \to X^* \qquad , \qquad \pi_2^{r*} : X^* \times A \to X^*$$

Now the maps bellow are Arens regular by [7, 4.4],

$$D^{**} : (A^{**}, \square) \to X^{***} \qquad , \qquad D^{**} : (A^{**}, \Diamond) \to X^{***}$$

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