



Some unified formulas involving generalized-Apostol-type-Gould-Hopper polynomials and multiple power sums



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Abstract

The article aims to introduce a new unified class of generalized Apostol type polynomials. Further, under a slight modification on the parameters associated to the generalized Apostol type and generalized Gould-Hopper polynomials and by the use of generating method, we introduce a new class of generalized-Apostol-type-Gould-Hopper polynomials. We state some quasi-monomial properties for a new class of extensions of generalized Apostol type polynomials as well as, some summation, multiplication and explicit formulae which connect this polynomial class with the λ -Stirling numbers of second kind and generalized Hurwitz zeta function. Some general symmetry identities involving multiple power sums are also established. The new class of polynomials contains as its special cases, not only the generalized-Gould-Hopper-Apostol-Bernoulli, Euler and Genocchi polynomials but many more known smaller classes of polynomials. Finally, these polynomials are framed within the context of generalized modified Milne-Thomson's polynomials.

Keywords: Generalized-Apostol-type-Gould-Hopper polynomials, monomiality principle, summation formula, multiplication formulae, symmetry identities.

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1. Introduction and preliminaries

The theory of generalized special functions serves as an analytic foundation for the majority of problems in mathematical physics that have been solved exactly and finds broad practical applications. As an example, the generalized Hermite polynomials have been exploited to deal with quantum mechanical and optical beam transport problems and arise in problems ranging from the theory of partial differential equations to the abstract group theory [8, 9]. We recall the following 2-variable forms of Hermite polynomials.

Definition 1.1. The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ [1] are defined

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by the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}.$$

Definition 1.2. The higher order Hermite polynomials or the Gould-Hopper polynomials (GHP) $H_n^{(j)}(x, y)$ are defined by the following generating function [11, p.58(6.3)]

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x, y) \frac{t^n}{n!}, \quad j > 2.$$

Definition 1.3. The generalized-Hermite polynomials (GHP) $H_n(x, y, c)$ are defined by the following generating function [37]:

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y, c) \frac{t^n}{n!}, \quad c > 1.$$

Definition 1.4. The generalized-Gould-Hopper polynomials (GGdHP) $H_n^{(j)}(x, y, c)$ are defined by the following generating function [41]:

$$c^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x, y, c) \frac{t^n}{n!}, \quad c > 1; \quad j > 2. \quad (1.1)$$

These polynomials play an important role in problems involving Laplace's equation in parabolic coordinates, in quantum mechanics and in probability theory and are shown to be solutions of classical and generalized heat equations.

Operational methods provide powerful techniques to solve problems both in classical and quantum mechanics. Differential equations have been the primary motivation for the introduction of these techniques. The operational techniques combined with the monomiality principle [5] open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of special polynomials.

Definition 1.5. According to the monomiality principle, there exist two operators \hat{M} and \hat{P} playing, respectively, the role of multiplicative and derivative operators for a polynomial set $\{p_n(x)\}_{n \in \mathbb{N}}$, that is, \hat{M} and \hat{P} satisfy the following identities, for all $n \in \mathbb{N}$:

$$\hat{M}\{p_n(x)\} = p_{n+1}(x), \quad \hat{P}\{p_n(x)\} = n p_{n-1}(x). \quad (1.2)$$

The polynomial set $\{p_n(x)\}_{n \in \mathbb{N}}$ is then called a quasi-monomial. If \hat{M} and \hat{P} have differential realizations, then the polynomial set $\{p_n(x)\}_{n \in \mathbb{N}}$ satisfy the following differential equation:

$$\hat{M}\hat{P}\{p_n(x)\} = n p_n(x).$$

Most of the properties of families of polynomials, recognized as quasi-monomial, can be deduced by using relevant multiplicative and derivative operators. The notion of quasi-monomiality has been exploited within different context. There is a continuous demand of solving problems by means of differential equations in research fields like classical and quantum optics. The aforementioned 2-variable forms of Hermite polynomials are all quasi-monomials and corresponding properties are derived using the rules associated with monomiality principle, for this see [6, 37, 41].

Various unified forms of Apostol-type polynomials, which are the unifications of the Apostol-Bernoulli, Euler and Genocchi polynomials are introduced and studied by many authors in a systematic manner, for this see [2, 3, 13, 19, 23, 24, 26, 28, 29, 34, 35, 38].

We recall the following generalized forms of the Apostol-Bernoulli, Euler and Genocchi polynomials introduced by many authors [14, 30, 31, 39].

Definition 1.6. Let $a, b, c \in \mathbb{R}^+$ with $a \neq b$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating function [39]:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; a, b, c; \lambda) \frac{t^n}{n!}, \quad |t| < \left| \frac{\ln(\lambda)}{\ln\left(\frac{a}{b}\right)} \right|. \quad (1.3)$$

Definition 1.7. Let $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating function:

$$\left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; a, b, c; \lambda) \frac{t^n}{n!}, \quad |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|.$$

Definition 1.8. Let $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. The generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating function [14]:

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x; a, b, c; \lambda) \frac{t^n}{n!}, \quad |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|. \quad (1.4)$$

We note that

$$\mathfrak{B}_n^{(\alpha)}(0; a, b, c; \lambda) := \mathfrak{B}_n^{(\alpha)}(a, b, c; \lambda), \quad \mathfrak{E}_n^{(\alpha)}(0; a, b, c; \lambda) := \mathfrak{E}_n^{(\alpha)}(a, b, c; \lambda), \quad \mathfrak{G}_n^{(\alpha)}(0; a, b, c; \lambda) := \mathfrak{G}_n^{(\alpha)}(a, b, c; \lambda), \quad (1.5)$$

are the generalized Apostol-Bernoulli, Euler and Genocchi numbers, each of order α .

For $b = c = e$, $a = 1$, we have Apostol-Bernoulli, Euler and Genocchi polynomials, each of order α , for this see [25, 27, 32].

Here, we introduce the following unified form of the generalized Apostol-Bernoulli, Euler and Genocchi polynomials.

Definition 1.9. Let $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$ and $n \in \mathbb{N}_0$. The generalized Apostol type polynomials (GATP) $\mathcal{F}_n^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating function:

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|, \quad (1.6)$$

where

$$\mathcal{F}_n^{(\alpha)}(0; a, b, c; \mu, \nu; \lambda) := \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda)$$

denotes the generalized Apostol type numbers of order α defined by

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}. \quad (1.7)$$

For $b = c = e$, $a = 1$, we have Luo and Srivastava [34] Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x; \lambda; \mu, \nu)$. In fact, from equations (1.4)-(1.6) and (1.7), we have

$$\begin{aligned} (-1)^\alpha \mathcal{F}_n^{(\alpha)}(x; a, b, c; 0, 1; -\lambda) &= \mathfrak{B}_n^{(\alpha)}(x; a, b, c; \lambda), \\ \mathcal{F}_n^{(\alpha)}(x; a, b, c; 1, 0; \lambda) &= \mathfrak{E}_n^{(\alpha)}(x; a, b, c; \lambda), \\ \mathcal{F}_n^{(\alpha)}(x; a, b, c; 1, 1; \lambda) &= \mathfrak{G}_n^{(\alpha)}(x; a, b, c; \lambda). \end{aligned} \quad (1.8)$$

To introduce the hybridized forms of generalized Apostol type polynomials and to study their properties is a new investigation. The process of combining two multi-variable forms of special polynomials

either by replacement technique or operational technique is called hybridization of polynomials, which shows that the properties of new hybridized polynomial lie within the context of parent polynomials. The multi-variable hybrid special polynomials are important as they possess important properties such as recurrence and explicit relations and functional and differential equations, summation formulae, symmetric and convolution identities etc.. The usefulness and potential for applications of various properties of multi-variable hybrid special polynomials in certain problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics arose the desire to introduce a new hybrid class of generalized-Apostol-type-Gould-Hopper polynomials.

In this article, a hybrid class of generalized-Apostol-type-Gould-Hopper polynomials is introduced and their properties are being characterized via different generating functions method. The differential operators, differential equations, summation, multiplication and explicit formula and symmetry identities for these polynomials are established. Certain illustrative special cases are considered and analogous properties for these special cases are deduced.

2. Generalized-Apostol-type-Gould-Hopper polynomials

In this section, a hybrid class of generalized-Apostol-type-Gould-Hopper polynomials (GATGdHP) is introduced by convoluting GGdHP and GATP. The GGdHATP are defined as discrete generalized Gould-Hopper convolution of generalized-Apostol type polynomials.

First, we introduce the generating function applying replacement technique for the generalized-Apostol-type-Gould-Hopper polynomials. For this, the following result is proved.

Theorem 2.1. *Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$ and $n \in \mathbb{N}_0$. The generalized-Apostol-type-Gould-Hopper polynomials are defined by the following generating function:*

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad c > 1; j > 2; |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|. \quad (2.1)$$

Proof. Expanding the exponential function $e^{(x \log c)t}$ and then replacing the powers of x , i.e., $(x \log c)^0, (x \log c)^1, (x \log c)^2, \dots, (x \log c)^n$ by the polynomials $H_0^{(j)}(x, y, c), H_1^{(j)}(x, y, c), \dots, H_n^{(j)}(x, y, c)$ in the l.h.s. and x by the polynomial $H_1^{(j)}(x, y, c)$ in the r.h.s. of equation (1.7) and after summing up the terms in the l.h.s. of the resultant equation, we have

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha \sum_{n=0}^{\infty} H_n^{(j)}(x, y, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}((H_1^{(j)}(x, y, c); a, b, c; \mu, \nu; \lambda)) \frac{t^n}{n!},$$

which on using equation (1.1) in the l.h.s. and denoting the resultant GATGdHP in the r.h.s. by ${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda)$, we are led to assertion (2.1). \square

Corollary 2.2. *Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$; $n \in \mathbb{N}_0$ and $j = 2$. The generalized-Apostol-type-Gould-Hopper polynomials are defined by the following generating function:*

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad c > 1; |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|. \quad (2.2)$$

Note 1. For $b = c = e$, $a = 1$, equations (2.1) and (2.2) yield the Apostol-type-Gould-Hopper polynomials ${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; \mu, \nu; \lambda)$ and Apostol-type-Hermite polynomials ${}_H\mathcal{F}_n^{(\alpha)}(x, y; \mu, \nu; \lambda)$, which for $x \rightarrow 2x$, $y = -1$ becomes the Apostol-type-Hermite polynomials ${}_H\mathcal{F}_n^{(\alpha)}(x; \mu, \nu; \lambda)$, for these polynomials see [19].

Remark 2.3. For the special case $\lambda \rightarrow -\lambda$; $\mu = 0$ and $\nu = 1$ and on use of $(-1)^\alpha {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; 0, 1; \lambda) = {}_H\mathcal{B}_n^{(\alpha,j)}(x, y; a, b, c; \lambda)$, we have generalized-Gould-Hopper-Apostol-Bernoulli polynomials defined by

$$\left(\frac{t}{\lambda b^t - a^t}\right)^\alpha c^{xt+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad c > 1; j > 2; |t| < \left| \frac{\ln(\lambda)}{\ln\left(\frac{a}{b}\right)} \right|.$$

Remark 2.4. For the special case $\mu = 1$ and $\nu = 0$ and on use of ${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; 1, 0; \lambda) = {}_H\mathcal{E}_n^{(\alpha,j)}(x, y; a, b, c; \lambda)$, we have generalized-Gould-Hopper-Apostol-Euler polynomials defined by

$$\left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{E}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad c > 1; j > 2; |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|.$$

Remark 2.5. For the special case $\mu = 1$ and $\nu = 1$ and on use of ${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; 1, 1; \lambda) = {}_H\mathcal{G}_n^{(\alpha,j)}(x, y; a, b, c; \lambda)$, we have generalized-Gould-Hopper-Apostol-Genocchi polynomials defined by

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha c^{xt+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \quad c > 1; j > 2; |t| < \left| \frac{\ln(-\lambda)}{\ln\left(\frac{a}{b}\right)} \right|.$$

Note 2. For $b = c = e$, $a = 1$, Remarks 2.3-2.5 give the Gould-Hopper-Apostol-Bernoulli, Euler and Genocchi polynomials, each of order α , which for $\lambda = 1$ yield Gould-Hopper-Bernoulli, Euler and Genocchi polynomials, each of order α , which again for $\alpha = 1$ yield Gould-Hopper-Bernoulli, Euler and Genocchi polynomials [17, 19].

Note 3. For $j = 2$, Remarks 2.3-2.5 give the generalized-Hermite-Apostol-Bernoulli, Euler and Genocchi polynomials, each of order α , which for $b = c = e$, $a = 1$ yield Hermite-Apostol-Bernoulli, Euler and Genocchi polynomials, each of order α , which again for $\lambda = 1$ yield Hermite-Bernoulli, Euler and Genocchi polynomials, each of order α , which further for $\alpha = 1$ reduce to Hermite-Bernoulli, Euler and Genocchi polynomials. Some of these polynomials are given in [16, 37].

Theorem 2.6. The generalized-Apostol-type-Gould-Hopper polynomials are defined by the following series expansion:

$${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) = n! \sum_{k=0}^n \sum_{s=0}^{\lfloor \frac{k}{j} \rfloor} \frac{\mathcal{F}_{n-k}^{(\alpha)}(a, b, c; \mu, \nu; \lambda) x^{k-js} y^s (\log c)^{k+s-js}}{(n-k)!(k-js)! s!}.$$

Proof. Using equations (1.1) and (1.8) in generating function (2.1) yields

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} H_k^{(j)}(x, y; c) \frac{t^k}{k!},$$

which on using the series expansion of generalized Gould-Hopper polynomials from [41] gives

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} k! \sum_{s=0}^{\lfloor \frac{k}{j} \rfloor} \frac{x^{k-js} y^s (\log c)^{k+s-js} t^k}{(k-js)! s! k!}.$$

On applying Cauchy product rule in above equation, we find

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} n! \sum_{k=0}^n \sum_{s=0}^{\lfloor \frac{k}{j} \rfloor} \frac{\mathcal{F}_{n-k}^{(\alpha)}(a, b, c; \mu, \nu; \lambda) x^{k-js} y^s (\log c)^{k+s-js} t^n}{(n-k)!(k-js)! s! n!},$$

which on equating the coefficients of same powers of t yields assertion (2.2). \square

In order to show that ATGGdHP are quasi-monomial, we prove the following result.

Theorem 2.7. *The generalized-Apostol-type-Gould-Hopper polynomials are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$M_{\mathcal{H}\mathcal{F}^{(\alpha,j)}} := (x \log c + jy(\log c)^{2-j} \partial_x^{j-1}) + \frac{\alpha \nu (\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x}) - \alpha (\lambda b^{(\log c)^{-1} \partial_x} \log b + a^{(\log c)^{-1} \partial_x} \log a)}{(\log c)^{-1} \partial_x (\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x})}, \quad (2.3)$$

$$P_{\mathcal{H}\mathcal{F}^{(\alpha,j)}} := (\log c)^{-1} \partial_x. \quad (2.4)$$

Proof. On differentiation of equation (2.1) with respect to x , we obtain the following identity:

$$(\log c)^{-1} \partial_x \left\{ \left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt+yt^j} \right\} = t \left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt+yt^j}. \quad (2.5)$$

Again, differentiating both sides of the equation (2.1) w.r.t. t , we find

$$\begin{aligned} & \left((x + jyt^{j-1}) \log c + \frac{\alpha \nu (\lambda b^t + a^t) - \alpha (\lambda b^t \log b + a^t \log a)}{t(\lambda b^t + a^t)} \right) \left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt+yt^j} \\ &= \sum_{n=0}^{\infty} \mathcal{H}\mathcal{F}_{n+1}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}, \end{aligned}$$

which on using identity (2.5) and equation (2.1) becomes

$$\begin{aligned} & \left((x \log c + jy(\log c)^{2-j} \partial_x^{j-1}) + \frac{\alpha \nu (\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x}) - \alpha (\lambda b^{(\log c)^{-1} \partial_x} \log b + a^{(\log c)^{-1} \partial_x} \log a)}{(\log c)^{-1} \partial_x (\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x})} \right) \\ & \times \sum_{n=0}^{\infty} \mathcal{H}\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{H}\mathcal{F}_{n+1}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

On equating the coefficients of same power of t in both sides of the above equation and in view of monomiality principle equation (1.2) yields assertion (2.3).

Using equation (2.1) in identity (2.5) yields

$$(\log c)^{-1} \partial_x \left\{ \sum_{n=0}^{\infty} \mathcal{H}\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \right\} = \sum_{n=0}^{\infty} \mathcal{H}\mathcal{F}_{n-1}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}.$$

On equating the coefficients of same power of t in both sides of the above equation and in view of monomiality principle equation (1.2) yields assertion (2.4). \square

To derive the differential equation for the ATGGdHP, the following result is proved.

Theorem 2.8. *The generalized-Apostol-type-Gould-Hopper polynomials satisfy the following differential equation:*

$$\begin{aligned} & \left(x \partial_x + jy(\log c)^{1-j} \partial_x^j + \frac{\alpha \nu (\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x}) - \alpha (\lambda b^{(\log c)^{-1} \partial_x} \log b + a^{(\log c)^{-1} \partial_x} \log a)}{(\lambda b^{(\log c)^{-1} \partial_x} + a^{(\log c)^{-1} \partial_x})} \right. \\ & \left. - n(\log c)^{-1} \partial_x \right) \mathcal{H}\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) = 0. \end{aligned} \quad (2.7)$$

Proof. Using of expressions of multiplicative and derivative operators given by equations (2.3) and (2.4) in equation (1.3) and on simplification, we get the required differential equation given by (2.7). \square

The series expansions, differential operators and differential equations for the above special cases and particular members can be easily obtained by choosing suitable values of parameters. Thus we omit them.

In the next section, we gave some unified formula for the generalized-Apostol-type-Gould-Hopper polynomials.

3. Summation, multiplication and explicit formula

There is a continuous demand of solving problems by means of formulas, functions equations, relations and identities in research fields like classical and quantum optics. These formulas, functional equations and identities arise in well-defined combinatorial contexts and they lead systematically to well-defined classes of functions. It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums involving special polynomials. The summation formula of hybrid type special polynomials of more than one variable often appear in applications ranging from electromagnetic processes to combinatorics, see for example [7, 15]. This provides motivation to establish some unified formula for the hybrid type generalized-Apostol-type-Gould-Hopper polynomials.

First, we establish some summation formula for the generalized-Apostol-type-Gould-Hopper polynomials. For this we prove the following theorems.

Theorem 3.1. For $\alpha, \beta \in \mathbb{Z}$, the following relation for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_n^{(\alpha \pm \beta, j)}(x, y; a, b, c; \mu, \nu; \lambda) = \sum_{k=0}^n \binom{n}{k} {}_H\mathcal{F}_{n-k}^{(\alpha, j)}(x, y; a, b, c; \mu, \nu; \lambda) \mathcal{F}_k^{(\pm \beta)}(a, b, c; \mu, \nu; \lambda). \quad (3.1)$$

Proof. We write generating function (2.1) in the following form:

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^{\alpha \pm \beta} c^{xt+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha \pm \beta, j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!},$$

which on using equations (1.8) and (2.1) becomes

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha \pm \beta, j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha, j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{F}_k^{(\pm \beta)}(a, b, c; \mu, \nu; \lambda) \frac{t^k}{k!}.$$

Using Cauchy product rule in above equation and then equating the coefficients of same powers of t in both sides of resultant equation yields assertion (3.1). \square

Theorem 3.2. The following implicit summation formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_n^{(\alpha, j)}(x+w, y; a, b, c; \mu, \nu; \lambda) = \sum_{k=0}^n \binom{n}{k} w^k (\log c)^k {}_H\mathcal{F}_{n-k}^{(\alpha, j)}(x, y; a, b, c; \mu, \nu; \lambda). \quad (3.2)$$

Proof. Replacing x by $x+w$ in generating relation (2.1), we have

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{(x+w)t+yt^j} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha, j)}(x+w, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!},$$

which can be written as

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t}\right)^\alpha c^{x+t+y} \exp(\log c \, wt) = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x+w, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}.$$

Using the generating function equation (2.1) and expanding the exponential in the l.h.s. of above the equation, we have

$$\left(\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!}\right) \left(\sum_{k=0}^{\infty} w^k (\log c)^k \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x+w, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!},$$

which on using Cauchy product rule in the l.h.s. and then equating the coefficients of like powers of t on both sides yields assertion (3.2). \square

Corollary 3.3. For $w = 1$ in equation (3.2), we have following implicit summation formula for the generalized-Apostol-type-Gould-Hopper polynomials:

$${}_H\mathcal{F}_n^{(\alpha,j)}(x+1, y; a, b, c; \mu, \nu; \lambda) = \sum_{k=0}^n \binom{n}{k} (\log c)^k {}_H\mathcal{F}_{n-k}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda).$$

Theorem 3.4. The following implicit summation formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_{n+m}^{(\alpha,j)}(z, y; a, b, c; \mu, \nu; \lambda) = \sum_{s,k=0}^{m,n} \binom{m}{s} \binom{n}{k} (\log c)^{s+k} (z-x)^{s+k} {}_H\mathcal{F}_{n+m-s-k}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda). \quad (3.3)$$

Proof. Replacing $t \rightarrow t+u$ in generating function (2.1) and using the following rule:

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{l,m=0}^{\infty} f(l+m) \frac{x^l y^m}{l! m!} \quad (3.4)$$

in the r.h.s. of the resultant equation, we find

$$\left(\frac{2^\mu (t+u)^\nu}{\lambda b^{t+u} + a^{t+u}}\right)^\alpha c^{x(t+u)+y(t+u)} = \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!}. \quad (3.5)$$

Rewriting equation (3.5) as:

$$\left(\frac{2^\mu (t+u)^\nu}{\lambda b^{t+u} + a^{t+u}}\right)^\alpha c^{y(t+u)} = c^{-x(t+u)} \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!}.$$

Replacing x by z in the above equation and then equating the resultant equation to the above equation, we find

$$\sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(z, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!} = c^{(z-x)(t+u)} \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!},$$

which on writing $c^{(z-x)(t+u)}$ in the form of exponential and then expanding the exponential in the r.h.s. gives

$$\begin{aligned} \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(z, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!} &= \sum_{N=0}^{\infty} \left(\log c (z-x)\right)^N \frac{(t+u)^N}{N!} \\ &\times \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!}. \end{aligned} \quad (3.6)$$

Again, on use of equation (3.4) in the r.h.s. of equation (3.6), we find

$$\sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(z, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!} = \sum_{k,s=0}^{\infty} (\log c)^{s+k} (z-x)^{s+k} \frac{t^k u^s}{k! s!} \times \sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!}. \quad (3.7)$$

Now, replacing $n \rightarrow n - k$ and $m \rightarrow m - s$ in the r.h.s. of equation (3.7), we find

$$\sum_{n,m=0}^{\infty} {}_H\mathcal{F}_{n+m}^{(\alpha,j)}(z, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} \sum_{k,s=0}^{n,m} (\log c)^{s+k} \frac{(z-x)^{s+k}}{k! s!} \times {}_H\mathcal{F}_{n+m-s-k}^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n u^m}{(n-k)! (m-s)!}. \quad (3.8)$$

Equating the coefficients of the same powers of t and u in both sides of equation (3.8), we are led to assertion (3.3). \square

Theorem 3.5. Let a, b , and c be positive integers by $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have

$${}_H\mathcal{F}_n^{(\alpha,j)}(x + \alpha, y; a, b, c; \mu, \nu; \lambda) = n! \sum_{k=0}^{\lfloor \frac{n}{j} \rfloor} \frac{y^k (\log c)^k {}_H\mathcal{F}_{n-jk}^{(\alpha)}(x; \frac{a}{c}, \frac{b}{c}, c; \mu, \nu; \lambda)}{(n-jk)! k!}. \quad (3.9)$$

Proof. Replacing x by $x + \alpha$ in generating relation (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x + \alpha, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{(x+\alpha)t + yt^j},$$

which can be written as

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x + \alpha, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \left(\frac{2^\mu t^\nu}{\lambda (\frac{b}{c})^t + (\frac{a}{c})^t} \right)^\alpha c^{xt} \exp(\log c \ y t^j).$$

Now, using the generating equation (1.7) and expanding the exponential in the r.h.s. of above the equation, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x + \alpha, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x; \frac{a}{c}, \frac{b}{c}, c; \mu, \nu; \lambda) \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} y^k (\log c)^k \frac{t^{jk}}{k!} \right),$$

which on applying Cauchy product rule in the l.h.s. and then equating the coefficients of like powers of t in both sides yields assertion (3.9). \square

Theorem 3.6. Let a, b and c be positive integers by $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have

$${}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathcal{F}_{n-k}^{(\alpha)}(a, b, c; \mu, \nu; \lambda) H_k^{(j)}(x, y, c). \quad (3.10)$$

Proof. By use of equations (1.8) and (1.1), we have

$$\left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^\alpha c^{xt + yt^j} = \left(\sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} H_k^{(j)}(x, y, c) \frac{t^k}{k!} \right).$$

Using Cauchy product rule in the r.h.s. of above equation and then equating the coefficients of like powers of t on both sides of the resultant equation, yields assertion (3.10). \square

Theorem 3.7. Let a , b , and c be positive integers by $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have

$${}_H\mathcal{F}_k^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) = k! \sum_{s=0}^{\lfloor \frac{k}{j} \rfloor} \sum_{n=0}^{k-sj} \frac{x^{k-sj-n} (\log c)^{k-s(j-1)-n} y^s}{(k-sj-n)! s! n!} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda). \quad (3.11)$$

Proof. By the use of equation (1.8) and expansions of $e^{(\log c)x^t}$ and $e^{(\log c)y^t^j}$ in generating equation (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} x^k (\log c)^k \frac{t^k}{k!} \sum_{s=0}^{\infty} y^s (\log c)^s \frac{t^{sj}}{s!},$$

which on applying Cauchy product rule becomes

$$\begin{aligned} \sum_{k=0}^{\infty} {}_H\mathcal{F}_k^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) \frac{t^k}{k!} &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^k \binom{k}{n} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) x^{k-n} (\log c)^{k-n} \right) \frac{t^k}{k!} \times \sum_{s=0}^{\infty} y^s (\log c)^s \frac{t^{sj}}{s!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{s=0}^{\lfloor \frac{k}{j} \rfloor} \sum_{n=0}^{k-sj} \binom{k-sj}{n} \mathcal{F}_n^{(\alpha)}(a, b, c; \mu, \nu; \lambda) x^{k-sj-n} \right. \\ &\quad \left. \times (\log c)^{k-s(j-1)-n} y^s \right) \frac{t^k}{(k-sj)! s!}. \end{aligned}$$

Equating the coefficients of like powers of t on both sides of the above equation, yields assertion (3.11). \square

Theorem 3.8. Let a , b and c be positive integers by $a \neq b$. Then for $x, y \in \mathbb{R}$ and $n \geq 0$, we have

$${}_H\mathcal{F}_k^{(\alpha,j)}(x, y; a, b, c; \mu, \nu; \lambda) = n! \sum_{p=0}^{\lfloor \frac{n}{j} \rfloor} \sum_{m=0}^{n-pj} \frac{(\log c)^m y^p (\log c)^p}{(n-pj-m)! p! m!} \mathcal{F}_{n-pj-m}^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda). \quad (3.12)$$

Proof. By the use of equation (1.7) and expansions of $e^{(\log c)t}$ and $e^{(\log c)y^t^j}$ in generating equation (2.1) with $x \rightarrow x+1$, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x+1, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} \sum_{m=0}^{\infty} (\log c)^m \frac{t^m}{m!} \sum_{p=0}^{\infty} y^p (\log c)^p \frac{t^{pj}}{p!},$$

which on applying Cauchy product rule becomes

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x+1, y; a, b, c; \mu, \nu; \lambda) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \mathcal{F}_{n-m}^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda) (\log c)^m \right) \frac{t^n}{n!} \left(\sum_{p=0}^{\infty} y^p (\log c)^p \frac{t^{pj}}{p!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{p=0}^{\lfloor \frac{n}{j} \rfloor} \sum_{m=0}^{n-pj} \binom{n-pj}{m} \mathcal{F}_{n-pj-m}^{(\alpha)}(x; a, b, c; \mu, \nu; \lambda) (\log c)^m y^p (\log c)^p \right) \frac{t^n}{(n-pj)! p!}. \end{aligned}$$

Equating the coefficients of like powers of t on both sides of the above equation, yields assertion (3.12). \square

Next, we find multiplication formula for the generalized-Apostol-type-Gould-Hopper polynomials and for their special cases. For this we prove the following results.

Theorem 3.9. Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following multiplication formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$$\begin{aligned} {}_H\mathcal{F}_n^{(\alpha, j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) &= r^{n-\nu\alpha} a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ &\times (-\lambda)^m {}_H\mathcal{F}_n^{(\alpha, j)} \left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \mu, \nu; \lambda^r \right) \text{ for } r \text{ odd,} \end{aligned} \quad (3.13)$$

$$\begin{aligned} {}_H\mathcal{F}_n^{(l, j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) &= \frac{(-1)^l 2^{\mu l} r^{n-\nu l}}{(n+1)(1-\nu)^l} a^{(r-1)lt} \sum_{\substack{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_{r-1} \leq l \\ \nu_1 + \nu_2 + \dots + \nu_{r-1} = l}} \binom{l}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ &\times (-\lambda)^m {}_H\mathcal{B}_{n+(1-\nu)l}^{(l, j)} \left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \text{ for } r \text{ even,} \end{aligned} \quad (3.14)$$

where $\nu_1 + 2\nu_2 + \dots + (r-1)\nu_{r-1} = m$.

Proof. We consider generating equation (2.1) in the following form:

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha, j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) \frac{t^n}{n!} = \frac{1}{r^{\nu\alpha}} \left(\frac{2^\mu (rt)^\nu}{\lambda^r b^{rt} + a^{rt}} \right)^\alpha \left(\frac{\lambda^r b^{rt} + a^{rt}}{\lambda b^t + a^t} \right)^\alpha \left(\frac{b}{a} \right)^{rxt+y(rt)^j},$$

which on simplifying becomes

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha, j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) \frac{t^n}{n!} &= \frac{1}{r^{\nu\alpha}} \left(\frac{2^\mu (rt)^\nu}{\lambda^r b^{rt} + a^{rt}} \right)^\alpha a^{(r-1)\alpha t} \left(\frac{\lambda^r \left(\frac{b}{a}\right)^{rt} + 1}{\lambda \left(\frac{b}{a}\right)^t + 1} \right)^\alpha \left(\frac{b}{a} \right)^{rxt+y(rt)^j} \\ &= \frac{1}{r^{\nu\alpha}} \left(\frac{2^\mu (rt)^\nu}{\lambda^r b^{rt} + a^{rt}} \right)^\alpha a^{(r-1)\alpha t} \left(\sum_{k=0}^{r-1} (-\lambda \left(\frac{b}{a}\right)^t)^k \right)^\alpha \left(\frac{b}{a} \right)^{rxt+y(rt)^j}. \end{aligned}$$

Now, on use of the following formula for the generalized multinomial identity [4]:

$$(1 + x_1 + x_2 + \dots + x_m)^\alpha = \sum_{\nu_1, \nu_2, \dots, \nu_m \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_m} x_1^{\nu_1} x_2^{\nu_2} \dots x_m^{\nu_m},$$

the above equation becomes

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha, j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) \frac{t^n}{n!} &= \frac{1}{r^{\nu\alpha}} \left(\frac{2^\mu (rt)^\nu}{\lambda^r b^{rt} + a^{rt}} \right)^\alpha a^{(r-1)\alpha t} \\ &\times \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} (-\lambda)^{\nu_1+2\nu_2+\dots+(r-1)\nu_{r-1}} \left(\frac{b}{a} \right)^{\nu_1+2\nu_2+\dots+(r-1)\nu_{r-1}} \left(\frac{b}{a} \right)^{rxt+y(rt)^j} \\ &= \frac{1}{r^{\nu\alpha}} \left(\frac{2^\mu (rt)^\nu}{\lambda^r b^{rt} + a^{rt}} \right)^\alpha a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} (-\lambda)^m \left(\frac{b}{a} \right)^{(x+\frac{m}{r})rt+y(rt)^j}. \end{aligned}$$

Now, using equation (2.1) in above equation, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \mu, \nu; \lambda \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(r^{n-\nu\alpha} a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} \right) \\ \times (-\lambda)^m \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \mu, \nu; \lambda^r \right) \right) \frac{t^n}{n!}.$$

Equation the coefficients of same power of t on both sides of above equation yields assertion (3.13). Again, with the help of following definition for the multinomial identity [4]:

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{\substack{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_m \leq n \\ \nu_1 + \nu_2 + \dots + \nu_m = n}} \binom{n}{\nu_1, \nu_2, \dots, \nu_m} x_1^{\nu_1} x_2^{\nu_2} \dots x_m^{\nu_m}$$

for r even, we can similarly prove assertion (3.14). The proof is completed. \square

In view of Remarks 2.3-2.5, we obtain the following consequences of Theorem 3.9.

Corollary 3.10. Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following multiplication formula for the generalized-Gould-Hopper-Apostol-Bernoulli polynomials holds true:

$${}_H\mathfrak{B}_n^{(\alpha,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \lambda \right) = r^{n-\alpha} a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ \times (-\lambda)^m {}_H\mathfrak{B}_n^{(\alpha,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \right).$$

Corollary 3.11. Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following multiplication formula for the generalized-Gould-Hopper-Apostol-Euler polynomials holds true:

$${}_H\mathfrak{E}_n^{(\alpha,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \lambda \right) = r^n a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ \times (-\lambda)^m {}_H\mathfrak{E}_n^{(\alpha,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \right) \text{ for } r \text{ odd,}$$

$${}_H\mathfrak{E}_n^{(l,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \lambda \right) = \frac{(-1)^l 2^l r^n}{(n+1)_l} a^{(r-1)lt} \sum_{\substack{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_{r-1} \leq l \\ \nu_1 + \nu_2 + \dots + \nu_{r-1} = l}} \binom{l}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ \times (-\lambda)^m {}_H\mathfrak{B}_{n+l}^{(l,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \right) \text{ for } r \text{ even.}$$

Corollary 3.12. Let $a, b, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following multiplication formula for the generalized-Gould-Hopper-Apostol-Genocchi polynomials holds true:

$${}_H\mathfrak{G}_n^{(\alpha,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \lambda \right) = r^{n-\alpha} a^{(r-1)\alpha t} \sum_{\nu_1, \nu_2, \dots, \nu_{r-1} \geq 0} \binom{\alpha}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ \times (-\lambda)^m {}_H\mathfrak{G}_n^{(\alpha,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \right) \text{ for } r \text{ odd,}$$

$${}_H\mathfrak{G}_n^{(l,j)} \left(rx, r^j y; a, b, \frac{b}{a}; \lambda \right) = (-1)^l 2^l r^{n-l} a^{(r-1)lt} \sum_{\substack{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_{r-1} \leq l \\ \nu_1 + \nu_2 + \dots + \nu_{r-1} = l}} \binom{l}{\nu_1, \nu_2, \dots, \nu_{r-1}} \\ \times (-\lambda)^m {}_H\mathfrak{B}_n^{(l,j)} \left(\left(x + \frac{m}{r}, y; a, b, \frac{b}{a}; \lambda^r \right) \right) \text{ for } r \text{ even.}$$

Further, we derive some explicit formulas involving some known functions. For this we recall the following definition.

Definition 3.13. Let $a, b \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $\nu \in \mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_\nu^n(x; a, b; \lambda)$ are defined by [38]:

$$\frac{(\lambda b^t - a^t)^\nu}{\nu!} b^{xt} = \sum_{n=0}^{\infty} \mathcal{S}_\nu^n(x; a, b; \lambda) \frac{t^n}{n!},$$

where $\mathcal{S}(n, \nu; a, b; \lambda) := \mathcal{S}_\nu^n(0; a, b; \lambda)$ are generalized λ -Stirling type numbers of second kind.

Definition 3.14. The polynomials $Y_n^\nu(x, \lambda; a)$ are defined by [24, 38]:

$$\left(\frac{t}{\lambda a^t - 1} \right)^\nu a^{xt} = \sum_{n=0}^{\infty} Y_n^\nu(x, \lambda; a) \frac{t^n}{n!} \quad (a \geq 1),$$

where $Y_n^\nu(\lambda; a) := Y_n^\nu(x, \lambda; a)$ are the corresponding numbers.

Definition 3.15. The generalized Hurwitz-Lerch Zeta function $\Phi_\mu(z, s, a)$ [12] is defined by

$$\Phi_\mu(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}.$$

Theorem 3.16. Let $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following explicit formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_{n-\nu\alpha}^{(-\alpha, j)}(x, y; a, b; \mu, \nu; \lambda) = \frac{(n-\nu\alpha)! \alpha!}{(2^\mu)^\alpha (n-jk)! k!} \sum_{k=0}^{\lfloor \frac{n}{j} \rfloor} (-1)^\alpha \mathcal{S}_\nu^{n-jk}(x; a, b; -\lambda) (\log b)^k y^k. \quad (3.15)$$

Proof. Replacing c by b in generating equation (2.1) yields

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(-\alpha, j)}(x, y; a, b; \mu, \nu; \lambda) \frac{t^n}{n!} &= \left(\frac{2^\mu t^\nu}{\lambda b^t + a^t} \right)^{-\alpha} b^{xt+yt^j} \\ &= \frac{\alpha!}{(2^\mu t^\nu)^\alpha} \left(\frac{(\lambda b^t + a^t)^\alpha}{\alpha!} b^{xt} \right) b^{yt^j} \\ &= \frac{\alpha!}{(2^\mu t^\nu)^\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^\alpha \mathcal{S}_\alpha^n(x; a, b; -\lambda) (\log b)^k y^k \frac{t^{n+jk}}{n! k!} \\ &= \frac{\alpha!}{(2^\mu t^\nu)^\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{j} \rfloor} (-1)^\alpha \mathcal{S}_\alpha^{n-jk}(x; a, b; -\lambda) (\log b)^k y^k \frac{t^n}{(n-jk)! k!}, \end{aligned}$$

which on equating the coefficients of same powers of t yields assertion (3.15). \square

Theorem 3.17. The following explicit formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_{n-\nu\alpha}^{(-\alpha, j)}(x, y; a, b; \mu, \nu; \lambda) = \frac{(n-\nu\alpha)! \alpha!}{(2^\mu)^\alpha} \sum_{k=0}^n \binom{n}{k} (-1)^\alpha \mathcal{S}(n-k, \alpha; a, b; -\lambda) H_k^{(j)}(x, y, b). \quad (3.16)$$

Theorem 3.18. Let $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following explicit formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$${}_H\mathcal{F}_n^{(\alpha, j)}(x, y; a, b; \mu, 1; \lambda) = \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{j} \rfloor} (-1)^\alpha y_{n-jk}^\alpha \left(x; \frac{b}{a}; -\lambda \right) (\log b)^k y^k \frac{t^n}{(n-jk)! k!}.$$

Proof. Replacing c by b in generating equation (2.1) yields

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x,y;a,b;\mu,1;\lambda) \frac{t^n}{n!} &= \left(\frac{2^\mu t}{\lambda b^t + a^t} \right)^\alpha b^{xt+yt^j} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \left(\left(\frac{t}{\lambda \left(\frac{b}{a}\right)^t + 1} \right)^\alpha b^{xt} \right) b^{yt^j} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^\alpha y_n^\alpha \left(x; \frac{b}{a}; -\lambda \right) (\log b)^k y^k \frac{t^{n+jk}}{n! k!} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{j} \rfloor} (-1)^\alpha y_{n-jk}^\alpha \left(x; \frac{b}{a}; -\lambda \right) (\log b)^k y^k \frac{t^n}{(n-jk)! k!}, \end{aligned}$$

which on equating the coefficients of same powers of t yields assertion (3.16). \square

Theorem 3.19. *The following explicit formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:*

$${}_H\mathcal{F}_n^{(\alpha,j)}(x,y;a,b;\mu,1;\lambda) = \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{k=0}^n \binom{n}{k} (-1)^\alpha y_{n-k}^\alpha \left(\frac{b}{a}; -\lambda \right) H_k^{(j)}(x,y,b).$$

Theorem 3.20. *Let $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$; $\lambda, \mu, \nu \in \mathbb{C}$; $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The following explicit formula for the generalized-Apostol-type-Gould-Hopper polynomials holds true:*

$${}_H\mathcal{F}_{n+\nu\alpha}^{(\alpha,j)}\left(x,y;a,b,\frac{b}{a};\mu,\nu;\lambda\right) = \left(\frac{2^\mu}{a^t} \right)^\alpha \frac{(n+\nu\alpha)!}{(n-jl)! l!} \sum_{l=0}^{\lfloor \frac{n}{j} \rfloor} \left(\log \frac{b}{a} \right)^{n-jl+l} \Phi_\alpha(\lambda, -n+jl, x) y^l. \quad (3.17)$$

Proof. We consider generating function (2.1) in the following form:

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha,j)}(x,y;a,b,\frac{b}{a};\mu,\nu;\lambda) \frac{t^{n-\nu\alpha}}{n!} &= \left(\frac{2^\mu}{a^t} \right)^\alpha \left(\lambda \left(\frac{b}{a} \right)^t + 1 \right)^{-\alpha} \left(\frac{b}{a} \right)^{xt+yt^j} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \lambda^k \left(\frac{b}{a} \right)^{kt} \left(\frac{b}{a} \right)^{xt+yt^j} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \lambda^k (\log \frac{b}{a})^n (k+x)^n \frac{t^n}{n!} \left(\frac{b}{a} \right)^{yt^j} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} (\log \frac{b}{a})^{n+l} \Phi_\alpha(\lambda, -n, x) \frac{t^n}{n!} \sum_{l=0}^{\infty} y^l \frac{t^{jl}}{l!} \\ &= \left(\frac{2^\mu}{a^t} \right)^\alpha \sum_{n=0}^{\infty} \sum_{l=0}^{\lfloor \frac{n}{j} \rfloor} (\log \frac{b}{a})^{n-jl+l} \Phi_\alpha(\lambda, -n+jl, x) y^l \frac{t^n}{(n-jl)! l!}, \end{aligned}$$

which on equating the coefficients of same powers of t in both sides of resultant equation yields assertion (3.17). \square

Next, we establish some symmetric identities for generalized-Apostol-type-Gould-Hopper polynomials.

4. Symmetric identities

In mathematics and statistics, sums of powers occur in a number of contexts. Many combinatorial problems can be solved with the aid of these power sums particularly occur in Euler's sum of powers conjecture, Fermat Catalan conjecture, Lander, Parkin and Selfridge conjectures, Prouhet-Tarry-Escott problem etc.. The power sum symmetric polynomials are considered to be building block for symmetric polynomials.

The symmetry identities related to classical and Apostol type polynomials are established by many authors [18, 20–22, 36]. This provides motivation to establish symmetry identities for the generalized-Apostol-type-Gould-Hopper polynomials.

In order to derive the symmetry identities for the GGdHATP ${}_H\mathcal{F}_n^{(\alpha)}(x, y; a, b, c; \mu, \nu; \lambda)$, we prove the following results.

Theorem 4.1. For all integers $l, m > 0$ and $n \geq 0$; $\alpha \geq 1$; $\lambda, \mu, \nu \in \mathbb{C}$, and $a, b, c \in \mathbb{R}^+$, the following symmetry identity for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} l^{n-j} m^{j+\nu} {}_H\mathcal{F}_{n-j}^{(\alpha)}(mx, m^j y; a, b, c; \mu, \nu; \lambda) \sum_{i=0}^j \binom{j}{i} a^{(l-1)mt} \mathcal{S}_i(l-1; -\lambda) \\ & \quad \times {}_H\mathcal{F}_{j-i}^{(\alpha-1)}(lX, l^j Y; a, b, c; \mu, \nu; \lambda) \\ & = \sum_{j=0}^n \binom{n}{j} m^{n-j} l^{j+\nu} {}_H\mathcal{F}_{n-j}^{(\alpha)}(lx, l^j y; a, b, c; \mu, \nu; \lambda) \sum_{i=0}^j \binom{j}{i} a^{(m-1)lt} \mathcal{S}_i(m-1; -\lambda) \\ & \quad \times {}_H\mathcal{F}_{j-i}^{(\alpha-1)}(mX, m^j Y; a, b, c; \mu, \nu; \lambda). \end{aligned} \tag{4.1}$$

Proof. Let

$$G(t) := \frac{2^{\mu(2\alpha-1)} t^{\nu(2\alpha-1)} c^{lmxt+y(lmt)^j} (\lambda b^{lmt} + a^{lmt}) c^{lmXt+Y(lmt)^j}}{(\lambda b^{lt} + a^{lt})^\alpha (\lambda b^{mt} + a^{mt})^\alpha},$$

which on rearranging the powers becomes

$$\begin{aligned} G(t) &= \frac{1}{l^\nu \alpha m^{\nu(\alpha-1)}} \left(\frac{2^\mu l^\nu t^\nu}{\lambda b^{lt} + a^{lt}} \right)^\alpha c^{lmxt+y(lmt)^j} \left(\frac{\lambda b^{lmt} + a^{lmt}}{\lambda b^{mt} + a^{mt}} \right) \\ & \quad \times \left(\frac{2^\mu m^\nu t^\nu}{\lambda b^{mt} + a^{mt}} \right)^{\alpha-1} c^{lmXt+Y(lmt)^j}. \end{aligned} \tag{4.2}$$

Since the expression (4.2) for $G(t)$ is symmetric in l and m , therefore we can expand $G(t)$ into series in two ways. First, we consider the following expansion:

$$G(t) = \frac{1}{l^\nu \alpha m^{\nu(\alpha-1)}} \left(\frac{2^\mu (lt)^\nu}{\lambda b^{lt} + a^{lt}} \right)^\alpha c^{lmxt+y(lmt)^j} \left(\frac{\lambda b^{lmt} + a^{lmt}}{\lambda b^{mt} + a^{mt}} \right) \left(\frac{2^\mu (mt)^\nu}{\lambda b^{mt} + a^{mt}} \right)^{\alpha-1} c^{lmXt+Y(lmt)^j},$$

which on using generating equation (2.1) and the following formula for the numbers $\mathcal{S}_k(n; \lambda)$:

$$\frac{\lambda e^{(n+1)t} - 1}{\lambda e^t - 1} = \sum_{k=0}^{\infty} \mathcal{S}_k(n; \lambda) \frac{t^k}{k!},$$

gives

$$G(t) = \frac{1}{l^\nu \alpha m^{\nu(\alpha-1)}} \left(\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(mx, m^j y; a, b, c; \mu, \nu; \lambda) \frac{(lt)^n}{n!} \right) \left(a^{(l-1)mt} \left(\log \frac{b}{a} \right)^i \sum_{i=0}^{\infty} \mathcal{S}_i(l-1; -\lambda) \frac{(mt)^i}{i!} \right)$$

$$\times \left(\sum_{j=0}^{\infty} {}_H\mathcal{F}_j^{(\alpha-1)}(lX, l^jY; a, b, c; \mu, \nu; \lambda) \frac{(mt)^j}{j!} \right),$$

which on using [40, p.890 Corollary 2] and Cauchy product rule gives

$$\begin{aligned} G(t) &= \frac{a^{(l-1)mt}}{l^{\nu\alpha} m^{\nu\alpha}} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} l^{n-j} m^{j+\nu} {}_H\mathcal{F}_{n-j}^{(\alpha)}(mX, m^jY; a, b, c; \mu, \nu; \lambda) \right. \\ &\quad \left. \times \sum_{i=0}^j \binom{j}{i} \left(\log \frac{b}{a} \right)^i \mathcal{S}_i(l-1; -\lambda) {}_H\mathcal{F}_{j-i}^{(\alpha-1)}(lX, l^jY; a, b, c; \mu, \nu; \lambda) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.3)$$

Similarly, we have

$$\begin{aligned} G(t) &= \frac{a^{(m-1)lt}}{m^{\nu\alpha} l^{\nu\alpha}} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} m^{n-j} l^{j+\nu} {}_H\mathcal{F}_{n-j}^{(\alpha)}(lX, l^jY; a, b, c; \mu, \nu; \lambda) \right. \\ &\quad \left. \times \sum_{i=0}^j \binom{j}{i} \left(\log \frac{b}{a} \right)^i \mathcal{S}_i(m-1; -\lambda) {}_H\mathcal{F}_{j-i}^{(\alpha-1)}(mX, m^jY; a, b, c; \mu, \nu; \lambda) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.4)$$

Equating the coefficients of same powers of t in r.h.s. of expansions (4.3) and (4.4), we are led to assertion (4.1). \square

Theorem 4.2. For all integers $l, m > 0$ and $n \geq 0$; $\alpha \geq 1$; $\lambda, \mu, \nu \in \mathbb{C}$ and $a, b \in \mathbb{R}^+$, the following symmetry identity for the generalized-Apostol-type-Gould-Hopper polynomials holds true:

$$\begin{aligned} &\sum_{p=0}^n \binom{n}{p} \sum_{i=0}^{l-1} \sum_{h=0}^{m-1} (-\lambda)^{i+h} l^p m^{n-p} {}_H\mathcal{F}_p^{(\alpha)}\left(mX + \frac{m}{l}i, m^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \\ &\quad \times {}_H\mathcal{F}_{n-p}^{(\alpha)}\left(lX + \frac{l}{m}h, l^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \\ &= \sum_{p=0}^n \binom{n}{p} \sum_{i=0}^{m-1} \sum_{h=0}^{l-1} (-\lambda)^{i+h} m^p l^{n-p} {}_H\mathcal{F}_p^{(\alpha)}\left(lX + \frac{l}{m}i, l^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \\ &\quad \times {}_H\mathcal{F}_{n-p}^{(\alpha)}\left(mX + \frac{m}{l}h, m^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right). \end{aligned} \quad (4.5)$$

Proof. Let

$$H(t) := \frac{2^{2\mu\alpha} t^{2\nu\alpha} \frac{b}{a}^{lmxt+y(lmt)^j} (\lambda^l b^{lmt} + a^{lmt})(\lambda^m b^{lmt} + a^{lmt}) \frac{b}{a}^{lmXt+Y(lmt)^j}}{(\lambda b^{lt} + a^{lt})^{\alpha+1} (\lambda b^{mt} + a^{mt})^{\alpha+1}},$$

which on rearranging the powers becomes

$$\begin{aligned} H(t) &:= \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} \left(\frac{2^\mu l^\nu t^\nu}{\lambda b^{lt} + a^{lt}} \right)^\alpha \frac{b^{lmxt+y(lmt)^j}}{a} \left(\frac{\lambda^l b^{lmt} + a^{lmt}}{\lambda b^{mt} + a^{mt}} \right) \left(\frac{2^\mu m^\nu t^\nu}{\lambda b^{mt} + a^{mt}} \right)^\alpha \\ &\quad \times \frac{b^{lmXt+Y(lmt)^j}}{a} \left(\frac{\lambda^m b^{lmt} + a^{lmt}}{\lambda b^{lt} + a^{lt}} \right). \end{aligned} \quad (4.6)$$

Since expression (4.6) for $H(t)$ is symmetric in l and m , therefore we can expand $H(t)$ into series in two ways. First, we consider the following expansion:

$$\begin{aligned} H(t) &:= \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} \left(\frac{2^\mu (lt)^\nu}{\lambda b^{lt} + a^{lt}} \right)^\alpha \frac{b^{lmxt+y(lmt)^j}}{a} \left(\frac{\lambda^l b^{lmt} + a^{lmt}}{\lambda b^{mt} + a^{mt}} \right) \left(\frac{2^\mu (mt)^\nu}{\lambda b^{mt} + a^{mt}} \right)^\alpha \\ &\quad \times \frac{b^{lmXt+Y(lmt)^j}}{a} \left(\frac{\lambda^m b^{lmt} + a^{lmt}}{\lambda b^{lt} + a^{lt}} \right). \end{aligned} \quad (4.7)$$

Now, using the series expansions for $\left(\frac{\lambda^l b^{lmt} + a^{lmt}}{\lambda b^{lt} + a^{lt}}\right)$ and $\left(\frac{\lambda^m b^{lmt} + a^{lmt}}{\lambda b^{lt} + a^{lt}}\right)$ in the r.h.s. of equation (4.7), we have

$$H(t) := \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} \left(\frac{2^\mu (lt)^\nu}{\lambda b^{lt} + a^{lt}}\right)^\alpha \frac{b^{lmxt+y(lmt)^j}}{a} \left(a^{(l-1)mt} \sum_{i=0}^{l-1} (-\lambda)^i \left(\frac{b}{a}\right)^{mti}\right) \\ \times \left(\frac{2^\mu (mt)^\nu}{\lambda b^{mt} + a^{mt}}\right)^\alpha \frac{b^{lmXt+Y(lmt)^j}}{a} \left(a^{(m-1)lt} \sum_{h=0}^{m-1} (-\lambda)^h \left(\frac{b}{a}\right)^{lth}\right).$$

Combining the terms in the above equation, we have

$$H(t) := \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} a^{2mlt-mt-lt} \sum_{i=0}^{l-1} (-\lambda)^i \left(\frac{2^\mu (lt)^\nu}{\lambda b^{lt} + a^{lt}}\right)^\alpha \frac{b^{(mx+\frac{m}{l}i)lt+m^jy(lt)^j}}{a} \\ \times \sum_{h=0}^{m-1} (-\lambda)^h \left(\frac{2^\mu (mt)^\nu}{\lambda b^{mt} + a^{mt}}\right)^\alpha \frac{b^{(lX+\frac{l}{m}h)mt+l^jY(mt)^j}}{a},$$

which on using equation (2.1) becomes

$$H(t) := \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} a^{2mlt-mt-lt} \left(\sum_{i=0}^{l-1} (-\lambda)^i \sum_{p=0}^{\infty} {}_H\mathcal{F}_p^{(\alpha)}\left(mx + \frac{m}{l}i, m^jy; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \frac{(lt)^p}{p!}\right) \\ \times \left(\sum_{h=0}^{m-1} (-\lambda)^h \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}\left(lX + \frac{l}{m}h, l^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \frac{(mt)^n}{n!}\right), \quad (4.8)$$

Applying the Cauchy product rule in the r.h.s. of equation (4.8), we find

$$H(t) := \frac{1}{l^{\nu\alpha}} \frac{1}{m^{\nu\alpha}} a^{2mlt-mt-lt} \sum_{n=0}^{\infty} \sum_{p=0}^n \binom{n}{p} \sum_{i=0}^{l-1} \sum_{h=0}^{m-1} (-\lambda)^{i+h} l^p m^{n-p} {}_H\mathcal{F}_p^{(\alpha)} \\ \times \left(mx + \frac{m}{l}i, m^jy; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) {}_H\mathcal{F}_{n-p}^{(\alpha)}\left(lX + \frac{l}{m}h, l^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \frac{t^n}{n!}. \quad (4.9)$$

Using the similar way, we obtain the second expansion of $H(t)$ as:

$$H(t) := \frac{1}{m^{\nu\alpha}} \frac{1}{l^{\nu\alpha}} a^{2mlt-mt-lt} \sum_{n=0}^{\infty} \sum_{p=0}^n \binom{n}{p} \sum_{i=0}^{m-1} \sum_{h=0}^{l-1} (-\lambda)^{i+h} m^p l^{n-p} {}_H\mathcal{F}_p^{(\alpha)} \\ \times \left(lx + \frac{l}{m}i, l^jy; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) {}_H\mathcal{F}_{n-p}^{(\alpha)}\left(mX + \frac{m}{l}h, m^jY; a, b, \frac{b}{a}; \mu, \nu; \lambda\right) \frac{t^n}{n!}. \quad (4.10)$$

Equating the coefficients of like powers of t in r.h.s. of expansions (4.9) and (4.10), we are led to assertion (4.5). \square

In view of Remarks 2.3-2.5, we can obtain the symmetry identities for the generalized-Gould-Hopper-Apostol-Bernoulli, Euler and Genocchi polynomials simply by choosing suitable values of parameters μ and ν . For the lack of space, we omit them.

Appendix A.

In 2006, Luo and Srivastava [33] introduced the Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x)$ of single variable defined by

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x) \frac{t^n}{n!} = f(t, \alpha) e^{xt}$$

and framed Apostol-Bernoulli and Euler polynomials within the context of Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x)$.

Recently, Dere and Simsek [10] gave a new class of the Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x, y)$ defined by

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y) \frac{t^n}{n!} = f(t, \alpha) e^{xt+h(t,y)}. \quad (\text{A.1})$$

In view of equation (A.1), we gave a unified form of generalized Milne-Thomson's polynomials $\Phi_n^\alpha(x, y; \mu, \nu; \lambda)$ defined by

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y; \mu, \nu; \lambda) \frac{t^n}{n!} = f(t, \alpha; \mu, \nu; \lambda) c^{xt+h(t,y)}.$$

Simply by choosing $f(t, \alpha; \mu, \nu; \lambda) = \left(\frac{2^{\mu} 1^{\nu}}{\lambda b^{t+\alpha t}}\right)^{\alpha}$ and $h(t, y) = yt^j$, we get generalized-Apostol-type-Gould-Hopper polynomials.

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