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Comparison of normal equation with an ABS approach for solving convex quadratic programs

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Abstract

In this paper, we present numerical results concerning a comparison between the normal equation approach and an ABS approach for computing the search direction of primal-dual infeasible interior point methods for solving convex quadratic programming problems (CQPs). Let m and n denote the number of constraints and the number of variables, respectively. The numerical results show that, when m/n is small, then the ABS approach needs a considerably less computing time. When m/n is close to one, then the normal equations approach is more efficient than the ABS approach.

Keywords: Infeasible interior point method, Primal-dual algorithms, Extended ABS algorithms, Search direction, Iteration free search vector.

1. Introduction

Consider the CQP of minimizing ${}^{1}/{}_{2}\, x^T Qx + c^T x$ subject to $Ax = b, x \geq 0$, where $c, x\, \epsilon R^n$, $Q \epsilon R^{n\times n}$ is symmetric positive semidefinite, $A \in R^{m\times n}$ and $b \in R^m$. Here, we assume that rank(A)=m and $m \leq n$. In the kth iteration of an IIPM for solving convex quadratic problems, the search direction is computed by solving the $(2n + m) \times (2n + m)$ system of linear equations [3],

$$\begin{pmatrix} -Q & A^T & I_n \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{pmatrix} \qquad (1)$$

where r_c^k, r_b^k and r_{xs}^k are given by $r_b^k = Ax^k - b, r_c^k = A^T\lambda^k + s^k - c, r_{xs}^k = -X^k S^k e + \sigma_k \mu_k \mathbf{1}$ and X^k and S^k denote the diagonal matrices whose diagonal elements are the components ofthe vectors x^k and s^k, respectively, and $\mathbf{1} = (1, \ldots, 1)^T \in R^n$. Moreover, $\sigma_k \in (0,1)$ and $\mu_k = \frac{(x^k)^T s^k}{n}$ are centering parameter and duality gap, respectively.

2. ABS approach

Let a_i^T, $1 \le i \le m$, denote rows of the matrix A, and consider the linear system $Ax = b$. Applying the ABS algorithm with $j_{i+1} = n - i$ to this system, starting with $\bar{H}_1 = I_n$ and $\bar{x}_1 = 0 \in R^n$, in the ith iteration, choose $\bar{z}_i$ so that $\bar{z}_i^T \bar{H}_i a_i \neq 0$, and let $\bar{p}_i = \bar{H}_i^T \bar{z}_i$. After computing $\bar{x}_{i+1}$, choose $\bar{G}_i \in R^{(n-i)\times(n-i+1)}$ so that $\bar{G}_i x = 0$ if and only if $x = \bar{H}_i a_i$, for some $\alpha \in R$, and let $\bar{H}_{i+1} = \bar{G}_i \bar{H}_i$. Consider the linear system $A^T y = 0$. In applying the ABS algorithm to this system, starting with $\hat{H}_1 = I_m$ and $\hat{x}_1 = 0 \in R^m$, in the ith iteration we choose $\hat{z}_i$ so that $\hat{z}_i^T \hat{H}_i^T A e_i \neq 0$, and let $\hat{p}_i = \hat{H}_i \bar{z}_i$. Then, we choose $\hat{w}_i$ so that $\hat{w}_i^T \hat{H}_i^T A e_i = 1$, and let $\hat{H}_{i+1} = \hat{H}_i - \hat{H}_i A e_i \hat{w}_i^T \hat{H}_i$. Let $\Pi^k = S^k (X^k)^{-1}$, $\bar{H} = \bar{H}_{m+1}$ and $B_k^0 = 0$. Define the matrices $B_k^j \in R^{n\times m}$ according to the following formula:

$$B_k^j = B_k^{j-1} - B_k^{j-1} A e_j \hat{w}_i^T \hat{H}_j + \bar{H}\left(Q^T + \Pi^k\right) e_j \hat{w}_j^T \hat{H}_j, \quad 1 \le j \le m. \qquad (2)$$

The solution of the first $2m + n$ equations of (1) is [2]:

$$\begin{pmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} P\lambda^k \\ -\tilde{A}\beta^k \\ -r_c^k - A^T \tilde{A}\beta^k \end{pmatrix},$$

where $P = (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_m)$, $\hat{P} = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m)$, $\lambda^k = \left(\lambda_1^k, \lambda_2^k, \dots, \lambda_m^k\right)$, $\beta^k = \left(\beta_1^k, \beta_2^k, \dots, \beta_m^k\right)$, and the ith component of the vectors λ^k and β^k are defined, using the properties of the ABS algorithm and as:

$$\lambda_1^k = \frac{-(r_b^k)^T e_i}{a_1^T \bar{p}_1}, \quad \lambda_i^k = \frac{-(r_b^k)^T e_i - \sum_{j=1}^{i-1} \lambda_j^k a_i^T \bar{p}_j}{a_i^T \bar{p}_i},$$

for $2 \le i \le m$, and

$$\beta_1^k = \frac{s_1^k e_1^T P\lambda^k - s_1^k e_1^T r_c^k + (r_{xs}^k)^T e_1 + x_1^k e_1^T Q P \lambda^k}{x_1^k e_1^T A^T \hat{p}_1},$$

$$\beta_i^k = \frac{(r_{xs}^k)^T e_i + s_i^k e_i^T P\lambda^k - x_i^k e_i^T \left(\sum_{j=1}^{i-1} \beta_j^k A^T \hat{p}_j\right) - x_j^k e_j^T r_c^k + x_j^k e_j^T Q P \lambda^k}{x_j^k e_j^T A^T \hat{p}_j},$$

for $2 \le i \le m$. Now, let A_m and A_{n-m} denote the first m columns and the last $n - m$ columns of A, respectively, $L = AP$ and $\hat{L}_m = A_m^T \hat{P}$. Then, we can also compute λ^k and β^k, by solving the linear systems $L\lambda^k = -r_b^k$ and $\hat{L}_m \beta^k = \hat{r}_m^k$, where $\hat{r}_m^k$ denotes the first m components of

$$\hat{r}_m^k = (X^k)^{-1} r_{xs}^k + \Pi^k P \lambda^k - r_c^k + Q P \lambda^k.$$

Note that L and $\hat{L}_m$ are nonsingular lower triangular matrices. To characterize the solution of (2), we first derive an efficient formula to compute B_k^m, as defined by (2). Then, using properties of the ABS algorithm, we have the solution of system (2) from the solution of the first $2m + n$ equations. Let $u_i^i = \hat{H}_i^T \hat{w}_i$, , for $2 \le i \le m$, and $u_j^i = (I_m - \hat{H}_i^T \hat{w}_i e_i^T A^T) u_j^{i-1}$, $1 \le j \le i - 1$. For $1 \le i \le m$, define $U_i = \sum_{t=1}^i e_t (u_t^i)^T$. It can be proved that [2], $B_k^j = \bar{H}\left(Q^T + \Pi^k\right) U_i$, $1 \le i \le m$. It is worth mentioning that the matrices U_i and $\bar{H}$ need to be computed only once in the first iteration of the IIPM, and Π^k is a diagonal matrix. Now, using the properties of the ABS algorithm [1], we construct the solution of system (1). For notational simplicity, let

$$F_s^k = \sum_{j=m+1}^{n} s_j^k e_j \hat{e}_{j-m}^T = \begin{pmatrix} 0 \\ S_{n-m}^k \end{pmatrix}, \qquad F_x^k = \sum_{j=m+1}^{n} x_j^k e_j \hat{e}_{j-m}^T = \begin{pmatrix} 0 \\ X_{n-m}^k \end{pmatrix},$$

$$(\bar{Z}^k)^T = \bar{H} F_s^k - B_k^m A F_x^k + \bar{H} Q^T F_x^k,$$

where X_{n-m}^k and S_{n-m}^k are diagonal matrices in $R^{(n-m)\times(n-m)}$ with the diagonal elements being the last $(n - m)$ elements of x_k and s_k, respectively, and $\hat{e}_i$ is the ith column of the identity matrix I_{n-m}. Let

$$(Z^k)^T = \left(H_{2m+n+1} \begin{pmatrix} s_{m+1}^k e_{m+1} \\ 0 \\ x_{m+1}^k e_{m+1} \end{pmatrix}, \ldots, H_{2m+n+1} \begin{pmatrix} s_n^k e_n \\ 0 \\ x_n^k e_n \end{pmatrix} \right).$$

We have

$$(Z^k)^T = \begin{pmatrix} \sum_{j=m+1}^{n} (s_j^k \overline{H} e_j - x_j^k B_k^m A e_j + x_j^k \overline{H} Q^T e_j) \hat{e}_{j-m}^T \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (Z^k)^T \\ 0 \\ 0 \end{pmatrix}.$$

On the other hand, the residual vector of system (1) at the solution of the first $2m + n$ equations is:

$$r^k = \begin{pmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{pmatrix} - \begin{pmatrix} -Q & A^T & I_n \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} P\lambda^k \\ \hat{P}\beta^k \\ -r_c^k - QP\lambda^k - A^T \hat{P}\beta^k \end{pmatrix}$$

Let $(r^k)_{n-m}$ denote the last $n - m$ components of the vector r^k. Moreover, let γ_k^* satisfy $\bar{Z}^k \gamma_k^* = (r^k)_{n-m}$. Then, the solution of (1) is:

$$\begin{pmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} P\lambda^k + \overline{H}^T \gamma_k^* \\ \hat{P}\beta^k + (B_k^m)^T \gamma_k^* \\ -r_c^k - QP\lambda^k - A^T \hat{P}\beta^k + QP\lambda^k - A^T (B_k^m)^T \gamma_k^* \end{pmatrix}.$$

3. Numerical results

There, we compare the computational work of the ABS approach with the normal equations. By performing simple algebraic calculations, it can be verified that the linear system (1) is equivalent to the so-called normal equations:

$$(AF^k X^k A^T)\Delta y^k = -AF^k r_{xs}^k - AF^k X^k r_c^k - r_b^k \qquad (4)$$

$$\Delta x^k = F^k X^k A^T \Delta y^k - F^k r_{xs}^k + F^k X^k r_c^k \qquad (5)$$

$$\Delta s^k = -r_c^k + Q\Delta x^k - A^T \Delta y^k \qquad (6)$$

Here, we investigate the practical efficiency of the ABS approach for computing the search direction in IIPM for solving convex quadratic programming problems. The numerical results show that the ABS approach helps to reduce the required computational work and computing time. To verify the practical efficiency of the ABS approach, we implemented two primal-dual infeasible interior point algorithms for solving the convex quadratic problems in the MATLAB environment. In one algorithm, we used the normal equations approach to compute the search direction and in the other we used the ABS approach to compute the search direction. We then solved the test problems by the two programs, and recorded the numerical results in Table 1. In Table 1, m, n, CTE, FI and TE denote the number of rows and columns of the matrix A , the computing time of the IIPM for solving the search direction, the computing time for the information that is computed only in the first iteration of the IIPM, the total computing time of the IIPM with the ABS approach, that is, FI+CTE, respectively, and CTN denotes the computing time of the IIPM with the normal equations approach. Moreover, AE and AN denote the average time for computing the search direction in one iteration of the IIPM for the ABS and normal equations approaches, respectively. As the numerical results of Table 1 show, when m/n is close to zero, the ABS approach needs considerably less computing time. For example, when m=100 and n=945, the computing time for the ABS approach reduces by about 60 percent. As m/n becomes larger, the efficiency of the ABS approach, in comparison with the normal equations approach, is reduced and the normal equations approach becomes more efficient. When m/n is close to one, then the normal equations approach is more efficient than the ABS approach. For example, when m=900 and n=915, the required computing time of the normal equation approach is about 50 percent less than that of the ABS approach. Moreover, in most of our generated test problems, the average time for computing the search direction by the ABS approach is less than that of the normal equations approach.

Table 1: Comparison of the ABS and the normal equations approach

m	n	FI	CTE	TE	CTN	AE	AN
100	945	7.75	52.31	60.10	180.45	1.02	3.28
10	340	0.25	0.25	0.5	9.78	0.05	0.16
110	863	6.91	41.61	48.51	162.53	0.81	2.56
231	1000	25.73	78.58	104.31	175.01	1.54	4.17
41	920	2.83	36.01	38.84	153.25	0.71	2.84
73	768	4.31	42.51	46.83	131.03	0.83	2.04
324	500	30.10	13.25	43.34	33.33	0.27	0.83
100	200	0.55	0.89	1.44	2.29	0.02	0.05
450	831	109.44	65.97	175.41	126.03	1.22	3.23
100	145	0.91	0.83	1.73	0.01	2.5	0.06
700	775	204.89	129.67	334.56	149.42	2.31	3.93
800	995	705.77	395.81	1101.58	249.94	14.40	7.56
200	223	5.97	2.73	8.70	5.84	0.04	0.12
100	101	0.94	0.34	1.28	0.75	0.08	0.02
900	915	491.10	92.37	584.35	268.58	1.78	6.71
921	922	453.55	65.59	519.14	339.28	1.26	6.92
731	734	258.80	38.10	296.89	151.64	0.75	3.61
621	626	132.86	23.89	156.75	87.95	0.49	2.25

4. Conclusion

We compared the normal equations approach and ABS approach for computing the search direction of IIPMs for solving CQPs.

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