

New scheme for nonlinear Schrödinger equations with variable coefficients



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Abstract

This paper proposes a numerical scheme for nonlinear Schrödinger equations with periodic variable coefficients and stochastic perturbation. The scheme is obtained by applying finite element method in spatial direction and finite difference scheme in temporal direction, respectively. The scheme is stable in the sense that it preserves discrete charge of the Schrödinger equations. The numerical examples verify the conservative property of the new scheme.

Keywords: Schrödinger equation, finite element method, finite difference scheme.

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1. Introduction

Schrödinger-type equations are important mathematical physical models in plasma physics, nonlinear optics, engineering, celestial mechanics, Bose-Einstein condensations ([14]). With the idea of chirp in optical soliton communication, they are used to study the propagation of picosecond optical pulses in single-mode fiber. They can describe the state of motion of outer electrons and reflect the characteristic wave-particle duality. Schrödinger-type equations are mathematical models for energy transfer in a monolayer molecular aggregate in the presence of thermal fluctuations. With average field theorem, they are applied to analyze the property of macroscopic wave function of aggregate in Bose-Einstein condensation.

The research for deterministic Schrödinger-type equations is well done. There are many investigations for deterministic PDEs, such as exp-function method [8], homotopy perturbation method [23], variational iteration method [22], collocation scheme [16], finite difference method [9] and so on. The numerical analysis for deterministic Schrödinger-type equations, such as symplectic schemes [17] and multi-symplectic schemes [20], pseudospectral method [4], compact method [6], finite volume scheme [7], collocation method [24], and conservative scheme [12, 19], can be found in the references therein.

Stochastic Schrödinger-type equations become a hot research topic recently. Well-posedness results for nonlinear Schrödinger equation (NSE) with linear Wiener noise is proved in [1]. Multi-symplectic scheme is proposed for stochastic NSE in [10]. An exponential scheme is developed to simulate quantum

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observables of NSE [15]. A semi-discrete scheme is considered to NSE with power nonlinearity [5]. The Strang-type splitting scheme is proposed to NSE with multiplicative noise [13]. A θ -scheme is analyzed for NSE with Stratonovich noise [3].

It is found that, finite element method is a kind of important numerical methods for NSE. A mixed finite-element method and the two-grid approach are applied to solve NSE [21]. Linearized Crank-Nicolson Galerkin method is studied for generalized NSE [18]. Continuous Galerkin methods on the space-time mesh are analyzed for NSE in [11]. Symplectic local discontinuous Galerkin method is proposed to NSE with multiplicative noise [2].

In this paper, we focus on the numerical investigations of conservative stochastic NSE. For conservative stochastic NSE, the reference of finite element method still lack. If we apply finite element method in space and finite difference scheme in time to conservative stochastic NSE, how does the obtained scheme behave? This motivates us to study the question in this paper. We try to observe the ability of our scheme in preserving conservative property of stochastic NSE.

In below we consider the initial-boundary problem of the following NSE with periodic variable coefficients and stochastic perturbation

$$\begin{cases} iu_t + \alpha(t)u_{xx} + \beta(t)|u|^2u = \epsilon u \circ \dot{\chi}, & x \in I = [-a, a], t \in (0, T], \\ u(x, 0) = \varphi(x), u(a, t) = u(-a, t) = 0, & x \in I, t \in [0, T], \end{cases} \quad (1.1)$$

where $i^2 = -1$, u_t and u_{xx} mean the first order partial derivative of u with respect to t and the second order partial derivative of u with respect to x , respectively. $\alpha(t)$ and $\beta(t)$ are real bounded functions, ϵ is a small real number, and \circ means Stratonovich product. $\dot{\chi}$ is a real-valued white noise which is delta correlated in time, either smooth or delta correlated in space. $\varphi(x)$ is a differential function. For NSE, $\alpha(t)u_{xx}$, $\beta(t)|u|^2u$, and $\epsilon u \circ \dot{\chi}$ are dispersion term, nonlinear term, and potential disturbance term, respectively. The balance between the effect of these terms can maintain the spatial profile of soliton waves. If $\epsilon = 0$, the system (1.1) is a deterministic system.

Proposition 1.1. *Under the periodic boundary condition, the solution of (1.1) satisfies the charge conservation law:*

$$\mathcal{N}(t) = \int_{-a}^a |u(x, t)|^2 dx = \mathcal{N}(0). \quad (1.2)$$

So we also say that the NSE system (1.1) is conservative.

Proof. By multiplying (1.1) by \bar{u} and integrating it with respect to x , we obtain that

$$i \int_I u_t \bar{u} dx + \alpha(t) \int_I u_{xx} \bar{u} dx + \beta(t) \int_I |u|^4 dx = \epsilon \int_I |u|^2 \circ \dot{\chi} dx.$$

According to the boundary condition, we derive that

$$i \int_I u_t \bar{u} dx - \alpha(t) \int_I |u_x|^2 dx + \beta(t) \int_I |u|^4 dx = \int_I \epsilon |u|^2 \circ \dot{\chi} dx.$$

Taking the imaginary part of above equation yields that

$$\int_I u_t \bar{u} + u \bar{u}_t dx = \frac{d}{dt} \int_I u \bar{u} dx = 0.$$

This implies the charge conservation law (1.2), which can be used to test the efficiency of numerical methods for the NSE system (1.1).

Denote $u = p + iq$. Then the NSE (1.1) can be written in the following form

$$\begin{cases} p_t + \alpha(t)q_{xx} + \beta(t)(p^2 + q^2)q = \epsilon q \circ \dot{\chi}, \\ -q_t + \alpha(t)p_{xx} + \beta(t)(p^2 + q^2)p = \epsilon p \circ \dot{\chi}. \end{cases} \quad (1.3)$$

□

2. New numerical scheme

In below, for simplicity, we apply the uniform mesh grids $\{x_k = -a + kh, t_n = n\tau\}$ with step-sizes $h = 2a/K$ and $\tau = T/M$. Numerical values of $u(x_k, t_n)$ are denoted by u_k^n .

The standard Sobolev space is denoted by

$$H_0^1(I) = \{\psi(x) | \psi, \psi' \in L^2(I), \psi(a) = \psi(-a) = 0, x \in I\}.$$

For any $\psi \in H_0^1(I)$, with the boundary condition and Green formula, we can derive the following weak equations of (1.3),

$$\begin{aligned} \int_I p_t \psi dx - \alpha(t) \int_I q_x \psi_x dx + \beta(t) \int_I (p^2 + q^2) q \psi dx &= \epsilon \int_I q \psi \circ \dot{\chi} dx, \\ - \int_I q_t \psi dx - \alpha(t) \int_I p_x \psi_x dx + \beta(t) \int_I (p^2 + q^2) p \psi dx &= \epsilon \int_I p \psi \circ \dot{\chi} dx. \end{aligned}$$

Additionally, we denote the finite element space by V_h . Accordingly, the discrete form of the weak equations is: to find $p_h, q_h \in V_h$, such that for any $\psi_h \in V_h$, the following equations

$$\int_I (p_h)_t \psi_h dx - \alpha(t) \int_I (q_h)_x (\psi_h)_x dx + \beta(t) \int_I (p_h^2 + q_h^2) q_h \psi_h dx = a_q^h, \tag{2.1}$$

$$- \int_I (q_h)_t \psi_h dx - \alpha(t) \int_I (p_h)_x (\psi_h)_x dx + \beta(t) \int_I (p_h^2 + q_h^2) p_h \psi_h dx = a_p^h \tag{2.2}$$

are satisfied.

In below, we will solve the discrete weak equations (2.1) and (2.2). We choose the space of segmental continuous linear function as V_h . Then V_h is the linear expansion space generated by $\phi_1, \phi_2, \dots, \phi_{K+1}$, where

$$\begin{aligned} \phi_1(x) &= \begin{cases} \frac{x_1-x}{h}, & x \in (x_0, x_1], \\ 0, & x \notin (x_0, x_1], \end{cases} & \phi_{K+1}(x) &= \begin{cases} \frac{x-x_K}{h}, & x \in (x_K, x_{K+1}], \\ 0, & x \notin (x_K, x_{K+1}], \end{cases} \\ \phi_k(x) &= \begin{cases} \frac{x-x_{k-1}}{h}, & x \in (x_{k-1}, x_k], \\ \frac{x_{k+1}-x}{h}, & x \in (x_k, x_{k+1}], \\ 0, & x \notin (x_{k-1}, x_{k+1}], \end{cases} & k &= 2, \dots, K. \end{aligned}$$

Suppose that

$$p_h = \sum_{k=1}^{K+1} a_k(t) \phi_k(x), \quad q_h = \sum_{k=1}^{K+1} b_k(t) \phi_k(x). \tag{2.3}$$

Replacing ψ_h by ϕ_j and inserting (2.3) into (2.1) and (2.2) yield that

$$AP' - [\alpha(t)B - \beta(t)D(P, Q) + \epsilon C]Q = 0, \tag{2.4}$$

$$AQ' + [\alpha(t)B - \beta(t)D(P, Q) + \epsilon C]P = 0, \tag{2.5}$$

where $P = (a_1, a_2, \dots, a_{K+1})^T$, $Q = (b_1, b_2, \dots, b_{K+1})^T$, $A = (a_{kj})$, $B = (b_{kj})$, $C = (c_{kj})$, and $D(P, Q) = (d_{kj})$ with the following notations

$$\begin{aligned} a_{kj} &= \int_I \phi_k \phi_j dx, \quad b_{kj} = \int_I \phi'_k \phi'_j dx, \quad c_{kj} = \int_I \phi_k \phi_j \circ \dot{\chi} dx, \\ d_{kj} &= \int_I \left(\left| \sum_{m=1}^{K+1} a_m(t) \phi_m \right|^2 + \left| \sum_{m=1}^{K+1} b_m(t) \phi_m \right|^2 \right) \phi_k \phi_j dx. \end{aligned}$$

Then we apply the following finite difference scheme to (2.4)-(2.5) and get the new scheme for the NSE system (1.3),

$$A \frac{P^{n+1} - P^n}{\tau} - [\alpha_n B - \beta_n D(P^n, Q^n) + \epsilon C^n] Q^{n+\frac{1}{2}} = 0, \tag{2.6}$$

$$A \frac{Q^{n+1} - Q^n}{\tau} + [\alpha_n B - \beta_n D(P^n, Q^n) + \epsilon C^n] P^{n+\frac{1}{2}} = 0, \tag{2.7}$$

where $\alpha_n = \alpha(t_n)$, $\beta_n = \beta(t_n)$, $P^{n+1/2} = (P^{n+1} + P^n)/2$, $Q^{n+1/2} = (Q^{n+1} + Q^n)/2$. The elements in

$D(P, Q), C$ are substituted by the numerical solutions to get $D(P^n, Q^n), C^n$.

Theorem 2.1. Denote $U^n = P^n + iQ^n$. Let $\|U^n\|^2 = h|U^n|^2 = h\sum U_k^n \overline{U_k^n}$. Then $\|U^n\|^2$ is the discrete charge invariant of the scheme (2.6)-(2.7), which implies the discrete charge conservation law of (1.1). So in this sense, we also say that the scheme (2.6)-(2.7) behaves conservatively and stably.

Proof. The scheme (2.6)-(2.7) is equivalent to

$$A \frac{U^{n+1} - U^n}{\tau} + i[\alpha_n B - \beta_n D(P^n, Q^n) + \epsilon C^n] U^{n+\frac{1}{2}} = 0, \tag{2.8}$$

where $\overline{U^{n+\frac{1}{2}}} = (U^{n+1} + U^n)/2$. This is a finite difference scheme to the NSE (1.1). By multiplying (2.8) with $\overline{U^{n+\frac{1}{2}}}$, we obtain that

$$A \frac{|U^{n+1}|^2 - |U^n|^2 + U^{n+1} \overline{U^n} - \overline{U^{n+1}} U^n}{2\tau} + i[\alpha_n B - \beta_n D(P^n, Q^n) + \epsilon C^n] |U^{n+\frac{1}{2}}|^2 = 0. \tag{2.9}$$

Considering the real symmetry and invertibility of A , the real part of above equation (2.9) implies that

$$\|U^n\|_2 = \|U^{n+1}\|_2.$$

□

3. Numerical results

We apply above scheme (2.6)-(2.8) to solve three NSEs with periodic boundary condition and test the conservative property of the scheme. We consider $\|U^n\|$ as the discrete charge of $N(t_n)$. The numerical residuals of $N(t)$ is measured by the error $\|U^n\| - \|U^0\|$.

Example 3.1. First we consider the following NSE with constant coefficients

$$\begin{cases} iu_t + u_{xx} + |u|^2 u = \epsilon u \circ \dot{\chi}, & x \in I = [-5, 5], t > 0, \\ u(x, 0) = \frac{\sqrt{2}}{2} \exp(i\frac{x}{2}) \operatorname{sech}(\frac{x}{2}), u(5, t) = u(-5, t), & x \in I, t > 0. \end{cases} \tag{3.1}$$

The step-sizes we used are $h = 1/8$ and $\tau = 1/100$. In Figure 1, we plot the discrete charge (left) and its residuals (right) for numerical solutions of (3.1) with $\epsilon = 0.4$, respectively. The figure verifies that our scheme preserves the discrete charge conservation law of (3.1) approximately. Numerical results are similar to other step-sizes h, τ and other perturbation size ϵ .

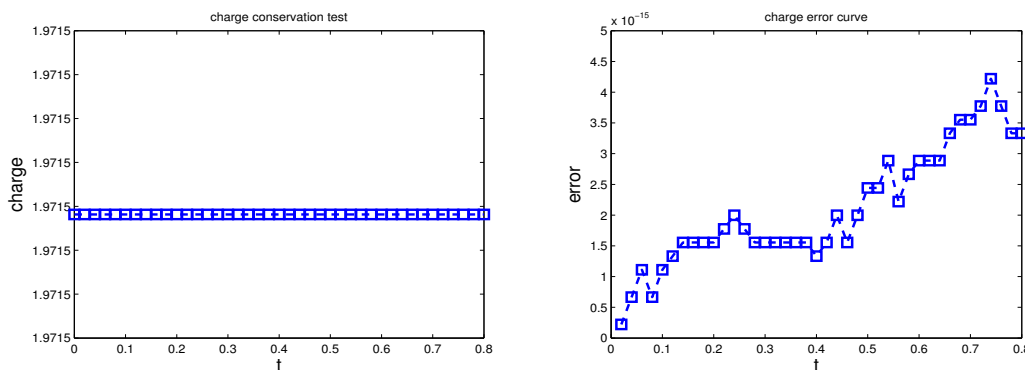


Figure 1: Discrete charge (left) and its residuals (right) for numerical solutions of the NSE (3.1).

Example 3.2. *Second, we consider the following NSE with variable coefficients*

$$\begin{cases} iu_t + \frac{1}{2} \cos(t)u_{xx} + \frac{\cos(t)}{\sin(t)+3}|u|^2u = \epsilon u \circ \dot{\chi}, & x \in I = [-5, 5], t > 0, \\ u(x, 0) = \frac{\sqrt{3}}{3} \exp\left(\frac{i(x^2-1)}{6}\right) \operatorname{sech}\left(\frac{x}{3}\right), u(5, t) = u(-5, t), & x \in I, t > 0. \end{cases} \quad (3.2)$$

The step-sizes we applied are $h = 1/8$ and $\tau = 1/100$. In Figure 2, we depict the discrete charge (left) and its residuals (right) for numerical solutions of (3.2) with $\epsilon = 0.5$, respectively. The figure shows that the method also preserves the discrete charge invariant of (3.2) approximately. To other step-sizes h, τ and other perturbation size ϵ , we obtain similar numerical results.

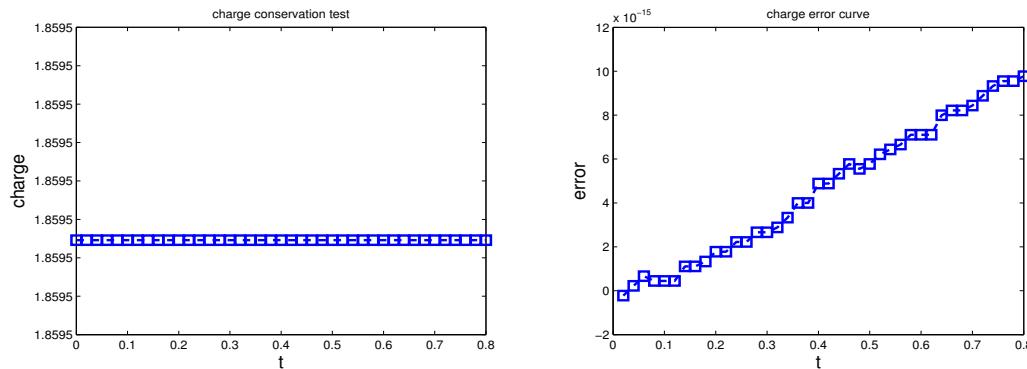


Figure 2: Discrete charge (left) and its residuals (right) for numerical solutions of the NSE (3.2).

Example 3.3. *Third, we consider the following NSE with variable coefficients*

$$\begin{cases} iu_t + \frac{1}{2}[\cos(t) + \sqrt{2} \cos(\sqrt{2}t)]u_{xx} + \frac{\cos(t) + \sqrt{2} \cos(\sqrt{2}t)}{\sin(t) + \sin(\sqrt{2}t) + 5}|u|^2u = \epsilon u \circ \dot{\chi}, & x \in I = [-5, 5], t > 0, \\ u(x, 0) = \frac{\sqrt{5}}{5} \exp\left(\frac{i(x^2-1)}{10}\right) \operatorname{sech}\left(\frac{x}{5}\right), u(5, t) = u(-5, t), & x \in I, t > 0. \end{cases} \quad (3.3)$$

The step-sizes we used are $h = 1/8$ and $\tau = 1/100$. In Figure 3, we plot the discrete charge (left) and its residuals (right) for numerical solutions of (3.3) with $\epsilon = 0.6$, respectively. The figure tells us that the method preserves the discrete charge conservative property of (3.3) approximately. It follows analogously for other step-sizes h, τ and other perturbation size ϵ .

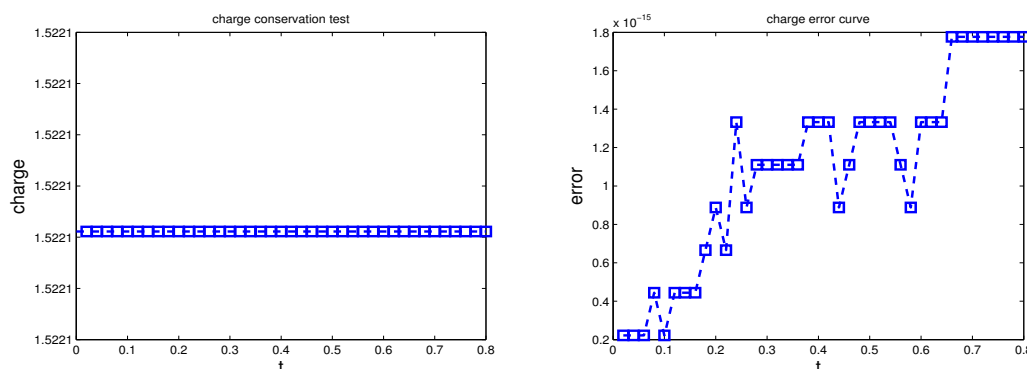


Figure 3: Discrete charge (left) and its residuals (right) for numerical solutions of the NSE (3.3).

In Figure 4, we plot the three-dimensional graph for imaginary parts of numerical solutions for Example 3.1 (left), Example 3.2 (middle), and Example 3.3 (right), respectively. Although the profile of soliton waves is maintained, the small noise produces stochastic fluctuations on the waves.

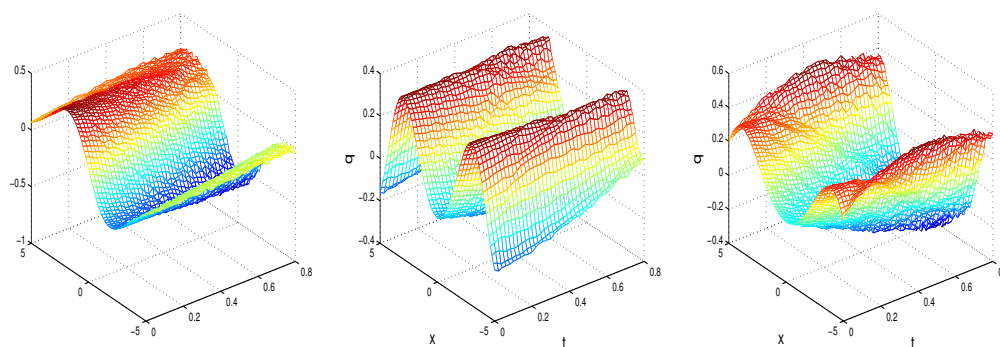


Figure 4: Imaginary parts graph for numerical solutions of the NSEs.

4. Conclusion

We combine finite element method and finite difference scheme to obtain a new scheme to NSEs. The stable scheme has conservative discrete charge for NSEs. We also investigate the scheme for three NSE examples with periodic variable coefficients and stochastic perturbation. Further discussion and research about the numerical analysis of stochastic NSEs will be followed.

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