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Hopf Bifurcation in Numerical Approximation for Price Reyleigh Equation with Finite Delay

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Abstract

The numerical approximation of Price Reyleigh equation is considered using delay as parameter. Fist, the delay difference equation obtained by using Euler method is written as a map.According to the theories of bifurcation for discrete dynamical systems, the conditions to guarantee the existence of Hopf bifurcation for numerical approximation are given. The relations of Hopf bifurcation between the continuous and the discrete are discussed. That when the Price Reyleigh equation has Hopf bifurcations at $r = r_0$, the numerical approximation also has Hopf bifurcations at $r_h + r_0 = O_h$ is proved.

Keywords: Price Reyleigh equation; Euler method; Hopf bifurcation; Numericalapproximation.

Introduction

In recent years, with the booming development of the economy of human society, price oscillation have raised the attention of many scholars. Many of them [8-10] have done systematic research to the price differential equation model in the market economy and have got many essential results[10].In 1997, the literature[1] studied the price reyleigh equation model without considering the effect of finite delay.

$$\ddot{x}(t) - \eta(ax^2(t) + \beta x(t) + \gamma)\dot{x}(t) + x(t) = 0 \qquad (0)$$

In which $\eta > 0$, a, β, γ are arbitrary constants.

The literatures [1, 2] give out the current situation about the research of the price reyleigh equation (0). The literature [1] gives the main economic conclusion about the equation (0).

And the study about the kinetic properties of the Price Reyleigh Equation with Finite Delay isn't quite much.

$$\ddot{x}(t)-\eta(ax^2(t)+\beta x(t)+\gamma)\dot{x}(t)+x(t-r)=0 \qquad (1)$$

Literature (9) discussed the stability of balance point and the existence of Hopf bifurcation using the τ - D partition method of exponential polynomial. Then get the calculation formula of the stability of the Hopf bifurcation direction and periodic solution choosing k as parameter as well as give the Hopf diagram in the $r-\gamma$ parameter plane completely, so "price has the hysteresis effect to supply, indicated by finite delay" can be concluded.

This text discussed the Hopf bifurcation in numerical approximation of the system (1) by choosing r as the bifurcation parameter, using the Euler method. The literature 10 to 13 took the lead in studying the Hopf bifurcation in numerical approximation of the finite delay Logistic equation and got satisfied results. What is called the numerical approximation is to examining whether its numerical solution can maintain the dynamic characteristic of the system while using the numerical method to achieve the discretization of system.

2. The existence of Hopf bifurcation for price reyleigh equation

Considering the price reyleigh equation

$$\ddot{x}(t)-\eta(ax^2(t)+\beta x(t)+\gamma)\dot{x}(t)+x(t-r)=0 \qquad (1)$$

System (1) is equivalent to the following second-order-finite-delay system

$$\begin{cases}\dot{x}(t)=y(t),\\ \dot{y}(t)=\eta\left[ax^2(t)+\beta x(t)+\gamma\right]y(t)-x(t-r),\end{cases} \qquad (2)$$

Let $\dot{x}=y$, then do the time conversion $t=rs$, and still note $x(rs),y(rs)$ as $x(t),y(t)$, therefore equation (2) can be transformed into its equivalent system

$$\begin{cases}\dot{x}(t)=ry(t)\\ \dot{y}(t)=r\eta\left[ax^2(t)+\beta x(t)+\gamma\right]y(t)-rx(t-1)\end{cases} \qquad (3)$$

Its linear part is

$$\begin{cases}\dot{x}(t)=ry(t)\\ \dot{y}(t)=-rx(t-1)+r\eta\gamma y(t)\end{cases} \qquad (4)$$

The characteristic equation of (4) is

$$\lambda^2-\eta\gamma r\lambda+r^2e^{-\lambda}=0 \qquad (5).$$

Lemma 1 Set r as a parameter, so when $r=r_0$, equation (3) exists Hopfbifurcation and r_0 satisfies following conditions:

$$\begin{cases} r_0 = -\dfrac{\eta\gamma\omega_0}{\sin\omega_0} \\ \tan\omega_0 = -\dfrac{\eta\gamma r_0}{\omega_0} \end{cases} \qquad (6)$$

(i) Equation (5) has a pair of conjugate complex roots $\lambda_{1,2} = \alpha(r) \pm i\beta(r)$, and the α, β here are real numbers, while $\alpha(r_0) = 0, \beta(r_0) = \omega_0 > 0$.

(ii) The roots of equation (5) in $r = r_0$ all have strictly negative real parts, except $\lambda(r_0), \bar{\lambda}(r_0)$.

(iii) $\alpha'(r_0) => 0$;

Proof: (i) If $\lambda = i\omega_0 (\omega > 0)$ is the root of (5), substitute λ for $i\omega_0$ in equation (5) separate the real and

Imaginary parts to get $\begin{cases} r^2 \cos\omega_0 = \omega_0{}^2 \\ r \sin\omega_0 = -\eta\gamma\omega_0 \end{cases}$ (*) .The result will be $r_0 = -\dfrac{\eta\gamma\omega_0}{\sin\omega_0}$.And ω_0 is the only solution satisfying $\tan\omega_0 = -\dfrac{\eta\gamma r_0}{\omega_0}$,which could be known from the conditions. So $\pm i\omega_0$ is the only pair of pure imaginary roots of the characteristic equation (5).

(ii) Assuming $\alpha \pm i\beta(r_0) = \alpha \pm i\omega_0$ is the root of (5), and $\alpha > 0, \beta > 0$, substitute $\lambda(r)$ for $\alpha(r) + i\beta(r)$ in equation (5), separate the real and imaginary parts to get

$$\begin{cases} h_1(\alpha, \beta) = \alpha^2 - \beta^2 - \eta\gamma r\alpha + r^2 e^{-\alpha} \cos\beta = 0 \\ h_2(\alpha, \beta) = 2\alpha\beta - \eta\gamma r\beta - r^2 e^{-\alpha} \sin\beta = 0 \end{cases}$$

The following will point out that it is impossible. On the one side, to random $\alpha > 0$,there is $r \sin\omega_0 = -\eta\gamma\omega_0$,according to (*).

$$h_2(\alpha, \omega_0) = 2\alpha\omega_0 - \eta\gamma r\omega_0 - r^2 e^{-\alpha} \sin\omega_0 = 2\alpha\omega_0 + r^2 \sin\omega_0 - r^2 e^{-\alpha} \sin\omega_0 > 0$$

$$h_{2\beta}{}'(\alpha, \beta) = 2\alpha - \eta\gamma r - r^2 e^{-\alpha} \cos\beta = 2\alpha + \frac{r^2}{\beta}(\sin\beta - \beta e^{-\alpha} \cos\beta) > 0$$

So $h_2(\alpha, \beta) = 0$ means $\beta < \omega_0$ is proved. On the other side,

$$h_1(\alpha, \omega_0) = \alpha^2 - \omega_0{}^2 - \eta\gamma r\alpha + r^2 e^{-\alpha} \cos\omega_0 = \alpha^2 - \omega_0{}^2 + r^2 \frac{\sin\omega_0}{\omega_0}\alpha + r^2 e^{-\alpha} \cos\omega_0 > 0$$

$$h_{1\beta}{}'(\alpha, \beta) = -2\beta - r^2 e^{-\alpha} \cos\beta < 0$$

So $h_1(\alpha, \beta) = 0$ means $\beta > \omega_0$ is proved, which is a contradiction. So equation (5) doesn't have complex root which contain positive real or imaginary part in $r = r_0$. Similarly, it doesn't have complex root which contain positive real part and negative imaginary part in $r = r_0$. So when $r = r_0$, the roots of characteristic equation (5) have strictly negative real parts except for a pair of pure imaginary roots $\pm i\omega_0$.

(iii) Because λ is the analytical function of r ,do derivation to equation (5) for r and noticing $\begin{cases} r^2 \cos\omega_0 = \omega_0{}^2 \\ r\sin\omega_0 = -\eta\gamma\omega_0 \end{cases}$ can get

$$\alpha'(r_0) = \frac{2\omega_0{}^4 + \eta^2\gamma^2 r_0{}^2\omega_0{}^2}{r_0\left[(-\eta\gamma r_0 - \omega_0{}^2)^2 + (2\omega_0 - \eta\gamma r_0\omega_0)^2\right]} = D(2\omega_0{}^4 + \eta^2\gamma^2 r_0{}^2\omega_0{}^2) > 0$$

In which $D = r_0\left[(-\eta\gamma r_0 - \omega_0{}^2)^2 + (2\omega_0 - \eta\gamma r_0\omega_0)^2\right]$This completes the prove.

3. Hopf Bifurcation in Numerical Approximation for Price Reyleigh Equation

Using the Euler Method [12] ($h = \frac{1}{m}, m \in Z_+$) , we get the numerical solution of equation (3)

$$\begin{cases} x_{n+1} = x_n + rhy_n \\ y_{n+1} = y_n - rhx_{n-m} + \eta\gamma rhy_n \end{cases} \qquad (7)$$

Introducing new vector $X_n = (x_n, y_n, x_{n-1}, y_{n-1}, \ldots x_{n-m}, y_{n-m})^T$, we can express (7) as

$$X_{n+1} = F(X_n, r) \qquad (8)$$

The $F(x) = (F_0, F_1, \ldots, F_m)^T$ is a vector-valued function with $2(m+1)$ dimensions, i.e.

$$F_k = \begin{cases} \begin{cases} x_n + rhy_n \\ y_n - rhx_{n-m} + \eta\gamma rhy_n \end{cases} & k = 0 \\ \begin{cases} x_n \\ y_n \end{cases} & 1 \le k \le m \end{cases}$$

Expand the equation (8) at $(0,0)$,

$$X_{n+1} = \widehat{A}X_n + \widehat{B}(X_n, X_n) + \widehat{C}(X_n, X_n, X_n) + \ldots \qquad (9)$$

Its linear part is $X_{n+1} = \widehat{A}X_n$ (10)

In which

$$\widehat{A}=\begin{bmatrix} A & 0 & \cdots & 0 & B \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}$$

I is a second order unit matrix, $A=\begin{pmatrix} 1 & rh \\ 0 & 1+\eta\gamma rh \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 \\ -rh & 0 \end{pmatrix}$

The characteristic equation of $\widehat{A}$ is

$$d_m(z,r,h)=z^{2m}(z-1)^2-\eta\gamma rhz^{2m}(z-1)+r^2h^2z^m=0 \qquad (11)$$

In order to facilitate the discussion about the bifurcation problem of the numerical solution in equation (3), we introduce equation

$$D(\mu,r,h)=\mu^2e^{2\mu}g^2(\mu h)-\eta\gamma r\mu e^{2\mu}g(\mu h)+r^2e^{\mu}=0 \qquad (12)$$

In which $g(x)=\dfrac{e^x-1}{x}$, providing $g(0)=1$

Just like the lemma 4.1 in literature [14], we can get lemma 2.

Lemma 2 if characteristic (5) satisfies condition (6), then $D(\mu,r,h)=0$ satisfies:

(i) $D(\mu,r,h)=0$ has a pair of conjugate complex roots $\mu_{1,2}=\sigma(r)\pm i\omega(r)$;

(ii) There exists $r_h=r_0+o(h)$, $\sigma(r_h)=0,\omega(r_h)\neq 0$;

(iii) $\left.\dfrac{d\sigma(r)}{dr}\right|_{r=r_h}>0$;

(iv) There exists $\varepsilon>0$ (nothing to do with r, h) to make for $h=\dfrac{1}{m}, m\in N$ .There exists

$(r,h)\in N(r_0,0)$

And $D(\mu,r,h)=0=\begin{cases} \mu=\sigma(r,h)\pm i\omega(r,h) \\ \operatorname{Re}\mu<-\varepsilon \end{cases}$.

Proof：(i-iii)Because $D(\mu, r, 0) = d(\mu, r)$, so $D(i\omega_0, r, 0) = d(i\omega_0, r)$.In $(i\omega_0, r_0, 0)$,

$\sigma'(r_0) = -\dfrac{d_r(\mu(r_0), r_0)}{d_\mu(\mu(r_0), r_0)}$, Therefore $d_\mu(i\omega_0, r_0) \neq 0$.By the implicit function theorem, in the

neighborhood of $(r_0, 0)$, there exists only one function $\sigma(r,h), \omega(r,h)$ making $\mu_{1,2} = \sigma(r) \pm i\omega(r)$.

Because $\sigma(r_0, 0) = 0, \sigma'(r_0, 0) \neq 0$, there exists $r = r_h$, making $\sigma(r_h) = 0,\ r_h = r_0 + o(h), \omega(r_h) \neq 0$, By the implicit function theorem again, in the neighborhood of $(r_0, 0)$, $\left.\dfrac{d\sigma(r)}{dr}\right|_{r=r_h} > 0$.

If $D(\mu, r, h) = 0$, then $D(\bar{\mu}, r, h) = 0$, so there exists a neighborhood of r_0, making $d(\mu, r) = 0$ hasonly one root $\mu_1(r)$, satisfying

To $r > 0$, there is $\mathrm{Re}(\mu_1(r)) > -\varepsilon, \quad \mathrm{Im}(\mu_1(r)) > 0,$ and $D(\bar{\mu}, r, h) = 0$ also has similar character.

Set $\{\mu_m, r_m, h_m\}$ to make $D(\mu_m, r_m, h_m) = 0,\ (r_m, h_m) \in N(r_0, 0),\ \lim_{m\to\infty} h_m = 0$, so $|\mu_m|$ is uniformlybounded. So there exists

m_j, to make $\mu_{m_j} \to \mu_0, r_{m_j} \to r_0, h_{m_j} \to 0$.By the continuity of $D(\mu_0, r_0, 0) = 0$, there exists $\mu_0 = i\omega_0, r_h = r_0$. So

$$D(\mu, r, h) = 0 = \begin{cases} \mu = \sigma(r,h) \pm i\omega(r,h) \\ \mathrm{Re}\,\mu < -\varepsilon \end{cases}.$$

Lemma 3 When $h = \dfrac{1}{m}$,the necessary and sufficient condition of $D(\mu, r, h) = 0$ has the root μ is (11) has the root $Z = e^{\frac{\mu}{m}}$

Proof：substitute $e^{\frac{\mu}{m}}$ for Z in（11）

$$\mu^2 e^{2\mu} g^2(\mu h) - kr\mu e^{2\mu} g(\mu h) + r^2 e^{\mu} = 0$$

So the lemma 3 is proved.

Lemma 4 $\left.\frac{d|z|}{dr}\right|_{r=r_h} \neq 0$

Proof： $Z = e^{\frac{\mu}{m}}$， $h = \frac{1}{m}$， $|z|^2 = z\overline{z}$， so there exists

$$\frac{d|z|^2}{dr} = z\frac{d\overline{z}}{dr} + \overline{z}\frac{dz}{dr} = he^{\mu h}e^{\overline{\mu}h}\frac{d\overline{\mu}}{dr} + he^{\mu h}e^{\overline{\mu}h}\frac{d\mu}{dr} = 2he^{(\mu+\overline{\mu})h}\frac{d\sigma(r,h)}{dr},$$

because $\left.\frac{d\sigma(r,h)}{dr}\right|_{r=r_h} > 0$， so $\left.\frac{d|z|}{dr}\right|_{r=r_h} > 0$.

Theorem 1 If differential equation （3） has Hopf bifurcation in $r = r_0$, so when step size h is sufficiently small, differential equation（8） will produce Hopf bifurcation in $r_h = r_0 + o(h)$.

Prove：We can learn by lemma 3 and 4 that to the step size $h = \frac{1}{m}$ $(m \geq m_0)$, in the neighborhood of r_0, if characteristic equation (5) has root, $Z = e^{\frac{\mu}{m}}$

is the root of（11）. if（5） have a pair of simple conjugate complex roots $\mu = \pm i\omega_0$, while other roots have strictly real parts. So the differential equation (8) have a pair of conjugate complex roots $e^{\pm\frac{i\omega_h}{m}}$ in $r_h = r_0 + o(h)$ $(h = \frac{1}{m})$, and $\left|e^{\pm\frac{i\omega_h}{m}}\right| = 1$, while other roots' modules less than 1, and $\left.\frac{d|z|}{dr}\right|_{r=r_h} > 0$.

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