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Null controllability of fractional stochastic delay integro-differential equations

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Abstract

Sufficient conditions for exact null controllability of semilinear fractional stochastic delay integro-differential equations in Hilbert space are established. The required results are obtained based on fractional calculus, semigroup theory, Schauder's fixed point theorem and stochastic analysis. In the end, an example is given to show the application of our results.

Keywords: Fractional calculus, null controllability, fractional stochastic delay integrodifferential equation, Schauder's fixed point theorem.

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1. Introduction

Stochastic differential and integro-differential equations have attracted great interest due to their applications in characterizing many problems in physics, biology, mechanics and so on. Fractional differential equations have received great attention due to their applications in many important applied fields such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics and so on, for more details (see [1–5, 9–12, 18, 21, 23, 25, 26]). On the other hand, the notion of controllability of dynamical systems is one of the fundamental concepts in mathematical control theory which plays pivotal role in many areas of science and engineering. The problem of controllability of nonlinear stochastic or deterministic system has been discussed by many authors (see [6, 8, 13–17, 24]). The problems of exact and exact null controllability are to be distinguished. Exact controllability enables to steer the system to arbitrary final state while exact null controllability means that the system can be steered to zero state.

In this paper, we investigate the exact null controllability of semilinear fractional stochastic delay

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integrodifferential equation in the following form

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + Bu(t) + F(t, x_t) + \int_0^t G(s, x_s)d\omega(s), \quad t \in J = [0, b], \\ x_0(\zeta) = \phi(\zeta), \quad \zeta \in [-r, 0], \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$, ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, A is the infinitesimal generator of an analytic compact semigroup of bounded linear operators $S(t)$, $t \geqslant 0$, on a separable Hilbert space H with inner product $\langle .,. \rangle$ and norm $\| \ . \ \|$. This means that there exists a $M \geqslant 1$ such that $\|S(t)\| \leqslant M$. The control function $u(\cdot)$ is given in $L_2(J, U)$, the Hilbert space of admissible control functions with U as a separable Hilbert space. The symbol B stands for a bounded linear operator from U into H. Here ω is an H-valued Wiener process associated with a positive, nuclear covariance operator Q, F is an H-valued map and G is a $L(K, H)$-valued map both defined on $J \times C_r$ (where K is a real separable Hilbert space and $L(K, H)$ is the space of all bounded, linear operators from K to H, we write simply $L(H)$ if $H = K$) and ϕ is an C_r-valued random variable independent of ω with finite second moment. Here $C_r = C([-r, 0], H)$ is a Banach space of all continuous functions $\phi : [-r, 0] \to H$ endowed with norm $\|\phi\| = \sup\{\|\phi(\vartheta)\| : -r \leqslant \vartheta \leqslant 0\}$. Also for $x(\cdot) \in C([-r, b], H)$ we have $x_t(\cdot) \in C_r$ for $t \in J$, $x_t(\vartheta) = x(t + \vartheta)$ for $\vartheta \in [-r, 0]$.

To the best of our knowledge, there is no work reported on null controllability of semilinear fractional stochastic delay integro-differential equations in Hilbert space of the form (1.1). Thus, we will make the first attempt to study such problem in this paper. The rest of this paper is organized as follows. In Section 2, we present some basic definitions and lemmas which are useful to prove the main results. In Section 3, we investigate the sufficient conditions for null controllability of semilinear fractional stochastic delay integro-differential equations in Hilbert space. In the final section, we consider an example to verify the theoretical results.

2. Preliminaries

In this section, we provide definitions, lemmas and notations necessary to establish our main result.

Definition 2.1 ([21, 23])**.** The fractional integral of order α with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \ \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([21, 23])**.** The Caputo fractional derivative of order α with the lower limit zero for a function f can be written as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ 0 \leqslant n-1 < \alpha < n.$$

If f is an abstract function with values in H, then the integrals appearing in the above definitions are taken in Bochner's sense.

Throughout this paper, $(H, \| \ . \ \|)$ and $(K, \| \ . \ \|_K)$ denote two real Hilbert spaces. Let (Ω, Υ, P) be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Upsilon_t : t \in J\}$ satisfying $\Upsilon_t \subset \Upsilon$. An H-valued random variable is an Υ-measurable function $x(t) : \Omega \to H$ and a collection of random variables $\Psi = \{x(t, \omega) : \Omega \to H | t \in J\}$ is called a stochastic process. Usually we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to H$ in the place of Ψ. Let $\beta_n(t)$ $(n = 1, 2, \dots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent over (Ω, Υ, P). Set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \ t \geqslant 0,$$

where λ_n, $(n = 1, 2, \dots)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1, 2, \dots)$ is a complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $\omega(t)$ is called Q-Wiener process.

We assume that $\Upsilon_t = \sigma\{\omega(s) : 0 \leqslant s \leqslant t\}$ is the σ-algebra generated by ω. For $\phi \in L(K,H)$ we define

$$\| \phi \|_Q^2 = Tr(\phi Q \phi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \phi e_n \|^2 .$$

If $\| \phi \|_Q^2 < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\phi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\| . \|_Q$ where $\| \phi \|_Q^2 = \langle \phi, \phi \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly-measurable, square-integrable, H-valued random variables, denoted by $L_2(\Omega,H)$, is a Banach space equipped with norm $\| x(\cdot) \|_{L_2(\Omega,H)} = (E \| x(.,\omega) \|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $E(x) = \int_\Omega x(\omega) dP$. An important subspace of $L_2(\Omega,H)$ is given by $L_2^0(\Omega,H) = \{x \in L_2(\Omega,H),$ x is Υ_0-measurable$\}$. Let $C([-r,b], L_2(\Omega,H))$ be the Banach space of all continuous maps from $[-r,b]$ into $L_2(\Omega,H)$ satisfying the condition $\sup_{t\in J} E \| x(t) \|^2 < \infty$. Let Y be the closed subspace of all continuous process x that belong to the space $C([-r,b], L_2(\Omega,H))$ consisting of Υ_t-adapted and measurable processes. One can prove that this is a Banach space when equipped with the norm

$$\| x \|_Y^2 = \sup_{t\in J}\{E \| x(t) \|^2 : x \in C([-r,b], L_2(\Omega,H))\}.$$

Theorem 2.3 ([19, Banach fixed point theorem]). *Let D be a closed subset of a Banach space X and let Φ be a contraction mapping from D to D, i.e.,*

$$\|\Phi(y) - \Phi(z)\| \leqslant \delta \|y - z\| \text{ for all } y, z \in D; \quad 0 < \delta < 1.$$

Then there exists a unique $z \in \Phi$ such that $\Phi(z) = z$.

Theorem 2.4 ([19, Schauder fixed point theorem]). *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $\Phi : D \to D$ a completely continuous operator (i.e., Φ is continuous and $\Phi(D)$ is relatively compact). Then Φ has at least one fixed point.*

Our aim is to study the exact null controllability problem for (1.1). First, we give the definitions of mild solution and exact null controllability for it.

Definition 2.5. We say $x \in C([-r,b], L_2(\Omega,H))$ is a mild solution to (1.1) if it satisfies that

$$\begin{cases} x(t) = S_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[F(s,x_s) + Bu(s) + \int_0^s G(\tau,x_\tau) d\omega(\tau)] ds, & t \in J, \\ x_0(\zeta) = \phi(\zeta), \quad \zeta \in [-r,0], \end{cases}$$

where

$$S_\alpha(t)x = \int_0^\infty \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \qquad T_\alpha(t)x = \alpha \int_0^\infty \theta \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta$$

with η_α a probability density function defined on $(0,\infty)$.

Remark 2.6. $\int_0^\infty \theta \eta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}$ (see[1, 7]).

Lemma 2.7 ([27]). *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(I) *for any fixed $x \in H$, $\| S_\alpha(t)x \| \leqslant M \| x \|$, $\| T_\alpha(t)x \| \leqslant \frac{\alpha M}{\Gamma(\alpha+1)} \| x \|$;*
(II) *$\{S_\alpha(t),\ t \geqslant 0\}$ and $\{T_\alpha(t),\ t \geqslant 0\}$ are strongly continuous;*
(III) *for every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators if $S(t),\ t > 0$ is compact.*

Definition 2.8. The system (1.1) is said to be exact null controllable on the interval J if for every ϕ and preassigned time b there exists a stochastic control $u \in L_2(J, U)$ such that the solution $x(t)$ of the system (1.1) satisfies $x(b) = 0$.

Define

$$L_0^b u = \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) B u(s) ds : \quad L_2(J, U) \to H,$$

$$N_0^b(y, F, G) = S_\alpha(b) y + \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s)[F(s) + \int_0^s G(\tau) d\omega(\tau)] ds : \quad H \times L_2(J, U) \to H,$$

Consider the fractional linear system

$$\begin{cases} {}^c D^\alpha y(t) = A y(t) + B u(t) + F(t) + \int_0^t G(s) d\omega(s), & t \in J = [0, b], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

associated with the system.

Definition 2.9. The system (2.1) is said to be exactly null controllable on J if $\operatorname{Im} L_0^b \supset \operatorname{Im} N_0^b$.

Remark 2.10. It is known that, (see[20]), system (2.1) is exactly null controllable if and only if there exists $\gamma > 0$ such that $\|(L_0^b)^*\|^2 \geqslant \gamma \|(N_0^b)^* y\|^2$ for all $y \in H$.

Lemma 2.11 ([21]). *Suppose that the linear system (2.1) is exactly null controllable on* J. *Then the linear operator* $(L_0)^{-1} N_0^b : H \times L_2(J, H) \to L_2(J, U)$ *is bounded and the control*

$$u(t) = -(L_0)^{-1}\Bigg[S_\alpha(b) y_0 + \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) F(s) ds + \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s)[\int_0^s G(\tau) d\omega(\tau)] ds\Bigg](t),$$

transfers the system (2.1) from y_0 *to* 0, *where* L_0 *is the restriction of* L_0^b *to* $[\ker L_0^b]^\perp$, $F \in L_2(J, H)$ *and* $G \in L_2(J, L(K, H))$.

3. Exact null controllability

In this section, we formulate sufficient conditions for exact null controllability of the system (1.1). For this purpose, we impose the following conditions on data of the problem.

(H1) The fractional linear system (2.1) is exactly null controllable on J.

(H2) The functions $F : J \times C_r \to H$ and $G : J \times C_r \to L(K, H)$ are continuous, for each $x \in H$ the functions $F(\cdot, x) : J \to H$ and $G(\cdot, x) : J \to L(K_Q, H)$ are strongly Υ_t-measurable and there exists functions $\lambda(\cdot) \in L_1(J, R^+)$ and $g(\cdot) \in L_1(C_r, R^+)$ be such that

$$E\|F(t, \phi)\|^2 \vee E\|G(t, \phi)\|_Q^2 \leqslant \lambda(t) g(\phi) \textit{ for } (t, \phi) \in J \times C_r.$$

In this paper $\|\lambda\| = \int_0^b \lambda(s) ds$ and $M_B = \|B\|$.

Theorem 3.1. *If the hypotheses*(H1)-(H2) *are satisfied, then the system* (1.1) *is exactly null controllable on* J *provided that there exists a constant* $r > 0$ *such that*

$$M^2 E\|\phi(0)\|^2 + \frac{\alpha^2 \|(L_0)^{-1}\|^2 M^4 M_B^2 b^{2\alpha-1}}{(2\alpha-1)\,\Gamma^2(\alpha+1)}\left[E\|\phi(0)\|^2 + \frac{\alpha^2 M^2 b^{2\alpha-1} \psi(r) \|\lambda\|}{(2\alpha-1)\Gamma^2(\alpha+1)}(1 + b\operatorname{Tr}(Q))\right] + \frac{\alpha^2 M^2 b^{2\alpha-1} \psi(r) \|\lambda\|}{(2\alpha-1)\Gamma^2(\alpha+1)}[1 + b\operatorname{Tr}(Q)] \leqslant r.$$

Proof. Using the hypothesis (H2) for an arbitrary $x(\cdot)$ define the operator Φ on Y as follows

$$(\Phi x)(t)=\begin{cases}\phi(t), \quad t\in[-r,0],\\ S_\alpha(t)\phi(0)+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)F(s,x_s)ds,\\ +\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)(\int_0^s G(\tau,x_\tau)d\omega(\tau))ds, \quad t\in J.\end{cases} \tag{3.1}$$

It will be shown that the operator Φ from Y into itself has a fixed point.

On the Banach space Y, we introduce the set

$$Y_r=\{x(\cdot)\in Y: x(t)=\phi(t),\ t\in[-r,0],\ \|x\|_Y^2\leqslant r \ \text{ for all }\ t\in[-r,b]\},$$

where $r>0$. The proof will be given in several steps.

Step 1. The control $u(\cdot)$ is bounded on Y_r. Indeed,

$$\begin{aligned}\|u\|_Y^2=\sup_{t\in J}E\|u\|^2&\leqslant\|(L_0)^{-1}\|^2\left\{M^2E\|\phi(0)\|^2+\frac{\alpha^2M^2}{\Gamma^2(\alpha+1)}\int_0^b(b-s)^{2\alpha-2}ds\int_0^b E\|F(s,x_s)\|^2ds\right\}\\&+\frac{\alpha^2M^2\|(L_0)^{-1}\|^2}{\Gamma^2(\alpha+1)}\int_0^b(b-s)^{2\alpha-2}ds\int_0^b\|E\int_0^s G(\tau,x_\tau)d\omega(\tau)\|^2ds\\&\leqslant\|(L_0)^{-1}\|^2M^2\left[E\|\phi(0)\|^2+\frac{\alpha^2M^2b^{2\alpha-1}\psi(r)\|\lambda\|}{(2\alpha-1)\Gamma^2(\alpha+1)}(1+bTr(Q))\right].\end{aligned} \tag{3.2}$$

Step 2. We show that Φ maps Y_r into itself. If $x(\cdot)\in Y_r$, from (3.1) and (3.2) for $t\in J$, we have

$$\begin{aligned}\|\Phi x\|_Y^2=\sup_{t\in J}E\|(\Phi x)(t)\|^2&\leqslant M^2E\|\phi(0)\|^2+\frac{\alpha^2M^2M_B^2b^{2\alpha-1}}{(2\alpha-1)\ \Gamma^2(\alpha+1)}\int_0^b E\|u(s)\|^2ds\\&+\frac{\alpha^2M^2b^{2\alpha-1}}{(2\alpha-1)\ \Gamma^2(\alpha+1)}\left[\int_0^t E\|F(s,x_s)\|^2ds+\int_0^t E\|\int_0^s G(\tau,x_\tau)d\omega(\tau)\|^2ds\right]\\&\leqslant M^2E\|\phi(0)\|^2+\frac{\alpha^2\|(L_0)^{-1}\|^2M^4M_B^2b^{2\alpha-1}}{(2\alpha-1)\ \Gamma^2(\alpha+1)}\left[E\|\phi(0)\|^2+\frac{\alpha^2M^2b^{2\alpha-1}\psi(r)\|\lambda\|}{(2\alpha-1)\Gamma^2(\alpha+1)}(1+bTr(Q))\right]\\&+\frac{\alpha^2M^2b^{2\alpha-1}\psi(r)\|\lambda\|}{(2\alpha-1)\Gamma^2(\alpha+1)}[1+bTr(Q)]\leqslant r.\end{aligned}$$

Hence, Φ maps Y_r into itself.

Step 3. The operator Φ maps Y_r into equicontinuous set of Y_r.

Let $0<t_1<t_2\leqslant b$. For each $x\in Y_r$, we have

$$\begin{aligned}&E\|(\Phi x)(t_2)-(\Phi)x(t_1)\|^2\\&\leqslant\sup_{t\in J}E\|[S_\alpha(t_2)-S_\alpha(t_1)]\phi(0)\|^2\\&+\|B\|^2\|(L_0)^{-1}\|^2E\|\int_0^{t_1}[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)-(t_1-s)^{\alpha-1}T_\alpha(t_2-s)]\{S_\alpha(b)\phi(0)\\&+\int_0^b(b-\zeta)^{\alpha-1}T_\alpha(b-\zeta)F(\zeta,x_\zeta)d\zeta+\int_0^b(b-\zeta)^{\alpha-1}T_\alpha(b-\zeta)(\int_0^\zeta G(\tau,x_\tau)d\omega(\tau))d\zeta\}ds\|^2\\&+\|B\|^2\|(L_0)^{-1}\|^2E\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)\{S_\alpha(b)\phi(0)\\&+\int_0^b(b-\zeta)^{\alpha-1}T_\alpha(b-\zeta)F(\zeta,x_\zeta)d\zeta+\int_0^b(b-\zeta)^{\alpha-1}T_\alpha(b-\zeta)(\int_0^\zeta G(\tau,x_\tau)d\omega(\tau))d\zeta\}ds\|^2\\&+\sup_{t\in J}E\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)[F(s,x_s)+\int_0^s G(\tau,x_\tau)d\omega(\tau)]ds\|^2\\&+\sup_{t\in J}E\|\int_0^{t_1}[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)-(t_1-s)^{\alpha-1}T_\alpha(t_1-s)]\left[F(s,x_s)+\int_0^s G(\tau,x_\tau)d\omega(\tau)\right]ds\|^2.\end{aligned} \tag{3.3}$$

From the above fact, we see that $E\|(\Phi x)(t_2)-(\Phi)x(t_1)\|^2$ tends to zero as $t_2\to t_1$. The equicontinuity for the cases $t_1<t_2\leqslant 0$ and $t_1\leqslant 0\leqslant t_2$ follows from the uniform continuity of ϕ on the interval $[-r,0]$.

Step 4. For arbitrary $t\in J$ the set $V(t)=\{(\Phi x)(t):x(\cdot)\in Y_r\}$ is relatively compact. In fact, the case where $t=0$ is trivial, since $V(0)=\{\phi(0)\}$. So, let $0<t\leqslant b$ be fixed. For $0<\xi\leqslant t$ and arbitrary $\delta>0$, take

$$
\begin{aligned}
(\Phi^{\xi,\delta}x)(t) &= \int_\delta^\infty \eta_\alpha(\theta)S(t^\alpha\theta)\phi(0)d\theta+\alpha\int_0^{t-\xi}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)[Bu(s)\\
&\quad +F(s,x_s)+\int_0^s G(\tau,x_\tau)d\omega(\tau)]dsd\theta\\
&= S(\xi^\alpha\delta)\int_\delta^\infty \eta_\alpha(\theta)S(t^\alpha\theta-\xi^\alpha\delta)\phi(0)d\theta\\
&\quad +\alpha S(\xi^\alpha\delta)\int_0^{t-\xi}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\xi^\alpha\delta)[Bu(s)\\
&\quad +F(s,x_s)+\int_0^s G(\tau,x_\tau)d\omega(\tau)]dsd\theta.
\end{aligned}
$$

Since $S(\xi^\alpha\delta)$, $\xi^\alpha\delta>0$ is a compact operator, the set $V^{\xi,\delta}(t)=\{(\Phi^{\xi,\delta}x)(t):x(\cdot)\in Y_r\}$ is a relative compact set in H for every ξ, $0<\xi<t$ and for all $\delta>0$. On the other hand, for every $x(\cdot)\in Y_r$ by (3.3) we have

$$
\begin{aligned}
\|\Phi x-\Phi^{\xi,\delta}x\|_Y^2 &= \sup_{t\in J} E\|(\Phi x)(t)-(\Phi^{\xi,\delta}x)(t)\|^2\\
&\leqslant \sup_{t\in J} E\|\int_\delta^\infty \eta_\alpha(\theta)S(t^\alpha\theta)\phi(0)d\theta\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_{t-\xi}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu(s)ds\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_{t-\xi}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)F(s,x_s)ds\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_{t-\xi}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\int_0^s G(\tau,x_\tau)d\omega(\tau)ds\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu(s)ds\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)F(s,x_s)ds\|^2\\
&\quad +\sup_{t\in J} E\|\alpha\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\int_0^s G(\tau,x_\tau)d\omega(\tau)ds\|^2.
\end{aligned}
$$

We see that, for each $x\in Y_r$, $\|\Phi x-\Phi^{\xi,\delta}x\|_Y^2\to 0$ as $\xi\to 0^+$ and $\delta\to 0^+$. Therefore, there are relative compact sets arbitrary close to the set $V(t)$. Hence, for each $t\in J$, the set $V(t)$ is relative compact in H (see [22]).

From the steps 2-4 and by the Ascoli-Arzela theorem, one can conclude that Φ is compact. On the other hand, it is easy to see that Φ is continuous on Y_r. Hence, Φ is a compact continuous operator on Y_r. From the Schauder fixed point theorem Φ has a fixed point. Thus, the system (1.1) is exactly null controllable on J. □

4. Example

We consider the following fractional stochastic integro-partial differential equation

$$
\begin{cases}
D_{0+}^\alpha x(t,z)=\frac{\partial^2}{\partial z^2}x(t,z)+u(t,z)+f(t,x(t-h,z))+\int_0^t g(s,x(s-h,z))d\omega(s), & t\in J,\ 0<z<1,\\
x_z(t,0)=x_z(t,1)=0,\ t\in J,\\
x(t,z)=\phi(t,z), t\in[-h,0],
\end{cases}
\tag{4.1}
$$

where $0 < \alpha < 1$ and $\omega(t)$ is Wiener process, $u \in L_2(0,b)$, and $H = L_2([0,1])$. Let $f : R \times R \to R$ and $g : R \times R \to L(R)$ are continuous. Also, let $A : H \to H$ be defined by $Ay = \frac{\partial^2}{\partial z^2} y$ with domain $D(A) = \{y \in H : y, \frac{\partial y}{\partial z}$ are absolutely continuous, and $\frac{\partial^2 y}{\partial z^2} \in H, y(0) = y(1) = 0\}$.

It is known that A is self-adjoin and has the eigenvalues $\lambda_n = -n^2\pi^2$, $n \in N$, with the corresponding normalized eigenvectors $e_n(z) = \sqrt{2}\sin(n\pi z)$. Furthermore, A generates an analytic compact semigroup of bounded linear operator $S(t)$, $t \geqslant 0$, on a separable Hilbert space H which is given by

$$S(t)y = \sum_{n=1}^{\infty} (y_n, e_n)e_n = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \sin(n\pi z) \int_0^1 \sin(n\pi\xi)y(\xi)d\xi, \ y \in H.$$

If $u \in L_2(J, H)$, then $B = I$, $B^* = I$. We consider the fractional linear system

$$\begin{cases} D^{\alpha}_{0+} y(t,z) = \frac{\partial^2}{\partial z^2} y(t,z) + u(t,z) + f(t,z) + g(t,z)d\omega(t), & t \in J, \ 0 < z < 1, \\ y_z(t,0) = y_z(t,1) = 0, \ t \in J, \\ y(t,z) = \phi(t,z), \quad -h \leqslant t \leqslant 0. \end{cases} \tag{4.2}$$

The system (4.2) is exact null controllability if there is a $\gamma > 0$, such that

$$\int_0^b \|B^*(b-s)^{\alpha-1}T^*_\alpha(b-s)y\|^2 ds \geqslant \gamma[\|S^*_\alpha(b)y\|^2 + \int_0^b \|(b-s)^{\alpha-1}T^*_\alpha(b-s)y\|^2 ds]$$

or equivalently

$$\int_0^b \|(b-s)^{\alpha-1}T_\alpha(b-s)y\|^2 ds \geqslant \gamma[\|S_\alpha(b)y\|^2 + \int_0^b \|(b-s)^{\alpha-1}T_\alpha(b-s)y\|^2 ds].$$

If $f = 0$ and $g = 0$ in (4.2), then the fractional linear system is exactly null controllable if

$$\int_0^b \|(b-s)^{\alpha-1}T_\alpha(b-s)y\|^2 ds \geqslant b\|S_\alpha(b)y\|^2.$$

Therefore,

$$\int_0^b \|(b-s)^{\alpha-1}T_\alpha(b-s)y\|^2 ds \geqslant \frac{b}{1+b}[\|S_\alpha(b)y\|^2 + \int_0^b \|(b-s)^{\alpha-1}T_\alpha(b-s)y\|^2 ds].$$

Hence, the linear fractional system (4.2) is exactly null controllable on $[0,b]$. So the hypothesis (H_1) is satisfied.

We define $F : J \times H \to H$, and $G : J \times H \to L(K, H)$ as follows:

$$F(t, x_t) = f(t, x(t-h,z)) \text{ and } G(t, x_t) = g(t, x(t-h,z)).$$

Assume F and G are continuous and there is a constant $0 < \beta < 1$ and a function $\nu \in L_2(J)$ such that

$$\|F(t,\theta)\| \vee \|G(t,\theta)\| \leqslant \nu(t)\|\theta\|^\beta$$

for all $(t,\theta) \in J \times H$ then the condition (H2) is satisfied. Hence, all the hypotheses of Theorem 3.1 are satisfied, so the fractional stochastic integro-partial differential system (4.1) is exact null controllable on $[0,b]$.

5. Conclusion

In this paper, we have presented, by using fractional calculus and Schauder's fixed point theorem, sufficient conditions for exact null controllability of semilinear fractional stochastic integro-differential equations in Hilbert spaces. We provided example in fractional stochastic integro-partial differential equation to illustrate our results.

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References

[1] O. Abu Arqub, *Solutions of time–fractional Tricomi and Keldysh equations of Dirichlet functions types in Hilbert space*, Numer. Methods Partial Differential Equations, **34** (2018), 1759–1780. 1, 2.6

[2] O. Abu Arqub, M. Al-Smadi, *Atangana–Baleanu fractional approach to the solutions of Bagley–Torvik and Painlevé equations in Hilbert space*, Chaos Solitons Fractals, **117** (2018), 161–167.

[3] O. Abu Arqub, M. Al-Smadi, *Numerical algorithm for solving time–fractional partial integrodifferential equations subject to initial and Dirichlet boundary conditions*, Numer. Methods Partial Differential Equations, **34** (2018), 1577–1597.

[4] O. Abu Arqub, B. Maayah, *Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana–Baleanu fractional operator*, Chaos Solitons Fractals, **117** (2018), 117–124.

[5] O. Abu Arqub, Z. Odibat, M. Al-Smadi, *Numerical solutions of time–fractional partial integrodifferential equations of Robin functions types in Hilbert space with error bounds and error estimates*, Nonlinear Dyn., **94** (2018), 1819–1834. 1

[6] H. M. Ahmed, *Controllability of fractional stochastic delay equations*, Lobachevskii J. Math., **30** (2009), 195–202. 1

[7] H. M. Ahmed, *On some fractional stochastic integrodifferential equations in Hilbert spaces*, Int. J. Math. Math. Sci., **2009** (2009), 8 pages. 2.6

[8] H. M. Ahmed, *Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space*, Adv. Difference Equ., **2014** (2014), 11 pages. 1

[9] H. M. Ahmed, M. M. El-Borai, *Hilfer fractional stochastic integro-differential equations*, Appl. Math. Comput., **331** (2018), 182–189. 1

[10] A. Ali, B. Samet, K. Shah, R. A. Khan, *Existence and stability of solution to a toppled systems of differential equations of non-integer order*, Bound. Value Probl., **2017** (2017), 13 pages.

[11] A. Ali, K. Shah, R. A. Khan, *Existence of solution to a coupled system of hybrid fractional differential equations*, Bull. Math. Anal. Appl., **9** (2017), 9–18.

[12] M. Al-Smadi, O. A. Arqub, *Computational algorithm for solving fredholm time–fractional partial integrodifferential equations of dirichlet functions type with error estimates*, Appl. Math. Comput., **342** (2019), 280–294. 1

[13] K. Balachandran, P. Balasubramaniam, J. P. Dauer, *Local null controllability of nonlinear functional differential systems in Banach space*, J. Optim. Theory Appl., **88** (1996), 61–75. 1

[14] K. Balachandran, J.-H. Kim, *Sample controllability of nonlinear stochastic integrodifferential systems*, Nonlinear Anal. Hybrid Syst., **4** (2010), 543–549.

[15] J. P. Dauer, P. Balasubramaniam, *Null controllability of semilinear integrodifferential systems in Banach spaces*, Appl. Math. Lett., **10** (1997), 117–123.

[16] J. P. Dauer, N. I. Mahmudov, *Exact null controllability of semilinear integrodifferential systems in Hilbert spaces*, J. Math. Anal. Appl., **299** (2004), 322–332.

[17] X. L. Fu, Y. Zhang, *Exact null controllability of non-autonomous functional evolution systems with nonlocal conditions*, Acta Math. Sci. Ser. B (Engl. Ed.), **33** (2013), 747–757. 1

[18] P. Kumama, A. Ali, K. Shah, R. A. Khan, *Existence results and Hyers–Ullam stability to a class of nonlinear arbitrary order differential equations*, J. Nonlinear Sci. Appl., **2017** (2017), 2986–2997. 1

[19] P. Kumlin, *A note on fixed point theory*, Chalmers and GU, Gothenburg, (2004). 2.3, 2.4

[20] C. M. Marle, *Measures et Probabilités*, Hermann, Paris, (1974). 2.10

[21] K. S. Miller, B. Ross, *An Introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, New York, (1993). 1, 2.1, 2.2, 2.11

[22] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, (1983). 3

[23] I. Podlubny, *Fractional differential equations*, Academic press, San Diego, (1999). 1, 2.1, 2.2

[24] R. Sakthivel, S. Suganya, S. M. Anthoni, *Approximate controllability of fractional stochastic evolution equations*, Comput. Math. Appl., **63** (2012), 660–668. 1

[25] W. R. Schneider, W. Wyss, *Fractional diffusion and wave equation*, J. Math. Phys., **30** (1989), 134–144. 1

[26] K. Shah, A. Ali, R. A. Khan, *Degree theory and existence of positive solutions to coupled systems of multi–point boundary value problems*, Bound. Value Probl., **2016** (2016), 12 pages. 1

[27] Y. Zhou, F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl., **59** (2010), 1063–1077. 2.7