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A Kantorovich variant of a generalized Bernstein operators

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Abstract

In this note we present a Kantorovich variant of the operators proposed by [X. Y. Chen, J. Q. Tan, Z. Liu, J. Xie, J. Math. Anal. Appl., **450** (2017), 244–261] based on non-negative parameters. Here, we prove an approximation theorem with the help of Bohman-Korovkin's principle and study the estimate of the rate of approximation by using the modulus of smoothness and Lipschitz type function for these operators. Also, we establish Voronovskaja type theorem and Korovkin type A-statistical approximation theorem of these operators.

Keywords: Global approximation, rate of convergence, modulus of continuity, A-statistical convergence, Kantorovich operators.

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1. Introduction

For $f \in C(I)$, with $I = [0,1]$, the classical Bernstein polynomials are defined as follows:

$$\mathcal{B}_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in I,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis.

Chen et al. [10] introduced a generalization of the Bernstein operators based on any fixed real α as follows:

$$T_n^{(\alpha)}(f; x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in I, \tag{1.1}$$

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where $p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right]x^{k-1}(1-x)^{n-k-1}$ and $n \geqslant 2$. They proved the rate of convergence, Voronovskaja type asymptotic formula and shape preserving properties for these operators. In the special case $\alpha = 1$ these operators reduce to the well-known Bernstein operators.

Razi [27] introduced a Bernstein-Kantorovich operator involving Pólya-Eggenberger distribution. He obtained the degree of approximation and the rate of convergence for these operators. Özarslan and Duman [26] considered a sequence of modified Bernstein-Kantorovich operators based on nonnegative parameter and studied the rate of convergence and global approximation of these operators. Abel and Heilmann [1] studied the complete asymptotic expansion of the Bernstein-Durrmeyer operators. Gonska and Paltanea [20] introduced genuine Bernstein-Durrmeyer operators involving one parameter family of linear positive operators and studied the simultaneous approximation for these operators. Cárdenas-Morales and Gupta [9] considered the two-parameter family of Bernstein-Durrmeyer type operators involving Polya distribution and obtained the approximation properties of these operators. Gupta et al. [21] proposed the Durrmeyer variant of the Baskakov operators based on inverse Pòlya-Eggenberger distribution and present the direct results. For Kantorovich-type generalizations of linear positive operators, we refer to some articles such as [2–4, 7, 8, 11, 12, 19, 22], etc..

Inspired by their work, for $f \in C(I)$, $a > 0$, and $\rho > 0$, we present the following Kantorovich type modification of the operators (1.1) as:

$$\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int_0^1 f\left(\frac{k + a\, t^\rho}{n + a}\right) dt, \quad x \in I, \qquad (1.2)$$

where $p_{n,k}^{(\alpha)}(x)$ is as given in (1.1). For the special case $a = 0$, these operators reduce to the operators (1.1).

In this article, we prove the basic approximation theorem for the operators (1.2) by using Bohman-Korovkin's theorem. We also obtain estimates of the rate of convergence involving modulus of continuity and Lipschitz function. Also, we discuss A-statistical convergence for these operators using Korovkin type statistical convergence theorem.

Let $e_i(x) = x^i, i = 0, 1, 2, \dots$.

Lemma 1.1. *For the operators $\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)$, we have*

(i) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_0;x) = 1$;

(ii) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_1;x) = \dfrac{nx}{(n+a)} + \dfrac{a}{(n+a)(1+\rho)}$;

(iii) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_2;x) = \dfrac{x^2(n^2-n+2\alpha-2)}{(n+a)^2} + \dfrac{x(n(\rho+2a+1)-2(1+\rho)(\alpha-1))}{(n+a)^2(1+\rho)} + \dfrac{a^2}{(n+a)^2(1+2\rho)}$;

(iv) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_3;x) = \dfrac{x^3(n-2)(n^2-n+6\alpha-6)}{(n+a)^3} + \dfrac{3x^2\big(a(n^2-n+2\alpha-2)+(1+\rho)(n^2+n-2n\alpha+6\alpha-6)\big)}{(n+a)^3(1+\rho)}$
$+ \dfrac{x\big(3a^2n(1+\rho)+3a(n-2\alpha+2)(1+2\rho)+(n-6\alpha+6)(1+3\rho+2\rho^2)\big)}{(n+a)^3(1+\rho)(1+2\rho)} + \dfrac{a^3}{(n+a)^3(1+3\rho)}$;

(v) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_4;x) = \dfrac{x^4(n-2)(n-3)(n(n-1)+12(\alpha-1))}{(n+a)^4}$
$+ \dfrac{2x^3(n-2)\big(2a(n^2-n+6\alpha-6)+3(n^2+n-2(n-6)\alpha-12)(1+\rho)\big)}{(n+a)^4(1+\rho)}$
$+ \dfrac{x^2\big(6a^2(n^2-n+2\alpha-2)(1+\rho)+12a(n^2+n-2n\alpha+6\alpha-6)(1+2\rho)+(n(7n-36\alpha+29)+86(\alpha-1))(1+\rho)(1+2\rho)\big)}{(n+a)^4(1+\rho)(1+2\rho)}$
$+ \dfrac{x\big(4a^3n(2\rho^2+3\rho+1)+6a^2(n-2\alpha+2)(3\rho^2+4\rho+1)+4a(n-6\alpha+6)(6\rho^2+5\rho+1)+(n-14\alpha+14)(6\rho^3+11\rho^2+6\rho+1)\big)}{(n+a)^4(1+\rho)(1+2\rho)(1+3\rho)}$
$+ \dfrac{a^4}{(n+a)^4(1+4\rho)}$.

Let $e_i^x(t) = (t-x)^i, i = 1, 2$.

Lemma 1.2. *For the operators $\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)$, we get*

(i) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_1^x(t);x) = \dfrac{-ax}{(n+a)} + \dfrac{a}{(n+a)(1+\rho)}$;

(ii) $\mathcal{K}_{n,\rho}^{\alpha,a}(e_2^x(t);x) = \dfrac{x^2(a^2+2\alpha-n-2)}{(n+a)^2} + \dfrac{x\left((1+\rho)(n-2\alpha+2)-2a^2\right)}{(n+a)^2(1+\rho)} + \dfrac{a^2}{(n+a)^2(1+2\rho)}.$

Proof. Using Lemma 1.1, the proof of this lemma readily follows. Hence, the details are missing. □

Remark 1.3. Let $\Theta_{n,\rho}^{\alpha,m,a} := \mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^m;x), m = 1,2,4$ be the central moments of $\mathcal{K}_{n,\rho}^{\alpha,a}$, we get

$$\lim_{n\to\infty} n\,\Theta_{n,\rho}^{\alpha,1,a}(x) = -ax + \dfrac{a}{(1+\rho)}, \quad \lim_{n\to\infty} n\,\Theta_{n,\rho}^{\alpha,2,a}(x) = x(1-x), \quad \lim_{n\to\infty} n^2\,\Theta_{n,\rho}^{\alpha,4,a}(x) = 3x^2(1-x)^2.$$

Lemma 1.4. *For* $f \in C(I)$, *we have*

$$\|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)\| \leqslant \|f\|.$$

Proof. Applying the definition (1.2) and Lemma 1.1, we get

$$\|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)\| \leqslant \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int_0^1 \left| f\left(\dfrac{k+a\,t^\rho}{n+a}\right) dt \right| dt \leqslant \|f\|\mathcal{K}_{n,\rho}^{\alpha,a}(e_0;x) = \|f\|.$$

□

Theorem 1.5. *Let* $f \in C(I)$. *Then* $\lim\limits_{n\to\infty} \mathcal{K}_{n,\rho}^{\alpha,a}(f;x) = f(x)$, *uniformly in* I.

Proof. Since $\mathcal{K}_{n,\rho}^{\alpha,a}(1;x) = 1$, $\mathcal{K}_{n,\rho}^{\alpha,a}(t;x) \to x$, $\mathcal{K}_{n,\rho}^{\alpha,a}(t^2;x) \to x^2$ as $n \to \infty$, uniformly in I. By Bohman-Korovkin's theorem, it follows that $\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)$ converges to $f(x)$ uniformly on I. □

2. Local approximation

We begin by recalling the first and second order modulus of smoothness of $f \in C(I)$ are defined, respectively, by

$$\omega(f,\delta) = \sup_{0<h\leqslant\delta} \sup_{x,x+h\in I} |f(x+h) - f(x)|,$$

and

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x,x+2h\in I} |f(x+2h) - 2f(x+h) + f(x)|.$$

Let us consider the following K-functional

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\| : g \in W^2\} \ (\delta > 0),$$

where $W^2 = \{g : g'' \in C(I)\}$ and $\|.\|$ is the uniform norm on $C(I)$. In [13], there exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \leqslant C\omega_2(f,\sqrt{\delta}). \tag{2.1}$$

Theorem 2.1. *For the operators* $\mathcal{K}_{n,\rho}^{\alpha,a}$, *there exists a positive constant* $C > 0$ *such that*

$$\mid \mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x) \mid \leqslant C\omega_2\left(f, \sqrt{\mu_{n,\rho}^{\alpha,a}(x)}\right) + \omega\left(f,(n+a)^{-1}\right),$$

where $\mu_{n,\rho}^{\alpha,a}(x) = \dfrac{x^2(2a^2+2\alpha-n-2)}{(n+a)^2} + \dfrac{x((1+\rho)(n+2-2\alpha)-4a^2)}{(n+a)^2(1+\rho)} + \dfrac{a^2(2+\rho(4+\rho))}{(n+a)^2(1+\rho)^2(1+2\rho)}$, $f \in C(I)$, *and* $x \in I$.

Proof. Let

$$\mathcal{T}_{n,\rho}^{\alpha,a}(f;x) = \mathcal{K}_{n,\rho}^{\alpha,a}(f;x) + f(x) - f\left(\dfrac{nx}{(n+a)} + \dfrac{a}{(n+a)(1+\rho)}\right). \tag{2.2}$$

Then, in view of Lemma 1.1, we have

$$\mathcal{T}_{n,\rho}^{\alpha,a}(1;x) = \mathcal{K}_{n,\rho}^{\alpha,a}(1;x) = 1$$

and

$$\mathcal{T}_{n,\rho}^{\alpha,a}(t;x) = \mathcal{K}_{n,\rho}^{\alpha,a}(t;x) + x - \left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) = x.$$

Let $g \in W^2$ and $t \in I$. By Taylor's formula we find

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Operating the operator $\mathcal{T}_{n,\rho}^{\alpha,a}$ on both sides of the above equation, we get

$$\mathcal{T}_{n,\rho}^{\alpha,a}(g;x) = g(x) + \mathcal{T}_{n,\rho}^{\alpha,a}\left(\int_x^t (t-u)g''(u)du; x\right)$$

$$= g(x) + \mathcal{K}_{n,\rho}^{\alpha,a}\left(\int_x^t (t-u)g''(u)du; x\right)$$

$$- \int_x^{\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}} \left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - u\right)g''(u)du.$$

Therefore,

$$\mid \mathcal{T}_{n,\rho}^{\alpha,a}(g;x) - g(x) \mid \leqslant \mathcal{K}_{n,\rho}^{\alpha,a}\left(\left|\int_x^t |t-u||g''(u)|du\right|; x\right)$$

$$+ \left|\int_x^{\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}} \left|\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - u\right||g''(u)|du\right|$$

$$\leqslant \left\{\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2;x) + \left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - x\right)^2\right\}\|g''\| \tag{2.3}$$

$$= \left\{\frac{x^2(2a^2 + 2\alpha - n - 2)}{(n+a)^2} + \frac{x((1+\rho)(n+2-2\alpha) - 4a^2)}{(n+a)^2(1+\rho)}\right.$$

$$\left. + \frac{a^2(2+\rho(4+\rho))}{(n+a)^2(1+\rho)^2(1+2\rho)}\right\}\|g''\|$$

$$= \mu_{n,\rho}^{\alpha,a}(x)\|g''\|.$$

Using Lemma 1.4, we get

$$\mid \mathcal{T}_{n,\rho}^{\alpha,a}(f;x) \mid \leqslant |\mathcal{K}_{n,\rho}^{\alpha,a}(f;x)| + |f(x)| + \left|f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right)\right| \leqslant 3\|f\|, \quad \forall f \in C(I). \tag{2.4}$$

Now, for $g \in W^2$ and $f \in C(I)$, using (2.3) and (2.4), we get

$$\mid \mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x) \mid \leqslant \left|\mathcal{T}_{n,\rho}^{\alpha,a}(f;x) - f(x) + f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x)\right|$$

$$\leqslant |\mathcal{T}_{n,\rho}^{\alpha,a}(f-g;x)| + |\mathcal{T}_{n,\rho}^{\alpha,a}(g;x) - g(x)| + |g(x) - f(x)|$$

$$+ \left|f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x)\right|$$

$$\leqslant 4\|f-g\| + \mu_{n,\rho}^{\alpha,a}(x)\|g''\| + \omega\left(f, (n+a)^{-1}\right).$$

Taking the infimum on the right hand side over all $g \in W^2$, we have

$$| \mathcal{K}^{\alpha,a}_{n,\rho}(f;x) - f(x) | \leqslant 4K_2 \left(f, \mu^{\alpha,a}_{n,\rho}(x) \right) + \omega \left(f, (n+a)^{-1} \right),$$

and using the relation (2.1), we get

$$| \mathcal{K}^{\alpha,a}_{n,\rho}(f;x) - f(x) | \leqslant M\omega_2 \left(f, \sqrt{\mu^{\alpha,a}_{n,\rho}(x)} \right) + \omega \left(f, (n+a)^{-1} \right),$$

hence the theorem is proved . $\qquad\square$

Let us consider the Lipschitz-type space with two parameters [25]. For $\beta_1 \geqslant 0, \beta_2 > 0$, we define

$$\mathrm{Lip}^{(\beta_1,\beta_2)}_M(\eta) := \left\{ f \in C(I) : |f(t) - f(x)| \leqslant M \frac{|t-x|^\eta}{(t+\beta_1 x^2 + \beta_2 x)^{\frac{\eta}{2}}} ; t \in I, x \in (0,1] \right\},$$

where $0 < \eta \leqslant 1$.

Theorem 2.2. *Let* $f \in \mathrm{Lip}^{(\beta_1,\beta_2)}_M(\eta)$. *Then for all* $x \in (0,1]$, *we have*

$$\left| \mathcal{K}^{\alpha,a}_{n,\rho}(f;x) - f(x) \right| \leqslant M \left(\frac{\Theta^{\alpha,2,a}_{n,\rho}(x)}{\beta_1 x^2 + \beta_2 x} \right)^{\eta/2},$$

where $\Theta^{\alpha,2,a}_{n,\rho}(x) = \mathcal{K}^{\alpha,a}_{n,\rho}((t-x)^2;x)$.

Proof. Let we prove the theorem for the case $0 < \eta \leqslant 1$, applying Holder's inequality with $p = \frac{2}{\eta}, q = \frac{2}{2-\eta}$,

$$\begin{aligned}
\left| \mathcal{K}^{\alpha,a}_{n,\rho}(f;x) - f(x) \right| &\leqslant \sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \int_0^1 \left| f \left(\frac{k+a\, t^\rho}{n+a} \right) - f(x) \right| dt \\
&\leqslant \sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \left(\int_0^1 \left| f \left(\frac{k+a\, t^\rho}{n+a} \right) - f(x) \right|^{\frac{2}{\eta}} dt \right)^{\frac{\eta}{2}} \\
&\leqslant \left\{ \sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \int_0^1 \left| f \left(\frac{k+a\, t^\rho}{n+a} \right) - f(x) \right|^{\frac{2}{\eta}} dt \right\}^{\frac{\eta}{2}} \left(\sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \right)^{\frac{2-\eta}{2}} \\
&= \left(\sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \int_0^1 \left| f \left(\frac{k+a\, t^\rho}{n+a} \right) - f(x) \right|^{\frac{2}{\eta}} dt \right)^{\frac{\eta}{2}} \\
&\leqslant M \left(\sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \int_0^1 \frac{\left(\frac{k+a\, t^\rho}{n+a} - x \right)^2}{\left(\frac{k+a\, t^\rho}{n+a} + \beta_1 x^2 + \beta_2 x \right)} dt \right)^{\frac{\eta}{2}} \\
&\leqslant \frac{M}{(\beta_1 x^2 + \beta_2 x)^{\frac{\eta}{2}}} \left(\sum_{k=0}^n p^{(\alpha)}_{n,k}(x) \int_0^1 \left(\frac{k+a\, t^\rho}{n+a} - x \right)^2 dt \right)^{\frac{\eta}{2}} \\
&= \frac{M}{(\beta_1 x^2 + \beta_2 x)^{\frac{\eta}{2}}} \mathcal{K}^{\alpha,a}_{n,\rho}((t-x)^2;x)^{\frac{\eta}{2}} = \frac{M}{(\beta_1 x^2 + \beta_2 x)^{\frac{\eta}{2}}} (\Theta^{\alpha,2,a}_{n,\rho}(x))^{\frac{\eta}{2}}.
\end{aligned}$$

$\square$

Theorem 2.3. *Let* $f \in C(I)$. *Then,*

$$\left| \mathcal{K}^{\alpha,a}_{n,\rho}(f;x) - f(x) \right| \leqslant 2\omega \left(f; \sqrt{\Theta^{\alpha,2,a}_{n,\rho}(x)} \right).$$

Proof. Using the following well-known property of modulus of continuity,

$$|f(t) - f(x)| \leqslant \omega(f, \delta) \left(\frac{(t-x)^2}{\delta^2} + 1 \right), \quad \text{for any } \delta > 0,$$

we get

$$\left| \mathcal{K}_{n,\rho}^{\alpha,a}(f; x) - f(x) \right| \leqslant \mathcal{K}_{n,\rho}^{\alpha,a}(|f(t) - f(x)|; x) \leqslant \left(1 + \frac{1}{\delta^2} \mathcal{K}_{n,\rho}^{\alpha,a}\left((t-x)^2; x \right) \right) \omega(f, \delta).$$

Applying Lemma 1.2, it follows that $\lim_{n\to\infty} \mathcal{K}_{n,\rho}^{\alpha,a}\left((t-x)^2 \right); x) = 0$, hence choosing $\delta = \sqrt{\Theta_{n,\rho}^{\alpha,2,a}(x)}$, we obtain the result. $\square$

Theorem 2.4. *For any* $f \in C^1(I)$ *and* $x \in [0, 1]$, *we have*

$$\left| \mathcal{K}_{n,\rho}^{\alpha,a}(f; x) - f(x) \right| \leqslant \left| \frac{-ax}{(n+a)} + \frac{a}{(n+a)(1+\rho)} \right| |f'(x)| + 2\sqrt{\Theta_{n,\rho}^{\alpha,2,a}(x)} \omega\left(f', \sqrt{\Theta_{n,\rho}^{\alpha,2,a}(x)} \right). \qquad (2.5)$$

Proof. Let $f \in C^1(I)$. For any $t \in [0, 1], x \in [0, 1]$, we have

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t \left(f'(u) - f'(x) \right) du.$$

Using $\mathcal{K}_{n,\rho}^{\alpha,a}(\cdot; x)$ on both sides of the above equation, we have

$$\mathcal{K}_{n,\rho}^{\alpha,a}(f(t) - f(x); q_n, x) = f'(x)\mathcal{K}_{n,\rho}^{\alpha,a}(t - x; x) + \mathcal{K}_{n,\rho}^{\alpha,a}\left(\int_x^t \left(f'(u) - f'(x) \right) du; x \right).$$

Using the well-known property of modulus of continuity $|f(t) - f(x)| \leqslant \omega(f, \delta) \left(\frac{|t-x|}{\delta} + 1 \right), \delta > 0$, we obtain

$$\left| \int_x^t \left(f'(u) - f'(x) \right) du \right| \leqslant \omega(f', \delta) \left(\frac{(t-x)^2}{\delta} + |t - x| \right),$$

it follows that

$$\left| \mathcal{K}_{n,\rho}^{\alpha,a}(f; x) - f(x) \right| \leqslant |f'(x)| \ \left| \mathcal{K}_{n,\rho}^{\alpha,a}(t - x; x) \right| + \omega(f', \delta) \left\{ \frac{1}{\delta} \mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2; x) + \mathcal{K}_{n,\rho}^{\alpha,a}(|t - x|; x) \right\}.$$

From Cauchy-Schwarz inequality, we have

$$\left| \mathcal{K}_{n,\rho}^{\alpha,a}(f; x) - f(x) \right| \leqslant |f'(x)| \left| \mathcal{K}_{n,\rho}^{\alpha,a}(t - x; x) \right| + \omega(f', \delta) \left\{ \frac{1}{\delta} \sqrt{\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2; x)} + 1 \right\} \sqrt{\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2; x)}.$$

Now, taking $\delta = \sqrt{\Theta_{n,\rho}^{\alpha,2,a}(x)}$, we obtain (2.5). $\square$

3. Global approximation

For $f \in C(I)$, the first and second order Ditzian-Totik moduli of smoothness are defined by

$$\omega_\psi(f, \delta) = \sup_{0<|h|\leqslant\delta} \sup_{x, x+h\psi(x)\in I} |f(x + \psi(x)h) - f(x)|$$

and

$$\omega_{2,\phi}(f, \sqrt{\delta}) = \sup_{0<|h|\leqslant\sqrt{\delta}} \sup_{x, x\pm h\phi(x)\in I} |f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x))|,$$

respectively and the corresponding K-functional is

$$K_{2,\phi}(f,\delta) = \inf\{\|f - g\| + \delta\|\phi^2 g''\| : g \in W^2(\phi)\}, \quad \delta > 0,$$

where $W^2(\phi) = \{g \in C(I) : g' \in AC[0,1], \phi^2 g'' \in C(I)\}$ and $g' \in AC[0,1]$ means that g' is absolutely continuous on I. It is well known (see [14]) $K_{2,\phi}(f,\delta) \sim \omega_{2,\phi}(f,\sqrt{\delta})$ which means that there exists an absolute constant $C > 0$ such that

$$C^{-1}\omega_{2,\phi}(f,\sqrt{\delta}) \leqslant K_{2,\phi}(f,\delta) \leqslant C\omega_{2,\phi}(f,\sqrt{\delta}), \tag{3.1}$$

where ϕ is an admissible step-weight function on I.

Theorem 3.1. *Let* $f \in C(I)$ *and* $x \in (0,1)$. *Then, there is an absolute constant* $C > 0$ *such that*

$$|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)| \leqslant C\omega_{2,\phi}\left(f, \frac{\sqrt{\mu_{n,\rho}^{\alpha,a}(x)}}{\phi(x)}\right) + \omega_{\psi_{\rho,a}}\left(f, (n+a)^{-1}\right),$$

where $\psi_{\rho,a}(x) = a(x(1+\rho)+1)$ *and* $\phi^2(x) = x(1-x)$.

Proof. Operating the operators $\mathcal{T}_{n,\rho}^{\alpha,a}$ defined in (2.2) to Taylor's expansion in a similar manner with the proof of Theorem 2.1, we get

$$|\mathcal{T}_{n,\rho}^{\alpha,a}(g;x) - g(x)| \leqslant \mathcal{K}_{n,\rho}^{\alpha,a}\left(\left|\int_x^t |t-u|\,|g''(u)|du\right|;x\right)$$

$$+ \left|\int_x^{\frac{nx}{(n+a)}+\frac{a}{(n+a)(1+\rho)}} \left|\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - u\right| |g''(u)|\,du\right|$$

$$\leqslant \|\phi^2 g''\|\mathcal{K}_{n,\rho}^{\alpha,a}\left(\left|\int_x^t \frac{|t-u|}{\phi^2(u)}du\right|;x\right)$$

$$+ \|\phi^2 g''\|\left|\int_x^{\frac{nx}{(n+a)}+\frac{a}{(n+a)(1+\rho)}} \frac{\left|\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - u\right|}{\phi^2(u)}du\right|.$$

Let $u = \beta x + (1-\beta)t, \beta \in [0,1]$. Since $\phi^2(x)$ is concave on I, it follows that $\phi^2(u) \geqslant \beta\phi^2(x) + (1-\beta)\phi^2(t)$ and hence

$$\frac{|t-u|}{\phi^2(u)} = \frac{\beta|x-t|}{\phi^2(u)} \leqslant \frac{\beta|x-t|}{\beta\phi^2(x) + (1-\beta)\phi^2(t)} \leqslant \frac{|x-t|}{\phi^2(x)}.$$

Thus,

$$|\mathcal{T}_{n,\rho}^{\alpha,a}(g;x) - g(x)| \leqslant \frac{\|\phi^2 g''\|}{\phi^2(x)}\left[\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2;x) + \left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - x\right)^2\right] \leqslant \frac{\|\phi^2 g''\|}{\phi^2(x)}\mu_{n,\rho}^{\alpha,a}(x).$$

Now,

$$|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)| \leqslant |\mathcal{T}_{n,\rho}^{\alpha,a}(f-g;x)| + |\mathcal{T}_{n,\rho}^{\alpha,a}(g;x) - g(x)| + |f(x) - g(x)|$$

$$+ \left|f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x)\right|$$

$$\leqslant 4\|f - g\| + \frac{\|\phi^2 g''\|}{\phi^2(x)}\mu_{n,\rho}^{\alpha,a}(x) + \left|f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x)\right|.$$

Taking infimum on the right hand side of the above inequality over all $g \in W^2(\phi)$, we get

$$|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)| \leqslant 4K_{2,\phi}\left(f, \frac{\mu_{n,\rho}^{\alpha,a}(x)}{4\phi^2(x)}\right) + \left|f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x)\right|.$$

Also,

$$\left| f\left(\frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)}\right) - f(x) \right| = \left| f\left(x + \psi_{\rho,a}(x)\frac{\mathcal{K}_{n,\rho}^{\alpha,a}((t-x);x)}{\psi_{\rho,a}(x)}\right) - f(x) \right|$$

$$\leqslant \omega_{\psi_{\rho,a}}\left(f, \frac{1}{\psi_{\rho,a}(x)}\left|\mathcal{K}_{n,\rho}^{\alpha,a}((t-x);x)\right|\right) \leqslant \omega_{\psi_{\rho,a}}\left(f, (n+a)^{-1}\right).$$

Thus,

$$\left|\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)\right| \leqslant 4K_{2,\phi}\left(f, \frac{\mu_{n,\rho}^{\alpha,a}(x)}{4\phi^2(x)}\right) + \omega_{\psi_{\rho,a}}\left(f, (n+a)^{-1}\right).$$

Now, applying (3.1), we obtain the required result. □

4. Voronovskaja type theorem

In this section we establish Voronvoskaja type result for the $\mathcal{K}_{n,\rho}^{\alpha,a}$ operators.

Theorem 4.1. *Let* $f \in C(I)$. *If* f'' *exists at a point* $x \in I$, *then we have*

$$\lim_{n\to\infty} n\left[\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)\right] = \left(-ax + \frac{a}{(1+\rho)}\right)f'(x) + \frac{1}{2}x(1-x)f''(x).$$

Proof. Applying Taylor's expansion, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\lim_{t\to x} \varepsilon(t,x) = 0$. By using the linearity of the operator $\mathcal{K}_{n,\rho}^{\alpha,a}$, we get

$$\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x) = \mathcal{K}_{n,\rho}^{\alpha,a}((t-x);x)f'(x) + \frac{1}{2}\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^2;x)f''(x) + \mathcal{K}_{n,\rho}^{\alpha,a}(\varepsilon(t,x)(t-x)^2;x).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$n\mathcal{K}_{n,\rho}^{\alpha,a}(\varepsilon(t,x)(t-x)^2;x) \leqslant \sqrt{\mathcal{K}_{n,\rho}^{\alpha,a}(\varepsilon^2(t,x);x)}\sqrt{n^2\mathcal{K}_{n,\rho}^{\alpha,a}((t-x)^4;x)}. \tag{4.1}$$

In view of Theorem 1.5, we get

$$\lim_{n\to\infty} \mathcal{K}_{n,\rho}^{\alpha,a}(\varepsilon^2(t,x);x) = \varepsilon^2(x,x) = 0. \tag{4.2}$$

Combining (4.1)-(4.2) and Remark 1.3, we have

$$\lim_{n\to\infty} n\mathcal{K}_{n,\rho}^{\alpha,a}(\varepsilon(t,x)(t-x)^2;x) = 0.$$

Hence

$$\lim_{n\to\infty} n\left[\mathcal{K}_{n,\rho}^{\alpha,a}(f;x) - f(x)\right] = \left(-ax + \frac{a}{(1+\rho)}\right)f'(x) + \frac{1}{2}x(1-x)f''(x).$$

Thus the theorem follows immediately by using Remark 1.3. □

5. A-statistical approximation of Korovkin type

Let $A = (a_{nk})$ be a non-negative infinite summability matrix. For a given sequence $x := (x)_n$, the A-transform of x denoted by $Ax : (Ax)_n$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$$

provided the series converges for each n. A is called to be regular if $\lim_n(Ax)_n = L$ whenever $\lim_n(x)_n = L$.

Then $x = (x)_n$ is said to be A-statistically convergent to L, i.e., $st_A - \lim_n (x)_n = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geqslant \epsilon} a_{nk} = 0$. If we replace A by C_1 then A is a Cesaro matrix of order one and A-statistical convergence is reduced to the statistical convergence. Similarly, if $A = I$, the identity matrix then A-statistical convergence is called ordinary convergence. Many researchers have examined the statistical convergence properties for several sequences and classes of positive linear operators (cf. [5, 6, 15–17, 23, 24]).

Gadjiev and Orhan [18] proved the following Bohman-Korovkin type approximation theorem for statistical convergence:

Theorem 5.1 ([18]). *If the sequence of positive linear operators* $\mathcal{P}_n : C[c, d] \to C[c, d]$ *satisfy the conditions* $st - \lim_n \|\mathcal{P}_n(e_i) - e_i\|_{C[c,d]} = 0$ *with* $e_i(t) = t^i, i = 0, 1, 2$, *then for any function* $f \in C[c, d]$, *we have*

$$st - \lim_n \|\mathcal{P}_n(f) - f\|_{C[c,d]} = 0.$$

It is to be observe that this result works well for a A-statistical convergence also.

Theorem 5.2. *Let* (a_{nk}) *be a non-negative regular summability matrix. For any* $f \in C(I)$, *we have*

$$st_A - \lim_n \| \mathcal{K}_{n,\rho}^{\alpha,a}(f) - f \|_{C(I)} = 0.$$

Proof. Let us consider $e_i(x) = x^i, i = 0, 1, 2$. It is sufficient to show that $st_A - \lim_n \|\mathcal{K}_{n,\rho}^{\alpha,a}(e_i; \cdot) - e_i\|_{C(I)} = 0$, for $i = 0, 1, 2$. It is clear that

$$st_A - \lim_n \|\mathcal{K}_{n,\rho}^{\alpha,a}(e_0; \cdot) - e_0\|_{C(I)} = 0.$$

From Lemma 1.1, we have

$$st_A - \lim_n \|\mathcal{K}_{n,\rho}^{\alpha,a}(e_1; \cdot) - e_1\|_{C(I)} = \left| \frac{nx}{(n+a)} + \frac{a}{(n+a)(1+\rho)} - x \right| \leqslant \left| \frac{-a}{(n+a)} \right| + \frac{a}{(n+a)(1+\rho)}.$$

For $\epsilon > 0$, let us define the following sets:

$$E := \left\{ n \in \mathbb{N} : \| \mathcal{K}_{n,\rho}^{\alpha,a}(e_1; \cdot) - e_1 \| \geqslant \epsilon \right\},$$

$$E_1 := \left\{ n \in \mathbb{N} : \left| \frac{-a}{(n+a)} \right| \geqslant \frac{\epsilon}{2} \right\},$$

$$E_2 := \left\{ n \in \mathbb{N} : \frac{a}{(n+a)(1+\rho)} \geqslant \frac{\epsilon}{2} \right\}.$$

Then, we obtain $E \subseteq E_1$ which implies that $\sum_{k \in E} a_{nk} \leqslant \sum_{k \in E_1} a_{nk} + \sum_{k \in E_2} a_{nk}$ and hence

$$st_A - \lim_n \|\mathcal{K}_{n,\rho}^{\alpha,a}(e_1; \cdot) - e_1\|_{C(I)} = 0.$$

In the similar manner it can be shown

$$st_A - \lim_n \|\mathcal{K}_{n,\rho}^{\alpha,a}(e_2; \cdot) - e_2\|_{C(I)}$$

$$\leqslant \left| \frac{x^2(n^2 - n + 2\alpha - 2)}{(n+a)^2} + \frac{x(n(\rho + 2a + 1) - 2(1+\rho)(\alpha - 1))}{(n+a)^2(1+\rho)} + \frac{a^2}{(n+a)^2(1+2\rho)} - x^2 \right|$$

$$\leqslant \left| \frac{a^2 + n + 2an - 2\alpha + 2}{(n+a)^2} \right| + \left| \frac{(n(\rho + 2a + 1) - 2(1+\rho)(\alpha - 1))}{(n+a)^2(1+\rho)} \right| + \frac{a^2}{(n+a)^2(1+2\rho)}.$$

For $\epsilon > 0$, let us define the following sets:

$$F := \left\{ n \in \mathbb{N} : \parallel \mathcal{K}^{\alpha,a}_{n,\rho}(e_2; \cdot) - e_2 \parallel \geqslant \epsilon \right\},$$
$$F_1 := \left\{ n \in \mathbb{N} : \left| \frac{a^2 + n + 2an - 2\alpha + 2}{(n+a)^2} \right| \geqslant \frac{\epsilon}{3} \right\},$$
$$F_2 := \left\{ n \in \mathbb{N} : \left| \frac{(n(\rho + 2a + 1) - 2(1+\rho)(\alpha-1))}{(n+a)^2(1+\rho)} \right| \geqslant \frac{\epsilon}{3} \right\},$$
$$F_3 := \left\{ n \in \mathbb{N} : \frac{a^2}{(n+a)^2(1+2\rho)} \geqslant \frac{\epsilon}{3} \right\}.$$

Then, we obtain $F \subseteq F_1 \cup F_2$ which implies that $\sum_{k \in F} a_{nk} \leqslant \sum_{k \in F_1} a_{nk} + \sum_{k \in F_2} a_{nk} + \sum_{k \in F_3} a_{nk}$ and hence

$$st_A - \lim_n \| \mathcal{K}^{\alpha,a}_{n,\rho}(e_2; \cdot) - e_2 \|_{C(I)} = 0.$$

This completes the proof of the theorem. $\qquad\square$

6. Conclusion

Motivated by Chen et al. [10], we gave a Kantorovich variant of the operators based on non-negative parameters. From here, we proved an approximation theorem with the help of Bohman-Korovkin's principle and studied the estimate of the rate of approximation by making use of the modulus of smoothness and Lipschitz type function for these operators. Finally we established Voronovskaja type theorem and Korovkin type A-statistical approximation theorem of these operators.

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