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RECONSTRUCTION OF THE STURM-LIOUVILLE OPERATORS WITH TRANSMISSION AND PARAMETER DEPENDENT BOUNDARY CONDITIONS

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Abstract

Inverse problems of recovering the coefficients of discontinuous Sturm-Liouville problems with the eigenvalue parameter linearly contained in one of the boundary conditions are studied:

1) From Weyl m- function.
2) From spectral data.

Keywords: Inverse Sturm-Liouville problem, Weyl m-Function, discontinuous and parameter dependent boundary conditions.

1. Introduction

We consider the Sturm-Liouville problem

(1.1) $$ly = -y'' + qy = \lambda y,$$

With the eigenparameter dependent boundary conditions:

(1.2) $$U(y) := y'(0) - hy(0) = 0,$$

(1.3) $$V(y) := (\lambda - H_1)y'(\pi) + (\lambda H - H_2)y(\pi) = 0,$$

And discontinuous conditions

(1.4) $$y(a+0) = \alpha_1 y(a-0),\ \ y'(a+0) = \alpha_1^{-1} y'(a-0) + \alpha_2 y(a-0),$$

Where $q(x) \in L_2(0,\pi)$ is a real-valued function, $h,\ H,\ H_1,\ H_2,\ a,\ \alpha_1,\ \alpha_2 \in \mathbb{R}$ and $r = HH_1 - H_2 > 0$. For simplicity we use the notation $L = L(q(x), a, \alpha_1, \alpha_2, h, H, H_1, H_2)$ for the problem $(1.1) - (1.4)$ Boundary value problems with discontinuities inside the interval often appear in Mathematics, mechanics, physics, geophysics and other branches of natural sciences. As a rule, such problems are connected with discontinuous material properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences. The inverse problem of recovering higher-order differential operators from the Weyl functions has been studied in [17]. In [1] the Sturm-Liouville problem with discontinuities in the case when an eigenparameter linearly appears not only in the differential equation but it also appears in both of the boundary conditions is investigated. Paper [14] is devoted to the study of inverse problems by (i) one spectrum and a sequence of norming constants; (ii) two spectra. We will first start in section 2 to obtain the spectral properties of L and study the asymptotic behavior of eigenvalues, eigenfunction and norming constants with discontinuity in an interior point on $(0,\pi)$. In section 3 we study the inverse problem of recovering the pair $L=L(q(x),a,\alpha_1,\alpha_2,h,H,H_1,H_2)$ of the from $(1.1)-(1.4)$ from the given Weyl function $\mathrm{M}(\lambda)$. For this purpose we will use the method of spectral mappings for Sturm-Liouville operators on interval $(0,\mathrm{a})\cup(\mathrm{a},\pi)$ and using the solution of the main equation, we provide algorithm 3.1 for the solution of the inverse problem. In section 4 we construction Sturm-Liouville equation with spectra data $\{\lambda_n,\gamma_n\}_{n\geq 0}$ by algorithm (4.1).

We refer to the somewhat complementary surveys in [1, 14, 3, 4, 8, 9, 11, 14, 15, 18] and [20] for further aspects of this field. For general background on inverse Sturm-Liouville problems we refer (e.g.) to the monographs [7, 10, 12, 17, 19] and [21].

2. Properties of the spectrum

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1.1) satisfying the initial conditions

(2.1) $$\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=h,$$

(2.2) $$\psi(\pi,\lambda)=\lambda-H_1, \qquad \psi'(\pi,\lambda)=-\lambda H+H_2$$

From the linear differential equations we obtain the Wronskian

(2.3) $$W(u,v):=u(x)v'(x)-u'(x)v(x)$$

Is constant on $x\in(0,\mathrm{a})\cup(\mathrm{a},\pi)$for two solutions $lu=\lambda u$, $lv=\lambda v$ satisfying the transmission conditions (1.4). Moreover, we set

(2.4) $$\chi(\lambda):=W(\varphi(\lambda),\psi(\lambda))=V(\varphi)=-U(\psi).$$

Then $\chi(\lambda)$ is an entire function whose roots $\lambda_n$coincide with the eigenvalues of L. Let the inner product in the Hilbert space $\mathrm{H}=\mathrm{L}_2(0,\pi)\oplus\mathrm{C}$ be define by

(2.5) $$\langle F,G\rangle_{\mathbb{H}}=\int_0^\pi \mathrm{F}_1(\mathrm{x})\overline{\mathrm{G}}_1(\mathrm{x})\mathrm{dx}+\frac{1}{\mathrm{r}}\mathrm{F}_2\overline{\mathrm{G}}_2,$$

Where

(2.6) $$\mathrm{F}=\begin{pmatrix}\mathrm{F}_1(\mathrm{x})\\ \mathrm{F}_2\end{pmatrix} \text{ and } \mathrm{G}=\begin{pmatrix}\mathrm{G}_1(\mathrm{x})\\ \mathrm{G}_2\end{pmatrix}\in\mathbb{H}.$$

We define the operator T acting in $\mathbb{H}$such that

(2.7) $$\mathrm{T}(\mathrm{F})=\begin{pmatrix}-\mathrm{F}''(\mathrm{x})+\mathrm{q}(\mathrm{x})\mathrm{F}_1(\mathrm{x})\\ \mathrm{H}_1\mathrm{F}_1'(\pi)+\mathrm{H}_2\,\mathrm{F}_1(\pi)\end{pmatrix}$$

With

(2.8) $$D(T)=\left\{F\in\mathbb{H}\left|\begin{array}{c}F_1(x),F_1'(x)\in AC\,[0,a)\cup(a,\pi],lF_1\in L_2(0,\pi),\\ F_1'(a+0)=\alpha_1^{-1}F_1'(a-0)+\alpha_2F_1(a-0),\\ F_1(a+0)=\alpha_1F_1(a-0),F_2=F_1'(\pi)+HF_1(\pi)\end{array}\right.\right\}$$

Denote

(2.9) $$\Phi_n(x)=\begin{pmatrix}\varphi(x,\lambda_n)\\ \varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\end{pmatrix}$$

It is easy to see that the set of functions $\{\Phi_n\},n\geq 0$ are orthogonal function, i.e.

$$\langle\Phi_n,\Phi_m\rangle=0,\quad n\neq m.$$

By attaching a subscript 1 or 2 to the functions φ and ψ, we mean to refer to the first subinterval [0, a) or to the second subinterval $(a,\pi]$. Therefore we see that

(2.10) $$\varphi(x,\lambda_n)=\begin{cases}\varphi_1(x,\lambda_n), & x<a,\\ \varphi_2(x,\lambda_n), & x>a.\end{cases}$$

Therefore, we define norming constants by

(2.11) $$\gamma_n=||\Phi_n||^2=\int_0^a\varphi_1^2(x,\lambda_n)dx+\int_a^\pi\varphi_2^2(x,\lambda_n)dx+\frac{\left(\varphi_2'(\pi,\lambda_n)+H\varphi_2(\pi,\lambda_n)\right)^2}{r}$$

Where $\varphi_1(x,\lambda_n)$ and $\varphi_2(x,\lambda_n)$ are defined in Theorem 2.2.

Remark 2.1 The numbers $\{\lambda_n,\gamma_n\}_{n\geq 0}$are called the spectral data of the problem (1.1)-(1.4).

Theorem 2.2 The following asymptotic forms hold

(2.12) $$\sqrt{\lambda_n}=\rho_n=\rho_{n-3}^0+\frac{\delta_n}{n}+\frac{\xi_n}{n}$$

(2.13) $$\varphi(x,\rho_n)=\begin{cases}-(\rho_{n-3}^0)^2\cos\rho_{n-3}^0x+O\left(\frac{1}{n}\right) & x<a\\ (\rho_{n-3}^0)^2[-\alpha^+\cos\rho_{n-3}^0x+\alpha^-\cos\rho_{n-3}^0(2a-x)]+O\left(\frac{1}{n}\right) & x>a\end{cases}$$

(2.14) $$\gamma_n=(\rho_{n-3}^0)^4\left[\frac{\pi-a}{2}((\alpha^+)^2+(\alpha^-)^2)+\frac{a}{2}\right]+O(n^3)$$

Where $\tau=Im\rho,\alpha^\pm=\frac{1}{2}\left(\alpha_1\pm\frac{1}{\alpha_1}\right),\xi_n=O(1),\delta_n\in l_\infty,\rho_n^0=n+h_n, and\ h_n\in l_\infty.$

Proof By using the similar proof of [17] we obtain

(2.15) $$\varphi(x,\rho)=\cos\rho x+\frac{h}{\rho}\sin\rho x+\frac{1}{\rho}\int_0^x\sin\rho(x-t)q(t)\varphi(t,\rho)dt\qquad x<a,$$

(2.16) $$\varphi(x,\rho)=\alpha^+\cos\rho x+\alpha^-\cos\rho(2a-x)+\left(h\alpha^++\frac{\alpha_2}{2}\right)\frac{\sin\rho x}{\rho}$$

$$+(h\alpha^--\frac{\alpha_2}{2})\frac{\sin\rho(2a-x)}{\rho}+\frac{1}{\rho}\int_a^x\sin\rho(x-t)q(t)\varphi(t,\rho)dt\qquad x>a,$$

(2.17) $\psi(x,\rho) = (H_1 - \rho_1)(\alpha^+ \cos\rho(\pi - x) - \alpha^- \cos\rho(x + \pi - 2a)) +$

$$+\left((H_2 - \rho^2 H)\alpha^+ + \frac{\alpha_2}{2}\right)\frac{\sin\rho(\pi - x)}{\rho} + \left((H_2 - \rho^2 H)\alpha^- - \frac{\alpha_2}{2}\right)\frac{\sin\rho(x + \pi - 2a)}{\rho} +$$

$$+\frac{1}{\rho}\int_a^\pi (\alpha^+ \sin\rho(x - t) + \alpha^- \sin\rho(x + t - 2a))q(t)\psi(t,\rho)dt$$

$$+\frac{1}{\rho}\int_x^a \sin\rho\,(t - x)q(t)\psi(t,\rho)\,dt \qquad x < a,$$

(2.18) $\psi(x,\rho) = \frac{H_2 - \rho^2 H_0}{\rho}\sin\rho(\pi - x) + \ (H_1 - \rho^2)\cos\rho(\pi - x)$

$$+\frac{1}{\rho}\int_x^\pi \sin\rho\,(t - x)q(t)\psi(t,\rho)dt \qquad x > a.$$

From $(2.15) - (2.18)$ we obtain

(2.19) $$\frac{\varphi(x,\rho)}{\rho^2} = \begin{cases} -\cos\rho x + O\left(\frac{1}{|\rho|}\exp|\tau|\,x\right) & x < a, \\ (-\alpha^+ \cos\rho x + \alpha^- \cos\rho(2a - x)) + O\left(\frac{1}{|\rho|}\exp|\tau|\,x\right) & x > a \end{cases}$$

(2.20) $$\frac{\psi(x,\rho)}{\rho^2} = \begin{cases} -\alpha^+ \cos\rho(\pi - x) - \alpha^- \cos\rho(x + \pi - 2a) + O\left(\frac{1}{|\rho|}\exp|\tau|\,(\pi - x)\right), & x < a, \\ -\cos\rho(\pi - x) + O\left(\frac{1}{|\rho|}\exp|\tau|\,(\pi - x)\right) & x > a\,. \end{cases}$$

From (2.4) and $(2.19) - (2.20)$ we get

(2.21) $\chi(\lambda) = \lambda^3\left(\alpha^+ \frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}} + \alpha^- \frac{\sin\sqrt{\lambda}(2a - \pi)}{\sqrt{\lambda}}\right) + O(\lambda^2 \exp|\tau|\,\pi).$

Denote

(2.22) $\chi_0(\lambda) = \alpha^+ \frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}} + \alpha^- \frac{\sin\sqrt{\lambda}(2a - \pi)}{\sqrt{\lambda}}$

And

(2.23) $G_n = \{\lambda \in \mathbb{C}\,;\ \lambda = \rho^2,\ |\rho| = |\rho_n^0 - \delta|\},$

Where δ is sufficiently small and ρ_n^0 are the zeros of $\chi_0(\lambda)$ except 0 . Since, $|\chi_0(\lambda)| \geq C(\lambda^{\frac{5}{2}}\exp|\tau|\pi)$ And $|\chi(\lambda) - \chi_0(\lambda)| = O(\lambda^2 \exp|\tau|\pi)$ for $\lambda \in G_n$ and large values n, using the Rouche's theorem, We establish that contour G_n. Consequently, in the annulus between G_n and G_{n+1}, χ has precisely one zero, namely ρ_n^2. Therefore, for the eigenvalue λ_n, the equality $\lambda_{n+3} = \rho_n^2$ is true . On the other hand, by using again the Rouche's theorem in $\gamma_\varepsilon = \{\lambda : |\lambda - \rho_n^0| < \varepsilon\}$ for sufficiently small ε, we get the asymptotic formulae $\rho_n = \rho_n^0 + \varepsilon_n$ ($\varepsilon_n = o(1)$) is valid for large n. Finally, the equality $\varepsilon_n = O\left(\frac{1}{n}\right)$ is taken from the well known formulae $\chi_0(\lambda)(\rho_n^0 + \varepsilon_n) = \chi_0'(\rho_n^0)\varepsilon_n + o(\varepsilon_n)$, This fact proves the equality (2.12) where δ_n is the same as θ_n in ([6], $p.240$). By using (2.12) in (2.19) and (2.13) in (2.11) we get (2.13) and (2.14) respectively. ■

3. Reconstruction by Weyl M-function

Using properties of the spectrum the Weyl m-function [13], we can write

(3.1) $$m(\lambda) = -\frac{\psi(0,\lambda)}{\chi(\lambda)}$$

Also asymptotic expansion have been obtained

(3.2) $$m(\lambda) = \frac{1}{\sqrt{-\lambda}} + O(\lambda^{-1})$$

Let $S(x,\lambda)$ be a solution of (1.1) subject to the initial conditions

$$S(0,\lambda) = 0, \qquad S'(0,\lambda) = 1$$

and the jump conditions (1.4). The function $\psi(x,\lambda)$ can be represented as

(3.3) $$\theta(x,\lambda) := \frac{\psi(x,\lambda)}{\chi(\lambda)} = S(x,\lambda) - m(\lambda)\varphi(x,\lambda).$$

Where the functions $\theta(x,\lambda)$ and $m(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem L. Now, we prove the uniqueness theorem for the solution of the inverse problem. We agree together a boundary value problem $\widetilde{L}$ of the same form but with different coefficients $\tilde{q}(x), \tilde{a}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{h}, \widetilde{H}, \widetilde{H}_1, \widetilde{H}_2$.

Theorem 3.1 $If\ M(\lambda) = \widetilde{M}(\lambda)$, then $L = \tilde{L}$, i.e , $q(x) = \tilde{q}(x)$, a.e, $a = \tilde{a}, \alpha_1 = \tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$, $h = \tilde{h}, H = \widetilde{H}, H_1 = \widetilde{H}_1, H_2 = \widetilde{H}_2$.

Proof. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)], j; k = 1,2$ by the formula

(3.4) $$P(x,\lambda)\begin{pmatrix} \tilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix}$$

Using (3.2) and (3.4) we have,

(3.6) $$\begin{cases} \varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda), \\ \Phi(x,\lambda) = P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda), \end{cases}$$

According to (3.1) and (3.5), for fixed x, the functions $P_{jk}(x,\lambda)$ are meromorphic Function in λ with simple poles in the points λ_n and $\tilde{\lambda}_n$. Denote $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$ where

$$G_\delta = \{\lambda : |\lambda - \lambda_n| > \delta, n = 1,2, \dots\}$$

And

$$\tilde{G}_\delta = \{\lambda : |\lambda - \tilde{\lambda}_n| > \delta, n = 1,2, \dots\}$$

From $\rho \in G_\delta$ and using the asymptotic form of $\Phi^v(x,\lambda)$ we get $|\Phi^v(x,\lambda)| \le C_\delta |\rho|^{v-3} \exp(-|\tau|x)$.

Thus it follows that

(3.7) $\quad |P_{11}(x,\lambda)| \le C_\delta\,, |P_{12}(x,\lambda)| \le C_\delta |\rho|^{-1}, \rho \in G_\delta^0.$

Using (3.1) and (3.5) we get

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{\varphi}'(x,\lambda) + \left(\widetilde{M}(\lambda) - M(\lambda)\right)\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda)$$
$$P_{12}(x,\lambda) = S(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{S}(x,\lambda) + \left(\widetilde{M}(\lambda) - M(\lambda)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).$$

Thus, if $M(\lambda) \equiv \widetilde{M}(\lambda)$, then for each fixed x, the function $P_{1k}(x,\lambda)$ are entire in

λ. Together with (3.7) this yields $P_{12}(x,\lambda) \equiv 0, P_{11}(x,\lambda) \equiv A(x)$. Using (3.6) we

Derive

(3.8) $\quad \varphi(x,\lambda) \equiv A(x)\tilde{\varphi}(x,\lambda), \;\; \Phi(x,\lambda) \equiv A(x)\widetilde{\Phi}(x,\lambda).$

From $W(\Phi(x,\lambda),\varphi(x,\lambda)) \equiv 1$ and similarly $W(\widetilde{\Phi}(x,\lambda),\tilde{\varphi}(x,\lambda)) \equiv 1$, we have $A(x) = 1$. So from (3.8) we obtain, $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \widetilde{\Phi}(x,\lambda)$ for all x and λ. Consequently, $L = \tilde{L}$. ■

Now we construct the solution of the inverse problem. For this work first we denote

$$(3.9) \quad \begin{cases} D(x,\lambda,\mu) := \dfrac{W(\varphi(x,\lambda),\varphi(x,\mu))}{\lambda-\mu} = \int_0^x \varphi(t,\lambda)\varphi(t,\mu)dt, & r(x,\lambda,\mu) = D(x,\lambda,\mu)\widehat{M}(\mu), \\ \widetilde{D}(x,\lambda,\mu) := \dfrac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\mu))}{\lambda-\mu} = \int_0^x \tilde{\varphi}(t,\lambda)\tilde{\varphi}(t,\mu)dt, & \tilde{r}(x,\lambda,\mu) = \widetilde{D}(x,\lambda,\mu)\widehat{M}(\mu), \end{cases}$$

From Theorem 2.2 we have

$$(3.10) \quad D(x,\lambda,\mu) = \begin{cases} \int_0^x \varphi_1(x,\rho_n)\varphi_1(x,\theta_n)dt, & x < a, \\ \int_0^a \varphi_1(x,\rho_n)\varphi_1(x,\theta_n)dt + \int_a^x \varphi_2(x,\rho_n)\varphi_2(x,\theta_n)dt, & x > a, \end{cases}$$

where ${\rho_n}^2 = \lambda_n$ and $\theta_n^2 = \mu_n$.

Lemma 3.2, Let $\rho = \sigma + i\tau$. The following estimates hold

(3.11) $\quad |D(x,\lambda,\mu)|, \left|\widetilde{D}(x,\lambda,\mu)\right| \le \frac{C_x \exp(|\tau|x)}{|\rho\pm\theta|+1}, \mu = \theta^2 \ge 0, \pm\theta\Re\rho \ge 0$

Proof, For definiteness, let $\theta \ge 0$ and $\sigma \ge 0$. All other cases can be treated in the same way. Take a fixed δ_0 . For $|\rho - \theta| \ge \delta_0$ we have by virtue of (3.9) and relation of $|\varphi^v(x,\lambda)| \le C|\rho|^v\, exp(|\tau\,|x)$, for $v = 0, 1$

(3.12) $\quad |D(x,\lambda,\mu)| = \left|\frac{W(\varphi(x,\lambda),\varphi(x,\mu))}{\lambda-\mu}\right| \le Cexp(|\tau|x)\,\frac{|\rho|+|\theta|}{|\rho^2-\theta^2|}.$

Where C positive constant

Since

$$\frac{|\rho| + |\theta|}{|\rho + \theta|} = \frac{\sqrt{\sigma^2+\tau^2} + \theta}{\sqrt{(\sigma + \theta)^2 + \tau^2}} \le \frac{\sqrt{\sigma^2+\tau^2} + \theta}{\sqrt{\sigma^2 + \tau^2 + \theta^2}} \le \sqrt{2}.$$

So, we have

(3.13) $$|D(x,\lambda,\mu)| \leq \frac{Cexp(|\tau|x)}{|\tau-\theta|}.$$

For $|\rho-\theta| \geq \delta_0$ we get

$$\frac{|\rho-\theta|+1}{|\rho-\theta|} \leq 1+\frac{1}{\delta_0}$$

And consequently

$$\frac{1}{|\rho-\theta|} \leq \frac{C_0}{|\rho-\theta|+1}, C_0=\frac{\delta_0+1}{\delta_0}.$$

Substituting this estimate into the right-hand side of (3.13) we obtain

$$|D(x,\lambda,\mu)| \leq \frac{C\exp(|\tau|x)}{|\rho-\theta|+1}$$

Therefore, (3.11) is proved for $|\rho-\theta| \geq \delta_0$. Analogously, for $|\rho-\theta| \leq \delta_0$, (3.11) is valid. ■

In the following figure we have the contour $\gamma = \gamma' \cup \gamma''$ where γ' is a bounded closed contour encircling the set $\{\lambda = \rho^2 : Im\rho \geq 0, \rho \neq 0 : \chi(\rho) = 0\}$ and γ'' is the two-sided cut along the arc $\{\lambda : \lambda > 0, \lambda \notin int\,\gamma'\}$.

Theorem 3.3 The following relations hold

(3.14) $$\tilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\frac{1}{2\pi i}\int_{\gamma}\tilde{r}(x,\lambda,\mu)\varphi(x,\mu)d\mu$$

(3.15) $$r(x,\lambda,\mu)-\tilde{r}(x,\lambda,\mu)+\frac{1}{2\pi i}\int_{\gamma}\tilde{r}(x,\lambda,\xi)r(x,\mu,\xi)d\xi=0.$$

The relation (3.14) is called the main equation of the inverse problem.

Proof. For $\lambda, \mu \in \gamma, \pm\mathrm{Re}\,\rho, \mathrm{Re}\,\theta \geq 0$,

(3.16) $$|r(x,\lambda,\mu)|,\ |\tilde{r}(x,\lambda,\mu)| \leq \frac{C_x}{|\mu|(|\rho\pm\theta|+1)}, |\varphi(x,\lambda)| \leq C.$$

Denote $J_{\gamma} = \{\lambda : \lambda \notin \gamma \cup \gamma'\}$. Consider the Contour $\gamma_R = \gamma \cap \{\lambda : |\lambda| \leq R\}$ with counter clockwise circuit, and also consider the contour $\gamma_R^0 = \gamma_R \cup \{\lambda : |\lambda| = R\}$ with clockwise circuit. By Cauchy's integral formula

$$P_{1k}(x,\lambda)-\delta_{1k}=\frac{1}{2\pi i}\int_{\gamma_R^0}\frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}d\mu, \quad \lambda \in int\,\gamma_R^0,$$

$$\frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma_R^0}\frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}d\xi, \qquad \mu,\lambda \in int\ \gamma_R^0$$

Using (3.7) we get

$$\lim_{R\to\infty}\int_{|\mu|=R}\frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}d\mu = 0, \lim_{R\to\infty}\int\frac{P_{jk}(x,\mu)}{(\lambda-\xi)(\xi-\mu)}d\xi = 0,$$

And consequently

(3.17) $\quad P_{1k}(x,\lambda) = \delta_{1k} + \int_\gamma \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}d\mu, \quad \lambda \in J_\gamma$

(3.18) $\quad \frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_\gamma \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}d\xi, \qquad \mu,\lambda \in int\ J_\gamma$

By virtue of (3.6) and (3.17),

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{\tilde{\varphi}(x,\lambda)P_{11}(x,\mu)+\tilde{\varphi}'(x,\lambda)P_{12}(x,\mu)}{\lambda-\mu}d\mu, \quad \lambda \in J_\gamma.$$

Taking (3.4) into account we get

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \Big[\tilde{\varphi}(x,\lambda)\left(\varphi(x,\lambda)\tilde{\Phi}'(x,\mu) - \Phi(x,\mu)\tilde{\varphi}'(x,\mu)\right)$$
$$+\tilde{\varphi}'(x,\lambda)\big(\Phi(x,\lambda)\tilde{\varphi}(x,\mu) - \varphi(x,\mu)\tilde{\Phi}(x,\mu)\big)\Big]\frac{d\mu}{\lambda-\mu}.$$

In view of (3.1), this yields (3.14). According to (3.18) and the proof of Lemma (1.6.3, in [6]) we arrive at

$$D(x,\lambda,\mu) - \widetilde{D}(x,\lambda,\mu) = \frac{1}{2\pi i}\int_\gamma\Big(\frac{W\big(\tilde{\varphi}(x,\lambda),\tilde{\Phi}(x,\xi)\big)W(\varphi(x,\xi),\varphi(x,\mu))}{(\lambda-\xi)-(\xi-\mu)}\Big) - \frac{1}{2\pi i}\int_\gamma\Big(\frac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\xi))W(\Phi(x,\xi),\varphi(x,\mu))}{(\lambda-\xi)-(\xi-\mu)}\Big)d\xi$$

In view of (3.1) and (3.9) this yields (3.15). ■

Theorem 3.4 For each fixed $x \in (0,a) \cup (a,\pi)$, the main equation (3.14) has a unique solution $\varphi(x,\lambda) \in C(\gamma)$. Where $C(\gamma)$ is a Banach space contained the continuous bounded functions $z(\lambda), \lambda \in \gamma$, with the norm $\| z \| = \sup_{\lambda\in\gamma}|z(\lambda)|$.

Proof For $0 < x < a$, we consider the following linear bounded operators in $C(\gamma)$

(3.19) $\qquad \tilde{A}z(\lambda) = z(\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)z(\mu)d\mu,$

(3.20) $\qquad Az(\lambda) = z(\lambda) - \frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)z(\mu)d\mu,$

From (3.19) and (3.20) we get

$$\tilde{A}A(\lambda) = z(\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)z(\mu)d\mu - \frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)z(\mu)d\mu$$

$$-\frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\xi)\left(\frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)z(\mu)d\mu\right)d\xi =$$

$$= z(\lambda)-\frac{1}{2\pi i}\int_\gamma\left(r(x,\lambda,\mu)-\tilde{r}(x,\lambda,\mu)+\frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\mu)r(x,\xi,\mu)d\xi\right)z(\mu)d\mu.$$

From (3.15) we have

$$A\tilde{A}z(\lambda) = z(\lambda), z(\lambda) \in C(\lambda).$$

Also we obtain analogously $A\tilde{A}z(\lambda) = z(\lambda)$.Thus,

$$\tilde{A}A = A\tilde{A} = E$$

Where E is the identity operator, Hence the operator $\tilde{A}$ has a bounded inverse operator, and the main equation (3.14) is uniquely solvable for each fixed $0 < x < a$. Analogously For $a < x < \pi$ relation (3.14) is uniquely solvable. ■

Theorem 3.5 The following relations hold

(3.21) $$q(x) = \tilde{q}(x) + \varepsilon(x),$$

(3.22) $$h = \tilde{h} - \varepsilon_0(0).$$

(3.23) $$H = \tilde{H} + \varepsilon_0(\pi),\ H2 = \tilde{H}_2 + \varepsilon_0(\pi)\tilde{H}_1,\ H_1 = \tilde{H}_1,$$

(3.24) $$\alpha_2 = \tilde{\alpha}_2 - (\frac{\alpha_1^4-1}{\alpha_1})\varepsilon_0(a-0),$$

Where

(3.25) $$\varepsilon_0(x) = \frac{1}{2\pi i}\int_\gamma \tilde{\varphi}(x,\mu)\varphi(x,\mu)M(\mu)d\mu,\ \ \varepsilon(x) = -2\varepsilon_0'(x)$$

Proof. By (3.9), (3.14) and (3.25) we get

(3.26) $$\tilde{\varphi}'(x,\lambda)-\varepsilon_0(x)\tilde{\varphi}(x,\lambda) = \varphi'(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)\varphi'(x,\mu)d\mu$$

(3.27) $$\tilde{\varphi}''(x,\lambda) = \varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde{r}\,(x,\lambda,\mu)\varphi''(x,\mu)d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma 2\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu)\varphi'(x,\mu)M(\mu)d\mu+\frac{1}{2\pi i}\int_\gamma 2\big(\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu)\big)'\varphi(x,\mu)M(\mu)d\mu$$

In (3.27) we replace the second derivatives by using equation(1.1), and so we replace $\varphi(x,\lambda)$ using (3.14).This yields

$$\tilde{q}(x)\tilde{\varphi}(x,\lambda) = q(x)\tilde{\varphi}(x,\lambda)+\frac{1}{2\pi i}\int_\gamma W\big(\varphi(x,\lambda),\varphi(x,\mu)\big)M(\mu)\varphi(x,\mu)d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma 2\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu)M(\mu)\varphi'(x,\mu)d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma(\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu))'M(\mu)\varphi(x,\mu)d\mu$$

After canceling terms with $\varphi'(x,\lambda)$ we arrive at (3.21). Taking $x = 0$ and $x = \pi$ in (3.26) and(3.14) we get $(3.22) - (3.23)$, also By virtue of (1.4), it from (3.25) that

(3.28) $$\varepsilon_0(a+0) = \alpha_1^2 \varepsilon_0(a-0).$$

And using (1.4), (3.26) and (3.28) we have (3.24).

Thus, we obtain the following algorithm for the solution of the inverse problem.

Algorithm 3.1 Let the function $M(\lambda)$ be given. Then

i) Choose $\tilde{L}$ such that $\tilde{q}(x) \in L_2(0,\pi), \tilde{h}, \tilde{H}, \tilde{H}_1, \tilde{H}_2, \tilde{\alpha}_1$ and $\tilde{\alpha}_2 \in \mathbb{R}$;

ii) Find $\varphi(x,\lambda)$ by solving equation (3.14);

iii) Construct $q(x)$ and h, H, H_1, H_2 via $(3.21) - (3.23)$;

iv) Construct α_2 by (3.24).

4. Reconstruction by spectral data

Let two sequences of real numbers $\{\lambda_n\}$ and $\{\gamma_n\}$, $(n \in \mathbb{Z}_+)$ with the following

Properties be given

$$\sqrt{\lambda_n} = \rho_n = \rho_{n-3}^0 + \frac{\delta_n}{n} + \frac{\xi_n}{n}$$

$$\gamma_n = (\rho_{n-3}^0)^4 \left(\frac{\pi - a}{2}((\alpha^+)^2 + (\alpha^-)^2) + \frac{a}{2}\right) + O(n^3)$$

Where

$$\xi_n = O(1), \delta_n \in l_\infty, \alpha^+ = \alpha_1 + \frac{1}{\alpha_1}, \ \alpha^- = \alpha_1 - \frac{1}{\alpha_1}.$$

Now we consider the inverse problem of recovering L from the spectral data$\{\lambda_n, \gamma_n\}_{n\geq 0}$. Let us choose a model boundary value problem $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H}, \tilde{H}_1, \tilde{H}_2, \tilde{\alpha}_1, \tilde{\alpha}_2)$ with real $\tilde{q}(x) \in L_2(0,\pi), \tilde{h}, \tilde{H}, \tilde{H}_1, \tilde{H}_2$ and $\tilde{r} := \tilde{H}\tilde{H}_1 - \tilde{H}_2 > 0$ such that $\omega = \tilde{\omega}$ and $\omega = h + H + \frac{1}{2}\int_0^\pi q(t)dt$.

Let

(4.1) $$a = \tilde{a} \quad \text{and} \quad \sum_{n=0}^{\infty} \xi_n |\rho_n| < \infty$$

Where $\xi_n = |\rho_n - \tilde{\rho}_n| + |\gamma_n - \tilde{\gamma}_n|$. Denote

$$\lambda_{n0} = \lambda_n, \lambda_{n1} = \tilde{\lambda}_n, \gamma_{n0} = \gamma_n, \gamma_{n1} = \tilde{\gamma}_n, \varphi_{ni}(x) = \varphi(x,\lambda_{ni}), \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x,\lambda_{ni})$$

$$\tilde{Q}_{kj}(x,\lambda) = \frac{W(\tilde{\varphi}(x,\lambda), \tilde{\varphi}_{kj}(x))}{\gamma_{kj}(\lambda - \lambda_{kj})} = \frac{1}{\gamma_{kj}} \int_0^x \tilde{\varphi}(t,\lambda), \tilde{\varphi}_{kj}(t) dt$$

$$\tilde{Q}_{ni,kj}(x,\lambda) = \tilde{Q}_{kj}(x,\lambda_{ni})$$

It follows from (2.15),(2.16) that

(4.2) $$|\varphi_{nj}^{\nu}(x)| \leq C(|\rho_n^0 + 1|)^{\nu}, |\tilde{\varphi}_{nj}^{\nu}(x)| \leq C(|\rho_n^0 + 1|)^{\nu}$$

(4.3) $\left|\tilde{Q}_{ni,kj}(x)\right| \leq \frac{C}{|\rho_n^0-\rho_k^0|+1}, \left|\tilde{Q}^{v+1}{}_{ni,kj}(x)\right| \leq C\left(|\rho_n^0| + |\rho_k^0| + 1\right)^v.$

Lemma 4.1 The following relation holds

(4.4) $\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \sum_{n=0}^{\infty}(\tilde{Q}_{k0}(x,\lambda)\varphi_{k0}(x) - \tilde{Q}_{k1}(x,\lambda)\varphi_{k1}(x))$

Proof By virtue of (4.1) we have

(4.5) $a = \tilde{a}, \alpha_1 = \tilde{\alpha}_1$

It follow from (2.15) and (2.16) that

(4.6) $|\varphi^v(x,\lambda) - \tilde{\varphi}^v(x,\lambda)| \leq C|\rho|^{v-1}\exp(|\tau|x).$

Similary,

(4.7) $\left|\psi^v(x,\lambda) - \tilde{\psi}^v(x,\lambda)\right| \leq C|\rho|^{v-1}\exp\left(|\tau|(T-x)\right).$

Denote $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$. From (3.1) and (4.7) we have

(4.8) $\left|\Phi^v(x,\lambda) - \tilde{\Phi}^v(x,\lambda)\right| \leq C_\delta|\rho|^{v-2}\exp(-|\tau|x), \quad \rho \in G_\delta^0 \ v = 0,1.$

Let $P(x,\lambda)$ be the matrix defined in Theorem 3.1 and $\Gamma = \{\lambda = u + iv: u = (2h)^{-2}v^2 - h^2\}$. be the image of the set $Im\rho = \pm h$ under the mapping $\lambda = \rho^2$.Denote $\Gamma_n = \Gamma \cap \{\lambda : |\lambda| \leq r_n\}$, and Γ_{n0}= $\Gamma_n \cup \{\lambda : |\lambda| = r_n, \lambda \notin int\,\Gamma\}$, $\Gamma_{n1} = \Gamma_n \cup \{\lambda : |\lambda| = r_n, \lambda \in int\,\Gamma\}$. Since for each fixed x, the functions P_{1k} are meromorphic in λ with simple poles λ_n and $\tilde{\lambda}_n$, we get by Cauchy theorem

(4.9) $P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\Gamma_{n0}} \frac{P_{1k}(x,\xi)-\delta_{1k}}{\xi-\lambda}d\xi, \ \ k = 1,2.$

Where $\lambda \in \Gamma_{n0}$, and δ_{jk} is the Kronecker's delta. Further, (3.2) and (3.5) imply

(4.10) $P_{11}(x,\lambda) = 1 + \left(\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda)\right)\tilde{\Phi}'(x,\lambda) - \left(\Phi(x,\lambda) - \tilde{\Phi}(x,\lambda)\right)\tilde{\varphi}'(x,\lambda)$

Also we can obtain

(4.11) $|P_{1k}(x,\lambda) - \delta_{1k}| \leq C_\delta|\rho|^{-1}, \rho \in G_\delta^0.$

By virtue of (4.11)

(4.12) $\lim_{n\to\infty}\frac{1}{2\pi i}\int_{|\xi|=r_n}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\xi-\lambda}d\xi = 0,$

And consequently, (4.9) yields

$$P_{1k}(x,\lambda) - \delta_{1k} = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\xi-\lambda}d\xi\,.$$

Substituting into (3.6) we obtain

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{\tilde{\varphi}(x,\lambda)P_{11}(x,\xi) + \tilde{\varphi}'(x,\lambda)P_{12}(x,\xi)}{\xi-\lambda}d\xi\,.$$

Taking (3.5) into account we calculate

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}(\tilde{\varphi}(x,\lambda)(\varphi(x,\xi)\tilde{\Phi}'(x,\xi) - \Phi(x,\xi)\tilde{\varphi}'(x,\xi))$$

$$+\tilde{\varphi}'(x,\lambda)\left(\Phi(x,\xi)\tilde{\varphi}(x,\xi)-\varphi(x,\xi)\tilde{\Phi}(x,\xi)\right)\frac{d\xi}{\lambda-\xi},$$

Or, in view of (3.1),

(4.13) $$\tilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\xi))}{\lambda-\xi}M(\xi)\varphi(x,\xi)d\xi\,.$$

Then we have

$$Res_{\xi=\lambda_{kj}}\frac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\xi))}{\lambda-\xi}M(\lambda)\varphi(x,\xi)=\tilde{Q}_{kj}(x,\lambda)\varphi_{kj}(x).$$

Now with calculation the integral in (4.13) by residue theorem we arrive at (4.4). ■

Let K be a set of indices $n=(n,i), n\geq 0, i=0,1$. For each fixed $x\in[0,\pi]$, we define the vector

$$\psi(x)=[\psi_u(x)]_{u\in K}=\begin{bmatrix}\psi_{n0}(x)\\ \psi_{n1}(x)\end{bmatrix}_{n\geq 0}=[\psi_{00},\psi_{01},\psi_{02,\ldots}]^T$$

by the formulae

$$\begin{bmatrix}\psi_{n0}(x)\\ \psi_{n1}(x)\end{bmatrix}_{n\geq 0}=\begin{bmatrix}\chi_n & -\chi_n\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_{n0}(x)\\ \varphi_{n1}(x)\end{bmatrix}=\begin{bmatrix}\chi_n(\varphi_{n0}(x)-\varphi_{n1}(x))\\ \varphi_{n1}(x)\end{bmatrix},$$

$$\chi_n=\begin{cases}\xi_n^{-1} & \xi_n\neq 0\\ 0 & \xi_n=0\end{cases}$$

we also define the block matrix

$$H(x)=[H_{u,v}(x)]_{u,v\in K}=\begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix}.\ n,k\geq 0, u=(n,i), v=(k,j)$$

By the formulae

$$\begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix}=\begin{bmatrix}\chi_n & -\chi_n\\ 0 & 1\end{bmatrix}\begin{bmatrix}Q_{n0,k0}(x) & Q_{n0,k1}(x)\\ Q_{n1,k0}(x) & Q_{n1,k1}(x)\end{bmatrix}\begin{bmatrix}\xi_k & 1\\ 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}\xi_n\chi_n(Q_{n0,k0}(x)-Q_{n1,k0}(x)) & \chi_n(Q_{n0,k0}(x)-Q_{n0,k1}(x)-Q_{n1,k0}(x)+Q_{n1,k1}(x))\\ \xi_k Q_{n1,k0}(x) & Q_{n1,k0}(x)-Q_{n1,k1}(x)\end{bmatrix}$$

Analogously we define $\tilde{\psi}(x), \tilde{H}(x)$ by replacing in the previous definitions, $\varphi_{ni}(x)$ by $\tilde{\varphi}_{ni}(x)$ and $Q_{ni,kj}(x)$ by $\tilde{Q}_{ni,kj}(x)$ also we have

(4.14) $$|\psi_{ni}^{v}(x)|\leq C(\rho_n^0+1)^v, |H_{ni,kj}(x)|\leq\frac{C\xi_k}{|\rho_n^0-\rho_k^0|+1}$$

Similary

(4.15) $$\left|\tilde{\psi}_{ni}^{v}(x)\right|\leq C(\rho_n^0+1)^v, |\tilde{H}_{ni,kj}(x)|\leq\frac{C\xi_k}{|\rho_n^0-\rho_k^0|+1}$$

Let us consider the Banach space m of bounded squencess $\alpha = [\alpha_u]_{u\in K}$ with the norm $||\alpha||_m = sup_{u\in K} |\alpha_{\mathrm{u}}|$. It follows from (4.14) and (4.15) that for each fixed $x \in [0,\pi]$, the operators $E + \widetilde{H}(x)$ and $E - \widetilde{H}(x)$ (here E is the identity operator), acting from m to m, is a linear bounded operator, and

$$||\widetilde{H}(x)|| \leq sup_n \sum_k \frac{\xi_k}{|\rho_n^0-\rho_k^0|+1} < \infty$$

Taking into account our notation, we can rewrite (4.4) in the form

$$\tilde{\psi}_{ni}(x,\lambda) = \psi_{ni}(x,\lambda) + \sum_{n=0}^{\infty}(\widetilde{H}_{ni,k0}(x)\psi_{k0}(x) - \widetilde{H}_{ni,k1}(x)\psi_{k1}(x))$$

Or

(4.16) $$\tilde{\psi}(x,\lambda) = \left(E + \widetilde{H}(x)\right)\psi(x)$$

Thus, for each fixed$x$, the vector $\psi(x) \in m$ is a solution of equation (4.16)in the Banach space m. Equation (4.16) is called the main equation of the inverse problem. solving (4.16) we find the vector $\psi(x)$ and consequently, the functions $\varphi_{ni}(x), n \geq 0, i = 0,1$. Since $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ are the solutions of (1.1), we can construct the function $q(x)$ by the formula

(4.17) $$q(x) = \lambda_{ni} + \frac{\varphi''_{ni}(x)}{\varphi_{ni}(x)} .$$

we get the coefficient h by

(4.18) $$h = \varphi'(0,\lambda_{ni}).$$

Also we obtain the coefficients H, H_1 and H_2 from the linear system of equations

(4.19) $$\begin{cases}(\lambda_{n0} - H_1)\,\varphi'_{n0}(\pi) + (\lambda_{n0}H - H_2)\varphi_{n0}(\pi) = 0 & n \geq 0,\\ \widetilde{\omega} = \omega\end{cases}$$

And finally we obtain

(4.20) $$\alpha_2 = \frac{\varphi'_{ni}(a+0)}{\varphi_{ni}(a-0)} - \frac{\varphi'_{ni}(a-0)}{\varphi_{ni}(a+0)}$$

Now, we get the following algorithm for the solution of the inverse problem of recovering L from the given spectral data $\{\lambda_n, \gamma_n\}_{n\geq 0}$.

Algorithm 4.1 Let the spectral data$\{\lambda_n, \gamma_n\}_{n\geq 0}$ be given. Then

i) Choose $\tilde{L}$ such that $\widetilde{\omega} = \omega$, and construct $\tilde{\psi}(x)$ and $\widetilde{H}(x)$;

ii) Find $\psi(x)$ by solving (4.16);

iii) Calculate $q(x), h, H, H_1$, and H_2 by (4.17), (4.18) and (4.19);

iv) Construct α_2 by (4.20).

Example 4.2 Take $\tilde{L} = L(\tilde{q}(x) = 0, a, \alpha_1, 0, 0, 0, \tilde{H}_2 < 0)$. Let $\{\lambda_n, \gamma_n\}_{n\geq 0}$. be the spectral data of $\tilde{L}$. Clearly,

$$\tilde{\lambda}_0 = 0, \tilde{\gamma}_0 = a + (\pi - a)\alpha_1, \tilde{\varphi}_{00}(x) = 1(x < a), \tilde{\varphi}_{00}(x) = \alpha_1(x > a).$$

Let $\lambda_n = \tilde{\lambda}_n (n \geq 0), \gamma_n = \tilde{\gamma}_n$, and $\gamma_0 > 0$ be an arbitrary positive number. Denote $A := \frac{1}{\gamma_0} - \frac{1}{\tilde{\gamma}_0}$, then (4.4) yields

$$\tilde{\varphi}_{00}(x) = \varphi_{00}(x)(1 + A\int_0^x \tilde{\varphi}_{00}^2(t)dt)$$

So, we have

$$\varphi_{00}(x) = \begin{cases} (1 + Ax),^{-1} & x < a, \\ \alpha_1 (B + A\alpha_1^2 x),^{-1} & x > a. \end{cases}$$

Where $B = 1 + Aa - A\alpha_1^2 a$. Using (4.17) and value $\lambda_{00} = 0$, it is easy to see that

$$q(x) = \begin{cases} 2A^2 (1 + Ax),^{-2} & x < a, \\ 2A^2\alpha_1^4 (B + A\alpha_1^2 x),^{-2} & x > a. \end{cases}$$

Also, we can obtain the following relations

$$h = -A, \qquad H = \frac{A\alpha_1^2}{(B + A\alpha_1^2\pi)}, \quad H_1 = \frac{\alpha_1}{(B + A\alpha_1^2\pi)} \qquad H_2 = \frac{-A\alpha_1^3}{(B + A\alpha_1^2\pi)^2}$$

So finally we obtain

$$\alpha_2 = \frac{\varphi'_{ni}(a+0)}{\varphi_{ni}(a-0)} - \frac{\varphi'_{ni}(a-0)}{\varphi_{ni}(a+0)} = \frac{A(\alpha_1^{-1} - \alpha_1^3)}{(1 + Aa)}.$$

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