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Coloring the d^th power of the Cartesian product of two cycles and two paths

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Abstract

The d^{th} power graph G^d is defined on the vertex set of a graph G in such a way that distinct vertices with distance at most d in G are joined by an edge. In this paper the chromatic number of the d^{th} power of the Cartesian product $C_m \Box C_n$ of two cycles is studied and some of the exact value of $\chi((C_m \Box C_n)^d)$ with conditions are determined. Also the chromatic number of the d^{th} power of grid $P_m \Box P_n$ with some conditions are determined and the exact value of $\chi((P_m \Box P_n)^d)$ for $n = 2, 3$ is obtained.

Keywords: Chromatic number, power graphs, distance d coloring, Cartesian product of cycles, Cartesian product of paths.
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1. Introduction

G is a simple graph. $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A proper k-coloring of a graph G is a mapping c from $V(G)$ to the set $\{0, 1, ..., k-1\}$ such that $c(u) \neq c(v)$ whenever uv is an edge in $E(G)$. The chromatic number of G, denoted by $\chi(G)$, is the smallest k for which there exists a k-coloring of G.
Given two graphs G and H, the Cartesian product of these two graphs, denoted by $G \Box H$, is defined by $V(G \Box H) = V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. The d^{th} power graph G^d of a graph G is given by $V(G^d) = V(G)$ and two vertices u and v are adjacent in G^d, if their distance (number of edges in a shortest uv-path) in G is at most d. A distance d-coloring of G is a proper vertex coloring of G^d. The

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investigation of distance coloring was initiated by Kramer and Kramer in 1969, for a survey see [4]. The square of particular graphs (planar graphs) is sudied by some authors (see e.g. [2, 5, 6]). In [3], Jamison and Matthews considered the interaction between coding theory and distance k colorings of Hamming graphs and found some bounds for the chromatic number of these graphs. Chiang and Yan studied the chromatic number of the square of Cartesian products of paths and cycles [1].

Also Sopena and Wu studied the chromatic number of the square of Cartesian product $C_m \square C_n$ of two cycles and showed that this value is at most 7 except when (m, n) is $(3, 3)$, in such case the value is 9 and when (m, n) is $(4, 4)$ or $(3, 5)$, the chromatic number is 8 [8]. In [7], Selvakumar and Nithya represented the chromatic number of some graphs, so that in this coloring no two vertices have distance two get the same color.

In this paper some of the exact value of $\chi((C_m \square C_n)^d)$ for special conditions are determined. Also the chromatic number of the d^{th} power of grid $P_m \square P_n$ with some conditions are calculated and the exact value of $\chi((P_m \square P_n)^d)$ for $n = 2, 3$ is obtained. In a connected graph, distance is a metric and the following Proposition is conclouded by this property.

Proposition 1.1. *Let G be a connected graph. For every vertices $u, v, w \in V(G)$*

$$d(u, v) \leq d(u, w) + d(w, v).$$

2. Chromatic number of $(C_m \square C_n)^d$ for special cases

In this section some exact value of $\chi((C_m \square C_n)^d)$ is determined. In the following we have obviously lemma.

Lemma 2.1. *For every vertices $u, v \in V(C_n)$, $n \geq 3$, we have*

$$d(u, v) \leq \left\lfloor \frac{n}{2} \right\rfloor .$$

We refer to the vertices of $C_m \square C_n$ as an $m \times$n array $[v_{ij}]$, where in each row we have a copy of C_n and in each column a copy of C_m. Also we denote by $d_G(u, v)$, the distance between vertices u and v in graph G. Using Lemma 2.1 and Proposition 1.1, the following Lemma can be concluded.

Lemma 2.2. *Let $G = C_m \square C_n, m, n \geq 3$. For every vertices $v_{ij}, v_{i'j'} \in V((C_m \square C_n)^d)$, $1 \leq i, i' \leq m$ and $1 \leq j, j' \leq n$, we have*

$$d_G\left(v_{ij}, v_{i'j'}\right) \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor .$$

Theorem 2.3. *For every positive integers m, n, d, $3 \leq m \leq n \leq d$, we have*

$$\chi((C_m \square C_n)^d) = mn.$$

Proof. Since $3 \leq m \leq n \leq d$, there is a positive integer $k \geq 0$ such that $m = n - k$. Let $G = C_m \square C_n$. For every vertices $v_{ij}, v_{i'j'} \in V((C_m \square C_n)^d)$, Lemma 2.2 implies $d_G\left(v_{ij}, v_{i'j'}\right) \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-k}{2} \right\rfloor \leq n \leq d$, thus $(C_m \square C_n)^d$ is a complete graph with mn vertices and proof is completed. $\square$

Consider $3 \leq m \leq d \leq n$. For $d=n$ the chromatic number of graph $(C_m \square C_n)^d$ is concluded by Theorem 2.3. Suppose $d<n$, thus there are positive integers l, k, such that $m=n-k$, $d=n-l$ and $k \geq l \geq 1$.

Theorem 2.4. *For every positive integers n, l, k, $n \geq 3$, $l \geq 1$, $k \geq 2l-1$, $n-k \geq 3$ and $n-k \leq n-l$, we have*

$$\chi((C_{n-k} \square C_n)^{n-l}) = n(n-k).$$

Proof. Let $G=C_{n-k} \square C_n$. For every vertices $v_{ij}, v_{i'j'} \in V((C_{n-k} \square C_n)^{n-l})$, we show that $d_G(v_{ij}, v_{i'j'}) \leq n-l$. Lemma 2.2 implies $d_G\left(v_{ij}, v_{i'j'}\right) \leq \left\lfloor \frac{n-k}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$, but $k \geq 2l-1$ and it can be concluded $d_G(v_{ij}, v_{i'j'}) \leq n-l$. Thus $(C_{n-k} \square C_n)^{n-l}$ is a complete graph with $n(n-k)$ vertices and proof is completed. □

Corollary 2.5. *For every positive integers n, k, $n \geq 3$, $k \geq 1$, $n-k \geq 3$, we have*

$$\chi((C_{n-k} \square C_n)^{n-1}) = n(n-k).$$

Theorem 2.6. *For every odd integer $n \geq 3$ and every positive integer $l \geq 1$ and $n-(2l-2) \geq 3$, we have*

$$\chi((C_{n-(2l-2)} \square C_n)^{n-l}) = n(n-(2l-2)).$$

Proof. For every vertices $v_{ij}, v_{i'j'} \in V((C_{n-(2l-2)} \square C_n)^{n-l})$, using Lemma 2.2, it can be cocluded that $d_G(v_{ij}, v_{i'j'}) \leq n-l$ such that $G=C_{n-(2l-2)} \square C_n$. Therefore $(C_{n-(2l-2)} \square C_n)^{n-l}$ is a complete graph with $n(n-(2l-2))$ vertices and proof is completed.

□

Theorem 2.7. *For every even integer $n \geq 6$, we have*

$$\chi((C_{n-2} \square C_n)^{n-2}) = \frac{n(n-2)}{2}.$$

Proof. Let $G=C_{n-2} \square C_n$ and A_j be a set consists vertices in column j. Consider $S=\cup_{j=1}^{\frac{n}{2}} A_j$. For every vertices $v_{ij}, v_{i'j'} \in S \subset V((C_{n-2} \square C_n)^{n-2})$, $1 \leq i, i' \leq n-2$, $1 \leq j, j' \leq \frac{n}{2}$, using Proposition 1.1, we have

$$d_G(v_{ij}, v_{i'j'}) \leq d_G(v_{ij}, v_{i'j}) + d_G(v_{i'j}, v_{i'j'}). \tag{2.1}$$

But $d_G(v_{i'j}, v_{i'j'}) \leq |j-j'| \leq \frac{n}{2}-1$ and vertices $v_{ij}, v_{i'j}$ belong to a copy of cycle C_{n-2}, thus using Lemma 2.1 and equation (2.1) implies $d_G(v_{ij}, v_{i'j'}) \leq \frac{n-2}{2} + \frac{n}{2} - 1 = n-2$. Therefore the induced subgraph of $(C_{n-2} \square C_n)^{n-2}$ with vertex set S is a clique, so

$$\chi((C_{n-2} \square C_n)^{n-2}) \geq \frac{n(n-2)}{2}.$$

The following is a $\frac{n(n-2)}{2}$-coloring of $(C_{n-2}\square C_n)^{n-2}$.

$$c\colon V((C_{n-2}\square C_n)^{n-2})\to\{0,1,2,...,\frac{n(n-2)}{2}-1\}$$

$$c(v_{ij})=\begin{cases} i+(n-2)(j-1) & (\text{mod } t) \quad 1\leq i\leq n, 1\leq j\leq \frac{n}{2} \\ (i+\frac{n-2}{2})+(n-2)(j-(\frac{n}{2}+1)) & (\text{mod } t) \quad 1\leq i\leq \frac{n-2}{2}, r\leq j\leq n \\ (i-\frac{n-2}{2})+(n-2)(j-(\frac{n}{2}+1)) & (\text{mod } t) \quad \frac{n-2}{2}+1\leq i\leq n-2, r\leq j\leq n \end{cases}$$

and $t=\frac{n(n-2)}{2}$, $r=\frac{n}{2}+1$.

In this coloring we use two patterns A and A' such that

$$\begin{array}{llll} 1 & 1+n-2 & ... & 1+(n-2)(\frac{n}{2}-1) \\ 2 & 2+n-2 & ... & 2+(n-2)(\frac{n}{2}-1) \\ . & . & . & . \\ . & . & . & . \\ \frac{n-2}{2} & \frac{n-2}{2}+n-2 & ... & \frac{n-2}{2}+(n-2)(\frac{n}{2}-1) \end{array}$$

pattern A

and

$$\begin{array}{llll} \frac{n-2}{2}+1 & \frac{n-2}{2}+1+n-2 & ... & \frac{n-2}{2}+1+(n-2)(\frac{n}{2}-1) \\ \frac{n-2}{2}+2 & \frac{n-2}{2}+2+n-2 & ... & \frac{n-2}{2}+2+(n-2)(\frac{n}{2}-1) \\ . & . & . & . \\ . & . & . & . \\ n-3 & 2n-5 & ... & \frac{n(n-2)}{2}-1 \\ n-2 & 2n-4 & ... & 0 \end{array}$$

pattern A'

A $\frac{n(n-2)}{2}$-coloring of $(C_{n-2}\square C_n)^{n-2}$ can be obtained by the following pattern:

$$\begin{array}{ll} A & A' \\ A' & A. \end{array}$$

□

Theorem 2.8. *For every positive integers $n, k, n\geq 3, k\geq 2$ and $n-k\geq 3$, we have*

$$\chi((C_{n-k}\square C_n)^{n-2})=\begin{cases} \frac{n(n-k)}{2} & k=2,\ n \text{ even} \\ n(n-k) & k=2,\ n \text{ odd} \\ n(n-k) & k\geq 3. \end{cases}$$

Proof. If $k=2$, for even and odd n, the result is concluded by Theorems 2.6 and 2.7, respectively and for $k\geq 3$, the result is obtained by Theorem 2.4. □

Cosider $3 \leq d \leq m \leq n$. For $d=m=n$ the chromatic number of graph $(C_m \Box C_n)^d$ is concluded by Theorem 2.3. Consider $3 \leq d < m = n$, in the following we show a lower bound for chromatic number of these graphs.

Theorem 2.9. *For every positive integers $n \geq 3, l \geq 1$ and $\lfloor \frac{n}{2} \rfloor - l + 1 \geq 1$, we have*

$$\chi((C_n \Box C_n)^{n-l}) \geq n(\left\lfloor \frac{n}{2} \right\rfloor - l + 1).$$

Proof. Let $G = C_n \Box C_n$ and A_j be a set consists vertices in column j. Consider $S = \cup_{j=1}^{M} A_j$ and $M = \lfloor \frac{n}{2} \rfloor - l + 1$. For every vertices $v_{ij}, v_{i'j'} \in S \subset V((C_n \Box C_n)^{n-2})$, $1 \leq i, i' \leq n$, $1 \leq j, j' \leq M$, using Proposition 1.1, we have

$$d_G(v_{ij}, v_{i'j'}) \leq d_G(v_{ij}, v_{i'j}) + d_G(v_{i'j}, v_{i'j'}). \tag{2.2}$$

But $d_G(v_{i'j}, v_{i'j'}) \leq |j - j'| \leq M - 1$ and vertices $v_{ij}, v_{i'j}$ belong to a copy of cycle C_n, thus using Lemma 2.1 and equation (2.2) implies $d_G(v_{ij}, v_{i'j'}) \leq \lfloor \frac{n}{2} \rfloor + M - 1 \leq n - l$. Therefore the induced subgraph of $(C_n \Box C_n)^{n-l}$ with vertex set S is a clique, so

$$\chi((C_n \Box C_n)^{n-l}) \geq n(\left\lfloor \frac{n}{2} \right\rfloor - l + 1).$$

□

Theorem 2.10. *For every positive integer $n \geq 3$, we have*

$$\chi((C_n \Box C_n)^{n-1}) = \begin{cases} \frac{n^2}{2} & n \text{ even} \\ n^2 & n \text{ odd.} \end{cases}$$

Proof. Suppose n is even, using Theorem 2.9, it can be concluded

$$\chi((C_n \Box C_n)^{n-1}) \geq \frac{n^2}{2}.$$

The following is a $\frac{n^2}{2}$-coloring of $(C_n \Box C_n)^{n-1}$.

$$c : V((C_n \Box C_n)^{n-1}) \to \{0, 1, 2, ..., \frac{n^2}{2} - 1\}$$

$$c(v_{ij}) = \begin{cases} i + n(j-1) & \pmod{\frac{n^2}{2}} \quad 1 \leq i \leq n, 1 \leq j \leq \frac{n}{2} \\ (i + \frac{n}{2}) + n(j - (\frac{n}{2} + 1)) & \pmod{\frac{n^2}{2}} \quad 1 \leq i \leq \frac{n}{2},\ \frac{n}{2} + 1 \leq j \leq n \\ (i - \frac{n}{2}) + n(j - (\frac{n}{2} + 1)) & \pmod{\frac{n^2}{2}} \quad \frac{n}{2} + 1 \leq i \leq n, \frac{n}{2} + 1 \leq j \leq n. \end{cases}$$

In this coloring we use two patterns A and A' such that

$$\begin{array}{llll} 1 & 1+n & ... & 1+n(\frac{n}{2}-1) \\ 2 & 2+n & ... & 2+n(\frac{n}{2}-1) \\ . & . & . & . \\ . & . & . & . \\ \frac{n}{2} & \frac{n}{2}+n & ... & \frac{n}{2}+n(\frac{n}{2}-1) \end{array}$$

pattern A

and

$$\begin{array}{llll}
\frac{n}{2}+1 & \frac{n}{2}+n+1 & ... & \frac{n}{2}+n(\frac{n}{2}-1)+1 \\
\frac{n}{2}+2 & \frac{n}{2}+n+2 & ... & \frac{n}{2}+n(\frac{n}{2}-1)+2 \\
. & . & . & . \\
. & . & . & . \\
n-1 & 2n-1 & ... & \frac{n^2}{2}-1 \\
n & 2n & ... & 0
\end{array}$$

pattern A'

A $\frac{n^2}{2}$-coloring of $(C_n \square C_n)^{n-1}$ can be obtained by the following pattern:

$$\begin{array}{ll}
A & A' \\
A' & A.
\end{array}$$

Let n be odd and $G{=}C_n \square C_n$. For every vertices $v_{ij}, v_{i'j'} \in V((C_n \square C_n)^{n-1})$, $1 \leq i, i', j, j' \leq n$, using Proposition 1.1 and Lemma 2.1, we have

$$d_G(v_{ij}, v_{i'j'}) \leq n-1.$$

Thus the graph $(C_n \square C_n)^{n-1}$ is a complete graph with n^2 vertices and proof is completed. □

Theorem 2.11. *For every even integer $n \geq 4$, we have*

$$\chi((C_{n-1} \square C_n)^{n-2}) = \frac{n(n-1)}{2}.$$

Proof. Let $G{=}C_{n-1} \square C_n$, and A_j be a set consists vertices in column j. Consider $S{=}\cup_{j=1}^{\frac{n}{2}} A_j$. For every vertices $v_{ij}, v_{i'j'} \in S \subset V((C_{n-1} \square C_n)^{n-2})$, $1 \leq i, i' \leq n-1$, $1 \leq j, j' \leq \frac{n}{2}$, we show that $d_G(v_{i'j}, v_{i'j'}) \leq n-2$. Using Proposition 1.1, we have

$$d_G(v_{ij}, v_{i'j'}) \leq d_G(v_{ij}, v_{i'j}) + d_G(v_{i'j}, v_{i'j'}). \tag{2.3}$$

If $1 \leq i, i' \leq \frac{n}{2}$ or $\frac{n}{2}+1 \leq i, i' \leq n-1$, the Equation (2.3) implies $d_G(v_{ij}, v_{i'j'}) \leq |i-i'| + |j-j'| \leq n-2$.
If $1 \leq i \leq \frac{n}{2}$ and $\frac{n}{2}+1 \leq i' \leq n-1$, we have $d_G(v_{ij}, v_{i'j}) \leq \frac{n}{2}-1$ and $d_G(v_{i'j}, v_{i'j'}) \leq \frac{n}{2}-1$, thus the Equation (2.3) implies $d_G(v_{ij}, v_{i'j'}) \leq n-2$. For $1 \leq i' \leq \frac{n}{2}$ and $\frac{n}{2}+1 \leq i \leq n-1$, it will be similarly proved that $d_G(v_{ij}, v_{i'j'}) \leq n-2$. Therefore the induced subgraph of $(C_{n-1} \square C_n)^{n-2}$ with vertex set S is a clique, so $\chi((C_{n-1} \square C_n)^{n-2}) \geq \frac{n(n-1)}{2}$.

The following is a $\frac{n(n-1)}{2}$-coloring of $(C_{n-1} \square C_n)^{n-2}$.

$$c{:}V((C_{n-1} \square C_n)^{n-2}) \to \{0, 1, 2, ..., \frac{n(n-1)}{2}-1\}$$

$$c(v_{ij}) = \begin{cases} i+n(j-1) & (\text{mod } t) \quad 1 \leq i \leq n-1, 1 \leq j \leq \frac{n}{2} \\ (i+\frac{n}{2}-1)+n(j-(\frac{n}{2}+1)) & (\text{mod } t) \quad 1 \leq i \leq \frac{n}{2},\ \frac{n}{2}+1 \leq j \leq n \\ (i-(\frac{n}{2}-1))+n(j-(\frac{n}{2}+1)) & (\text{mod } t) \quad \frac{n}{2}+1 \leq i \leq n-1,\ \frac{n}{2}+1 \leq j \leq n \end{cases}$$

and $t{=}\frac{n(n-1)}{2}$.

In this coloring we use two patterns A and A' such that

$$\begin{array}{llll} 1 & 1{+}n & ... & 1{+}n(\frac{n}{2}{-}1) \\ 2 & 2{+}n & ... & 2{+}n(\frac{n}{2}{-}1) \\ . & . & . & . \\ . & . & . & . \\ \frac{n}{2} & \frac{n}{2}{+}n & ... & \frac{n}{2}{+}n(\frac{n}{2}{-}1) \end{array}$$

pattern A

and

$$\begin{array}{llll} \frac{n}{2}{+}1 & \frac{n}{2}{+}n{+}1 & ... & \frac{n}{2}{+}n(\frac{n}{2}{-}1)+1 \\ \frac{n}{2}{+}2 & \frac{n}{2}{+}n{+}2 & ... & \frac{n}{2}{+}n(\frac{n}{2}{-}1)+2 \\ . & . & . & . \\ . & . & . & . \\ n{-}2 & 2n{-}3 & ... & \frac{n(n-1)}{2}{-}1 \\ n{-}1 & 2n{-}2 & ... & 0 \end{array}$$

pattern A'

A $\frac{n(n-1)}{2}$-coloring of $(C_{n-1}\square C_n)^{n-2}$ can be obtained by the following pattern:

$$\begin{array}{ll} A & A' \\ A' & A. \end{array}$$

□

3. Chromatic number of $(P_m\square P_n)^d$ for special cases

Similar to graph $C_m\square C_n$, We refer to the vertices of $P_m\square P_n$ as an $m{\times}n$ array $[v_{ij}]$. Using Proposition 1.1, the following Lemma can be concluded.

Lemma 3.1. *Let $G = P_m\square P_n$. For every vertices $v_{ij}, v_{i'j'} \in V((P_m\square P_n)^d)$, $1 \le i, i' \le m$ and $1 \le j, j' \le n$, we have*

$$d_G(v_{ij}, v_{i'j'}){\le}|i{-}i'| + |j{-}j'|.$$

Theorem 3.2. *For every positive integers m, n, d, $d - (n - 2) \ge m \ge 2$, $n \ge 2$, we have*

$$\chi((P_m\square P_n)^d) = mn.$$

Proof. Let $G{=}P_m\square P_n$. For every vertices $v_{ij}, v_{i'j'}{\in}V((P_m\square P_n)^d)$, $1{\le}i, i'{\le}m$ and $1{\le}j, j'{\le}n$, using Lemma 3.1, therefore it can be concluded $d_G(v_{ij}, v_{i'j'}){\le}d$, thus $(P_m\square P_n)^d$ is a complete graph and proof is completed. □

Theorem 3.3. *For every positive integers m, n, d, $m \ge d - n + 3 \ge 3$, $m \ge n \ge 2$, we have*

$$\chi((P_m\square P_n)^d){\ge}nd{-}(n{-}1)(n{-}2).$$

Proof. Let $G{=}P_m\square P_n$, $A_k{=}\{v_{1k}, v_{2k}, ..., v_{d-(n-3)k}\}$, $1{\le}k{\le}n{-}1$ and

$$A_n{=}\{v_{2n}, v_{3n}, ..., v_{d-n+2\ n}\}.$$

Consider $S{=}\ (\cup_{k=1}^{n-1}A_j){\cup}A_n$. For every vertices $v_{ij}, v_{i'j'}{\in}S$, using Lemma 3.1 therefore it can be concluded $d_G(v_{ij}, v_{i'j'}){\le}d$. Thus the induced subgraph H of $(P_m\square P_n)^d$ with vertex set S is a complete graph, so $\chi((P_m\square P_n)^d){\ge}\chi(H) = nd{-}(n{-}1)(n{-}2)$. □

Theorem 3.4. *For every positive integers n, d, $d-n+3 \geq 3$, $n \geq 2$, we have*

$$\chi((P_{d-n+3} \square P_n)^d) = nd-(n-1)(n-2).$$

Proof. Theorem 3.3 implies, $\chi((P_{d-n+3} \square P_n)^d) \geq nd-(n-1)(n-2)$. The following is a $(nd-(n-1)(n-2))$-coloring of $(P_{d-n+3} \square P_n)^d$.

$$c:V((P_{d-n+3} \square P_n)^d) \rightarrow \{1, 2, ..., nd-(n-1)(n-2)\}$$

$$c(v_{ij}) = \begin{cases} i+(d-n+3)(j-1) & (i,j) \neq (1,n), (d-n+3,n) \\ d-n+3 & (i,j) = (1,n) \\ 1 & (i,j) = (d-n+3,n). \end{cases}$$

□

4. Chromatic number of graphs $(P_m \square P_n)^d$ for small n

Theorem 4.1. *For every positive integers m, d, we have*

$$\chi((P_m \square P_2)^d) = \begin{cases} 2m & d \geq m \geq 2 \\ 2d & m \geq d+1 \geq 3. \end{cases}$$

Proof. If $d \geq m \geq 2$, Theorem 3.2 implies $\chi((P_m \square P_2)^d) = 2m$. Suppose that $m \geq d+1 \geq 3$, thus Theorem 3.3 implies $\chi((P_m \square P_2)^d) \geq 2d$. The following is a $2d$-coloring of this graph.

$$c\colon V((P_m \square P_2)^d) \rightarrow \{0, 1, 2, ..., 2d-1\}$$

$$c(v_{ij}) = i+d(j-1) (\mathrm{mod}\ 2d).$$

□

Theorem 4.2. *For every positive integer $d \geq 3$, we have*

$$\chi((P_{d+1} \square P_3)^d) = 3d-1.$$

Proof. The function c is a $(3d-1)$-coloring of $(P_{d+1} \square P_3)^d$.

$$c\colon V((P_{d+3} \square P_3)^d) \rightarrow \{1, 2, ..., 3d-1\}$$

$$c(v_{ij}) = \begin{cases} i+d(j-1) & 1 \leq i \leq d, 1 \leq j \leq 3, (i,j) \neq (1,3), (d,3) \\ d & (i,j) = (1,3) \\ 1 & (i,j) = (d,3) \\ 2d+1 & (i,j) = (d+1,1) \\ 3d-1 & (i,j) = (d+1,2) \\ 2 & (i,j) = (d+1,3). \end{cases}$$

Now, we show that coloring of this graph with less than $3d-1$ colors is impossible. Suppose there is a $(3d-2)$-coloring of this graph. Consider

$$\begin{aligned} A_1 &= \{v_{11}, v_{21}, ..., v_{d1}\}, \\ A_2 &= \{v_{12}, v_{22}, ..., v_{d2}\}, \\ A_3 &= \{v_{23}, v_{33}, ..., v_{d-13}\}, \end{aligned}$$

and $S{=}\cup_{k=1}^{3}A_k$. The induced subgraph of $(P_{d+1}\Box P_3)^d$ with vertex set S is a complete graph and for coloring of this subgraph we need $3d{-}2$ colors. Without loss of generality, suppose these vertices are colored in the following manner.

$$\begin{array}{lll} 1 & d{+}1 & \bullet \\ 2 & d{+}2 & 2d{+}1 \\ 3 & d{+}3 & 2d{+}2 \\ . & . & . \\ . & . & . \\ . & . & . \\ d{-}1 & 2d{-}1 & 3d{-}2 \\ d & 2d & \bullet \\ \bullet & \bullet & \bullet \end{array}$$

Consider vertex v_{13}. This vertex is adjacent to all vertices in S except v_{d1}. Therefore in $(3d{-}2)$-coloring it should be colored with the color assigned to vertex v_{d1} i.e. color d. Similarly, vertex v_{d3} is adjacent to all vertices in S except v_{11} and it should be colored with the color 1. Now consider vertex v_{d+12}. This vertex is adjacent to all vertices in $V((P_{d+1}\Box P_3)^d)$ except vertices $v_{11}$and v_{13}. Thus in this coloring, it should be colored with colors allocated to vertices v_{11} or v_{13} i.e. colors 1 or d, but these colors exsist in vertices neighbor to v_{d+12}. Therefore, it is impossible and proof is completed. □

Theorem 4.3. *For every positive integers m, d, we have*

$$\chi((P_m\Box P_3)^d) = \begin{cases} 3m & d{-}1{\geq}m{\geq}3 \\ 3d{-}2 & m{=}d{\geq}3 \\ 3d{-}1 & m{\geq}d{+}1{\geq}4. \end{cases}$$

Proof. If $d{-}1{\geq}m{\geq}3$ or $m{=}d{\geq}3$, thus the results can be concluded from Theorem 3.2 and Theorem 3.4, respectively. Now suppose that $m{\geq}d{+}1{\geq}4$. Since $(P_{d+1}\Box P_3)^d$ is a subgraph of $(P_m\Box P_3)^d$, Theorem 4.2 implies $\chi((P_m\Box P_3)^d){\geq}3d{-}1$. The following is a $(3d{-}1)$-coloring of this graph.

$$c{:}V((P_m\Box P_2)^d){\to}\{0, 1, 2, ..., 3d{-}2\}$$

$$c(v_{ij}) = i{+}d(j{-}1)(\text{mod } \ 3d{-}1).$$

□

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