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The Gopala-Hemachandra universal code determined by straight lines

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Abstract

Variation on the Fibonacci universal code, known as Gopala-Hemachandra (or GH) code, is mainly used in data compression and cryptography as it is a self-synchronizing code. In 2010, Basu and Prasad showed that Gopala-Hemachandra code $GH_a(n)$ exists for $-20 \leqslant a \leqslant -2$ and $1 \leqslant n \leqslant 100$ as well as there are m consecutive non-existing Gopala-Hemachandra codewords in $GH_{-(4+m)}(n)$ column where $1 \leqslant m \leqslant 16$. In this paper, we have introduced GH code straight line in two-dimensional space where each integral point (a, n) on the GH code straight line represents a unique GH codeword. GH code straight lines confirm the existence of GH codewords for any integer $n \geqslant 1$ and integer $a \leqslant -2$. Moreover, for a given parameter (a, n), we have introduced two methods to check whether GH codeword exists or not.

Keywords: Fibonacci numbers, Fibonacci coding, Gopala-Hemachandra sequence, Gopala-Hemachandra code, Zeckendorf's representation.

2010 MSC: 94B25, 11B39, 11T71.

1. Introduction

The Fibonacci sequence is a sequence of positive integers whose terms are defined by the recurrence relation $F(n) = F(n-1) + F(n-2)$, $\forall n > 2$ with $F(1) = 1$ and $F(2) = 2$. The Fibonacci universal code encodes positive integers into binary codewords. The rule of encoding is performed by using the Zeckendorf's representation of positive integers. The Zeckendorf's theorem states that every positive integer has a unique representation as the sum of non-consecutive Fibonacci numbers [9]. Let, n be a positive integer which can be written as a sum of Fibonacci numbers as $n = \sum_{k=1}^{l} a_k F(k)$ where $a_k \in \{0, 1\}$ such that the string $a_1 a_2 a_3 \cdots a_l$ does not contain any consecutive 1's, where $F(k)$ is the k^{th} Fibonacci number. For an example, 14 can be represented as $F(1) + F(2) + F(3) + F(5)$ or $F(1) + F(4) + F(5)$ or $F(1) + F(6)$ but the Zeckendorf's representation will be $F(1) + F(6)$. So, the Zeckendorf's representation gives a unique binary string say $a_1 a_2 a_3 \cdots a_l$ for a positive integer n. Then $a_1 a_2 a_3 \cdots a_l 1$ will be the Fibonacci representation of n. The codewords end with 11 and have no other consecutive 1's in between it. The Fibonacci

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universal code is a uniquely decodable binary code of variable size since it is a prefix code. The main disadvantage of this representation is that the size n of the set of integers has to be known in advance since it determines the code size as $1 + \lfloor \log_2 n \rfloor$. Since Fibonacci codes do not have consecutive 1 bits, they restricts the number of binary patterns available for such codes and in comparison, they are longer than the other codes. But they perform better in contrast to the Elias code [3] until the number of source messages is not very large. The Fibonacci universal code has the additional attribute of robustness, which manifests itself by the local containment of errors.

Fibonacci universal code is much significant than other universal codes because it has the inherent property to recover data from a damaged stream easily. In channel coding, if a single bit is changed, the data comes after it will not be correctly identified. On the other hand, with the Fibonacci universal coding, a changed bit may cause one token to be read as two, or cause two tokens to be misread as one, but reading a 0 from the stream will stop the errors from propagating further. Since the only stream that has no 0 in it is a stream of 11 tokens, the total edit distance between a stream damaged by a single bit error and the original stream is at most three.

Around the year 1150, Hemachandra introduced the Gopala-Hemachandra (GH) sequence. It is worth noting that Gopala studied these numbers in about the year 1135. The GH sequences are correlated with the Fibonacci sequence have probable applications in cryptography and coding theory presented in several studies [1, 8].

In 2007, Thomas [8] proposed Variant Fibonacci sequence which was also called Gopala-Hemachandra (GH) sequence denoted by $VF_a(n)$. He had shown that the Variant Fibonacci sequence cannot be generated for all values of a. He gave a table in which he showed that for $a = -5$ there is no Zeckendorf's representation for $n = 5, 12$. But he did not give a specific range for a.

In 2010, Basu and Prasad [1] invented an improvement of [8] in which they showed the existence of the Gopala-Hemachandra code $GH_a(n)$ for $-20 \leqslant a \leqslant -2$, and $1 \leqslant n \leqslant 100$. Besides, they described that there are at most m consecutive non-existing GH codewords in a column of $GH_{-(4+m)}$ for $1 \leqslant m \leqslant 16$, which are given in Tables 1, 2, 3, and 4.

In this paper, we extend the availability of GH universal code. Here we define a GH code straight line to obtain a GH code. Thereafter, we propose two methods to check the existence of GH code for a given parameter (a, n), where $a \leqslant -2$ and n is any positive integer.

In this paper, we use the following two properties.

Property 1.1. *In the Zeckendorf's representation of any two consecutive positive integers, it is not possible that both the coefficients of* $F(1)$ *are* 1.

Property 1.2. *In the Zeckendorf's representation of any two consecutive positive integers* n *and* $n+1$*, if the coefficient of* $F(1)$ *is* 0 *for both, then the coefficient of* $F(2)$ *is* 1 *for* n*, and* 0 *for* $n+1$.

In Section 2, we discuss Gopala-Hemachandra sequence and codes and establish a relation between Gopala-Hemachandra sequence and the Fibonacci sequence. In Section 3, we introduce GH code straight line and give some properties on the Gopala-Hemachandra universal code. Two methods with illustrations are given in Section 4, which provide the existence or non-existence of GH code corresponding to a parameter (a, n). In the final section, we conclude the article.

2. Gopala-Hemachandra sequence and codes

The sequence $\{a,\ b,\ a+b,\ a+2b,\ 2a+3b, \ldots\}$ for any $a,\ b \in \mathbb{Z}$, named as Gopala-Hemachandra (GH) sequence is the more general variant Fibonacci sequence. For the pair of values $a = 1$ and $b = 2$, it represents the Fibonacci sequence. The historical details of these sequences are discussed in [4, 5, 7].

The variant of GH coding scheme is obtained by second order variant GH sequence $VF_a(k)$ such that $b = 1 - a$, $VF_a(1) = a$, $VF_a(2) = 1 - a$ and for $k \geqslant 3$,

$$VF_a(k) = VF_a(k-1) + VF_a(k-2).$$

In 1960, Daykin [2] proved that only the standard Fibonacci sequence $F(k)$ gives a unique Zeckendorf's representation for all positive integers.

Table 1: GH Code ($1 \leqslant n \leqslant 50,\ -2 \leqslant a \leqslant -11$).

n	GH_{-2}	GH_{-3}	GH_{-4}	GH_{-5}	GH_{-6}	GH_{-7}	GH_{-8}	GH_{-9}	GH_{-10}	GH_{-11}
1	0011	0011	0011	0011	0011	0011	0011	0011	0011	0011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	011	100011	100011	100011	100011	100011	100011	100011	100011	100011
4	00011	011	101011	101011	101011	101011	101011	101011	101011	101011
5	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
6	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A
7	01011	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A
8	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A	N/A
9	0000011	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A
10	0010011	010011	1010011	1000011	001011	000011	00011	011	N/A	N/A
11	1001011	0000011	01011	1010011	1000011	001011	000011	00011	011	N/A
12	0100011	0010011	010011	N/A	1010011	1000011	001011	000011	00011	011
13	10100011	1001011	0000011	01011	N/A	1010011	1000011	001011	000011	00011
14	00000011	10000011	0010011	010011	N/A	N/A	1010011	1000011	001011	000011
15	00100011	10100011	1001011	0000011	01011	N/A	N/A	1010011	1000011	001011
16	10010011	0001011	10000011	0010011	010011	N/A	N/A	N/A	1010011	1000011
17	01000011	00000011	10100011	1001011	0000011	01011	N/A	N/A	N/A	1010011
18	10101011	00100011	0100011	10000011	0010011	010011	N/A	N/A	N/A	N/A
19	00001011	10010011	0001011	10100011	1001011	1000011	01011	N/A	N/A	N/A
20	00101011	0101011	00000011	N/A	10000011	0010011	010011	N/A	N/A	N/A
21	01010011	10101011	00100011	0100011	10100011	1001011	0000011	01011	N/A	N/A
22	101000011	00010011	10010011	0001011	N/A	10000011	0010011	010011	N/A	N/A
23	000000011	00001011	10001011	00000011	N/A	10100011	1001011	0000011	01011	N/A
24	001000011	00101011	0101011	00100011	0100011	N/A	10000011	0010011	010011	N/A
25	100100011	100000011	01000011	10010011	0001011	N/A	10100011	1001011	0000011	01011
26	100010011	01010011	00010011	10001011	00000011	N/A	N/A	10000011	0010011	010011
27	101010011	01001011	00001011	10101011	00100011	0100011	N/A	10100011	1001011	0000011
28	000010011	000000011	00101011	0101011	10010011	0001011	N/A	N/A	10000011	0010011
29	001010011	001000011	100000011	01000011	10001011	00000011	N/A	N/A	10100011	1001011
30	010100011	100100011	101000011	00010011	10101011	00100011	0100011	N/A	N/A	10000011
31	101001011	100010011	01010011	00001011	N/A	10010011	0001011	N/A	N/A	10100011
32	000001011	101010011	01001011	00101011	0101011	10001011	00000011	N/A	N/A	N/A
33	001001011	000100011	000000011	100000011	01000011	10101011	00100011	0100011	N/A	N/A
34	100101011	000010011	001000011	101000011	00010011	N/A	10010011	0001011	N/A	N/A
35	010001011	001010011	100100011	N/A	00001011	N/A	10001011	00000011	N/A	N/A
36	1010000011	100001011	100010011	01010011	00101011	0101011	10101011	00100011	0100011	N/A
37	0000000011	101001011	101010011	01001011	100000011	01000011	N/A	10010011	0001011	N/A
38	0010000011	010010011	010000011	000000011	101000011	00010011	N/A	10001011	00000011	N/A
39	1001000011	000001011	000100011	001000011	N/A	00001011	N/A	10101011	00100011	0100011
40	1000100011	001001011	000010011	100100011	N/A	00101011	0101011	N/A	10010011	0001011
41	1010100011	100101011	001010011	100010011	01010011	100000011	01000011	N/A	10001011	00000011
42	0000100011	1000000011	100001011	101010011	01001011	101000011	00010011	N/A	10101011	00100011
43	0010100011	010001011	101001011	N/A	000000011	N/A	00001011	N/A	N/A	10010011
44	0101000011	000101011	010100011	010000011	001000011	N/A	00101011	0101011	N/A	10001011
45	1010010011	0000000011	010010011	000100011	100100011	N/A	100000011	01000011	N/A	10101011
46	0000010011	0010000011	000001011	000010011	100010011	01010011	101000011	00010011	N/A	N/A
47	0010010011	1001000011	001001011	001010011	101010011	01001011	N/A	00001011	N/A	N/A
48	1001010011	010101011	100101011	100001011	N/A	000000011	N/A	00101011	0101011	N/A
49	1000001011	1010100011	1000000011	101001011	N/A	001000011	N/A	100000011	01000011	N/A
50	1010001011	0001000011	1010000011	N/A	010000011	100100011	N/A	101000011	00010011	N/A

Theorem 2.1. *Let* $F(k)$ *be the Fibonacci sequence with* $F(0) = F(1) = 1$. *Then for* $k \geqslant 1$,

$$VF_a(k) = F(k-2) - aF(k-4).$$

Proof. Consider, the GH sequence $VF_a(k)$ as $\{a, 1-a, 1, 2-a, 3-a, 5-2a, 8-3a, 13-5a, 21-8a, \ldots\}$. Therefore, we have $VF_a(1) = a = 0 - a(-1) = F(-1) - aF(-3) = F(1-2) - aF(1-4)$ and $VF_a(2) = 1 - a(1) = F(0) - aF(-2) = F(2-2) - aF(2-4)$, according to [6]. Thus the result is true for $k = 1$ and 2.

Let the result is true for $k = 1, 2, 3, \ldots, m$. Then, $VF_a(m-1) = F(m-3) - aF(m-5)$ and $VF_a(m) = F(m-2) - aF(m-4)$. Therefore, $VF_a(m+1) = VF_a(m) + VF_a(m-1) = F(m-3) - aF(m-5) + F(m-2) - aF(m-4) = (F(m-3) + F(m-2)) - a(F(m-5) + F(m-4)) = F(m-1) - aF(m-3) = F(\overline{m+1} - 2) - aF(\overline{m+1} - 4)$.

Hence by the second principle of mathematical induction, we have

$$VF_a(k) = F(k-2) - aF(k-4), \quad k \geqslant 1.$$

□

3. Some properties on GH universal code for $a \leq -2$

The GH codewords for $n = 1, 2, 3, 4$ always exist and are 0011, 10011, 100011, 101011 respectively for any $a \leqslant -2$ since binary representation of 1, 2, 3, and 4 are independent of a.

Table 2: GH Code ($51 \leqslant n \leqslant 100,\ -2 \leqslant a \leqslant -11$).

n	GH_{-2}	GH_{-3}	GH_{-4}	GH_{-5}	GH_{-6}	GH_{-7}	GH_{-8}	GH_{-9}	GH_{-10}	GH_{-11}
51	0000001011	0000100011	010001011	010100011	000100011	100010011	01010011	N/A	00001011	N/A
52	0010001011	0010100011	000101011	010010011	000010011	101010011	01001011	N/A	00101011	0101011
53	0101010011	1000010011	0000000011	000001011	001010011	N/A	000000011	N/A	100000011	01000011
54	0100001011	1010010011	0010000011	001001011	100001011	N/A	001000011	N/A	101000011	00010011
55	0001001011	0100100011	1001000011	100101011	101001011	N/A	100100011	N/A	N/A	00001011
56	0000101011	0000010011	1000100011	1000000011	N/A	010000011	100010011	01010011	N/A	00101011
57	0010101011	0010010011	010101011	1010000011	N/A	000100011	101010011	01001011	N/A	100000011
58	10000000011	1001010011	0100000011	N/A	010100011	000010011	N/A	000000011	N/A	101000011
59	0100101011	1000001011	0001000011	010001011	010010011	001010011	N/A	001000011	N/A	N/A
60	00000000011	0100010011	0000100011	000101011	000001011	100001011	N/A	100100011	N/A	N/A
61	00100000011	0001010011	0010100011	0000000011	001001011	101001011	N/A	100010011	01010011	N/A
62	10010000011	0000001011	1000010011	0010000011	100101011	N/A	010000011	101010011	01001011	N/A
63	01000000011	0010001011	1010010011	1001000011	1000000011	N/A	000100011	N/A	000000011	N/A
64	10101000011	1001001011	0101000011	1000100011	1010000011	N/A	000010011	N/A	001000011	N/A
65	00001000011	1000101011	0100100011	1010100011	N/A	010100011	001010011	N/A	100100011	N/A
66	00101000011	1010101011	0000010011	010101011	N/A	010010011	100001011	N/A	100010011	01010011
67	10000100011	0001001011	0010010011	0100000011	010001011	000001011	101001011	N/A	101010011	01001011
68	10100100011	0000101011	1001010011	0001000011	000101011	001001011	N/A	010000011	N/A	000000011
69	00000100011	0010101011	1000001011	0000100011	0000000011	100101011	N/A	000100011	N/A	001000011
70	00100100011	10000000011	1010001011	0010100011	0010000011	1000000011	N/A	000010011	N/A	100100011
71	10010100011	0101001011	0100010011	1000010011	1001000011	1010000011	N/A	001010011	N/A	100010011
72	01000100011	0100101011	0001010011	1010010011	1000100011	N/A	010100011	100001011	N/A	101010011
73	10100010011	00000000011	0000001011	N/A	1010100011	N/A	010010011	101001011	N/A	N/A
74	00000010011	00100000011	0010001011	0101000011	N/A	N/A	000001011	N/A	010000011	N/A
75	00100010011	10010000011	1001001011	0100100011	010101011	010001011	001001011	N/A	000100011	N/A
76	10010010011	10001000011	1000101011	0000010011	0100000011	000101011	100101011	N/A	000010011	N/A
77	01000010011	10101000011	0101010011	0010010011	0001000011	0000000011	1000000011	N/A	001010011	N/A
78	10101010011	00010000011	0100001011	1001010011	0000100011	0010000011	1010000011	N/A	100001011	N/A
79	00001010011	00001000011	0001001011	1000001011	0010100011	1001000011	N/A	010100011	101001011	N/A
80	00101010011	00101000011	0000101011	1010001011	1000010011	1000100011	N/A	010010011	N/A	010000011
81	01010010011	10000100011	0010101011	N/A	1010010011	1010100011	N/A	000001011	N/A	000100011
82	01001010011	10100100011	10000000011	0100010011	N/A	N/A	N/A	001001011	N/A	000010011
83	00000001011	01001000011	10100000011	0001010011	N/A	N/A	010001011	100101011	N/A	001010011
84	00100001011	00000100011	0101001011	0000001011	0101000011	010101011	000101011	1000000011	N/A	100001011
85	10010001011	00100100011	0100101011	0010001011	0100100011	0100000011	0000000011	1010000011	N/A	101001011
86	10001001011	10010100011	00000000011	1001001011	0000010011	0001000011	0010000011	N/A	010100011	N/A
87	10101001011	10000010011	00100000011	1000101011	0010010011	0000100011	1001000011	N/A	010010011	N/A
88	00001001011	10100010011	10010000011	1010101011	1001010011	0010100011	1000100011	N/A	000001011	N/A
89	00101001011	00010100011	10001000011	0101010011	1000001011	1000010011	1010100011	N/A	001001011	N/A
90	01010001011	00000010011	10101000011	0100001011	1010001011	1010010011	N/A	N/A	100101011	N/A
91	10100101011	00100010011	01000000011	0001001011	N/A	N/A	N/A	010001011	1000000011	N/A
92	00000101011	10010010011	00010000011	0000101011	N/A	N/A	N/A	000101011	1010000011	N/A
93	00100101011	01010100011	00001000011	0010101011	0100010011	N/A	010101011	0000000011	N/A	010100011
94	10010101011	10101010011	00101000011	10000000011	0001010011	0101000011	0100000011	0010000011	N/A	010010011
95	01000101011	00010010011	10000100011	10100000011	0000001011	0100100011	0001000011	1001000011	N/A	000001011
96	00010101011	00001010011	10100100011	N/A	0010001011	0000010011	0000100011	1000100011	N/A	001001011
97	000000000011	00101010011	01010000011	0101001011	1001001011	0010010011	0010100011	1010100011	N/A	100101011
98	001000000011	10000001011	01001000011	0100101011	1000101011	1001010011	1000010011	N/A	N/A	1000000011
99	01010101011	01010010011	00000100011	00000000011	1010101011	1000001011	1010010011	N/A	010001011	1010000011
100	100010000011	01001010011	00100100011	00100000011	N/A	1010001011	N/A	N/A	000101011	N/A

Throughout this paper, we take a, n and j as integers. Now, we discuss the following properties of the GH code by considering (a, n) as a point in xy-plane and with the help of Tables 1, 2, 3, and 4.

Proposition 3.1. *For $a \leqslant -2$, there are four straight lines $y + 0x = 0 + j$ for $j = 1, 2, 3, 4$ such that the four points $(a, 1 - 0a)$, $(a, 2 - 0a)$, $(a, 3 - 0a)$, and $(a, 4 - 0a)$ lie on these lines for $j = 1, 2, 3, 4$, respectively which give the respective GH codewords* 0011, 10011, 100011, *and* 101011.

Proposition 3.2. *For $a \leqslant -2$, there are six straight lines $y + x = 0 + j$ for $j = 1, 2, \ldots, 6$ such that the six points $(a, 1 - a)$, $(a, 2 - a), \ldots, (a, 6 - a)$ lie on these lines for $j = 1, 2, \ldots, 6$, respectively which give the respective GH codewords* 011, 00011, 000011, 001011, 1000011, *and* 1010011.

Proposition 3.3. *For $a \leqslant -2$, there are seven straight lines $y + 2x = 2 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 3 - 2a)$, $(a, 4 - 2a), \ldots, (a, 9 - 2a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 01011, 010011, 0000011, 0010011, 1001011, 10000011, *and* 10100011.

Proposition 3.4. *For $a \leqslant -2$, there are seven straight lines $y + 3x = 5 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 6 - 3a)$, $(a, 7 - 3a), \ldots, (a, 12 - 3a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 0100011, 0001011, 00000011, 00100011, 10010011, 10001011, *and* 10101011.

Proposition 3.5. *For $a \leqslant -2$, there are seven straight lines $y + 4x = 7 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 8 - 4a)$, $(a, 9 - 4a), \ldots, (a, 14 - 4a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 0101011, 01000011, 00010011, 00001011, 00101011, 100000011, *and* 101000011.

Table 3: GH Code $(1 \leqslant n \leqslant 50,\ -12 \leqslant a \leqslant -20)$.

n	GH_{-12}	GH_{-13}	GH_{-14}	GH_{-15}	GH_{-16}	GH_{-17}	GH_{-18}	GH_{-19}	GH_{-20}
1	0011	0011	0011	0011	0011	0011	0011	0011	0011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	100011	100011	100011	100011	100011	100011	100011	100011	100011
4	101011	101011	101011	101011	101011	101011	101011	101011	101011
5	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
6	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
7	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
8	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
9	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
10	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
11	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
12	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
13	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
14	00011	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
15	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A
16	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A
17	1000011	001011	000011	00011	011	N/A	N/A	N/A	N/A
18	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A
19	N/A	1010011	1000011	001011	000011	00011	011	N/A	N/A
20	N/A	N/A	1010011	1000011	001011	000011	00011	011	N/A
21	N/A	N/A	N/A	1010011	1000011	001011	000011	00011	011
22	N/A	N/A	N/A	N/A	1010011	1000011	001011	000011	00011
23	N/A	N/A	N/A	N/A	N/A	1010011	1000011	001011	000011
24	N/A	N/A	N/A	N/A	N/A	N/A	1010011	1000011	001011
25	N/A	N/A	N/A	N/A	N/A	N/A	N/A	1010011	1000011
26	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	1010011
27	01011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
28	010011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
29	0000011	01011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
30	0010011	010011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
31	1001011	0000011	01011	N/A	N/A	N/A	N/A	N/A	N/A
32	10000011	0010011	010011	N/A	N/A	N/A	N/A	N/A	N/A
33	10100011	1001011	0000011	01011	N/A	N/A	N/A	N/A	N/A
34	N/A	10000011	0010011	010011	N/A	N/A	N/A	N/A	N/A
35	N/A	10100011	1001011	0000011	01011	N/A	N/A	N/A	N/A
36	N/A	N/A	10000011	0010011	010011	N/A	N/A	N/A	N/A
37	N/A	N/A	10100011	1001011	0000011	01011	N/A	N/A	N/A
38	N/A	N/A	N/A	10000011	0010011	010011	N/A	N/A	N/A
39	N/A	N/A	N/A	10100011	1001011	0000011	01011	N/A	N/A
40	N/A	N/A	N/A	N/A	10000011	0010011	010011	N/A	N/A
41	N/A	N/A	N/A	N/A	10100011	1001011	0000011	01011	N/A
42	0100011	N/A	N/A	N/A	N/A	10000011	0010011	010011	N/A
43	0001011	N/A	N/A	N/A	N/A	10100011	1001011	0000011	01011
44	00000011	N/A	N/A	N/A	N/A	N/A	10000011	0010011	010011
45	00100011	0100011	N/A	N/A	N/A	N/A	10100011	1001011	0000011
46	10010011	0001011	N/A	N/A	N/A	N/A	N/A	10000011	0010011
47	10001011	00000011	N/A	N/A	N/A	N/A	N/A	10100011	1001011
48	10101011	00100011	0100011	N/A	N/A	N/A	N/A	N/A	10000011
49	N/A	10010011	0001011	N/A	N/A	N/A	N/A	N/A	10100011
50	N/A	10001011	00000011	N/A	N/A	N/A	N/A	N/A	N/A

Proposition 3.6. *For $a \leqslant -2$, there are seven straight lines $y + 5x = 10 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 11 - 5a)$, $(a, 12 - 5a), \ldots, (a, 17 - 5a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 01010011, 01001011, 000000011, 001000011, 100100011, 100010011, *and* 101010011.

Proposition 3.7. *For $a \leqslant -2$, there are six straight lines $y + 6x = 13 + j$ for $j = 1, 2, \ldots, 6$ such that the six points $(a, 14 - 6a)$, $(a, 15 - 6a), \ldots, (a, 19 - 6a)$ lie on these lines for $j = 1, 2, \ldots, 6$, respectively which give the respective GH codewords* 010000011, 000100011, 000010011, 001010011, 100001011, *and* 101001011.

Proposition 3.8. *For $a \leqslant -2$, there are seven straight lines $y + 7x = 15 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 16 - 7a)$, $(a, 17 - 7a), \ldots, (a, 22 - 7a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 010100011, 010010011, 000001011, 001001011, 100101011, 1000000011, *and* 1010000011.

Proposition 3.9. *For $a \leqslant -2$, there are seven straight lines $y + 8x = 18 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 19 - 8a)$, $(a, 20 - 8a), \ldots, (a, 25 - 8a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 010001011, 000101011, 0000000011, 0010000011, 1001000011, 1000100011, *and* 1010100011.

Proposition 3.10. *For $a \leqslant -2$, there are seven straight lines $y + 9x = 20 + j$ for $j = 1, 2, \ldots, 7$ such that the seven points $(a, 21 - 9a)$, $(a, 22 - 9a), \ldots, (a, 27 - 9a)$ lie on these lines for $j = 1, 2, \ldots, 7$, respectively which give the respective GH codewords* 010101011, 0100000011, 0001000011, 0000100011, 0010100011, 1000010011, *and* 1010010011.

Table 4: GH Code ($51 \leqslant n \leqslant 100,\ -12 \leqslant a \leqslant -20$).

n	GH_{-12}	GH_{-13}	GH_{-14}	GH_{-15}	GH_{-16}	GH_{-17}	GH_{-18}	GH_{-19}	GH_{-20}
51	N/A	10101011	00100011	0100011	N/A	N/A	N/A	N/A	N/A
52	N/A	N/A	10010011	0001011	N/A	N/A	N/A	N/A	N/A
53	N/A	N/A	10001011	00000011	N/A	N/A	N/A	N/A	N/A
54	N/A	N/A	10101011	00100011	0100011	N/A	N/A	N/A	N/A
55	N/A	N/A	N/A	10010011	0001011	N/A	N/A	N/A	N/A
56	0101011	N/A	N/A	10001011	00000011	N/A	N/A	N/A	N/A
57	01000011	N/A	N/A	10101011	00100011	0100011	N/A	N/A	N/A
58	00010011	N/A	N/A	N/A	10010011	0001011	N/A	N/A	N/A
59	00001011	N/A	N/A	N/A	10001011	00000011	N/A	N/A	N/A
60	00101011	0101011	N/A	N/A	10101011	00100011	0100011	N/A	N/A
61	100000011	01000011	N/A	N/A	N/A	10010011	0001011	N/A	N/A
62	101000011	00010011	N/A	N/A	N/A	10001011	00000011	N/A	N/A
63	N/A	00001011	N/A	N/A	N/A	10101011	00100011	0100011	N/A
64	N/A	00101011	0101011	N/A	N/A	N/A	10010011	0001011	N/A
65	N/A	100000011	01000011	N/A	N/A	N/A	10001011	00000011	N/A
66	N/A	101000011	00010011	N/A	N/A	N/A	10101011	00100011	0100011
67	N/A	N/A	00001011	N/A	N/A	N/A	N/A	10010011	0001011
68	N/A	N/A	00101011	0101011	N/A	N/A	N/A	10001011	00000011
69	N/A	N/A	100000011	01000011	N/A	N/A	N/A	10101011	00100011
70	N/A	N/A	101000011	00010011	N/A	N/A	N/A	N/A	10010011
71	01010011	N/A	N/A	00001011	N/A	N/A	N/A	N/A	10001011
72	01001011	N/A	N/A	00101011	0101011	N/A	N/A	N/A	10101011
73	000000011	N/A	N/A	100000011	01000011	N/A	N/A	N/A	N/A
74	001000011	N/A	N/A	101000011	00010011	N/A	N/A	N/A	N/A
75	100100011	N/A	N/A	N/A	00001011	N/A	N/A	N/A	N/A
76	100010011	01010011	N/A	N/A	00101011	0101011	N/A	N/A	N/A
77	101010011	01001011	N/A	N/A	100000011	01000011	N/A	N/A	N/A
78	N/A	000000011	N/A	N/A	101000011	00010011	N/A	N/A	N/A
79	N/A	001000011	N/A	N/A	N/A	00001011	N/A	N/A	N/A
80	N/A	100100011	N/A	N/A	N/A	00101011	0101011	N/A	N/A
81	N/A	100010011	01010011	N/A	N/A	100000011	01000011	N/A	N/A
82	N/A	101010011	01001011	N/A	N/A	101000011	00010011	N/A	N/A
83	N/A	N/A	000000011	N/A	N/A	N/A	00001011	N/A	N/A
84	N/A	N/A	001000011	N/A	N/A	N/A	00101011	0101011	N/A
85	N/A	N/A	100100011	N/A	N/A	N/A	100000011	01000011	N/A
86	010000011	N/A	100010011	01010011	N/A	N/A	101000011	00010011	N/A
87	000100011	N/A	101010011	01001011	N/A	N/A	N/A	00001011	N/A
88	000010011	N/A	N/A	000000011	N/A	N/A	N/A	00101011	0101011
89	001010011	N/A	N/A	001000011	N/A	N/A	N/A	100000011	01000011
90	100001011	N/A	N/A	100100011	N/A	N/A	N/A	101000011	00010011
91	101001011	N/A	N/A	100010011	01010011	N/A	N/A	N/A	00001011
92	N/A	010000011	N/A	101010011	01001011	N/A	N/A	N/A	00101011
93	N/A	000100011	N/A	N/A	000000011	N/A	N/A	N/A	100000011
94	N/A	000010011	N/A	N/A	001000011	N/A	N/A	N/A	101000011
95	N/A	001010011	N/A	N/A	100100011	N/A	N/A	N/A	N/A
96	N/A	100001011	N/A	N/A	100010011	01010011	N/A	N/A	N/A
97	N/A	101001011	N/A	N/A	101010011	01001011	N/A	N/A	N/A
98	N/A	N/A	010000011	N/A	N/A	000000011	N/A	N/A	N/A
99	N/A	N/A	000100011	N/A	N/A	001000011	N/A	N/A	N/A
100	010100011	N/A	000010011	N/A	N/A	100100011	N/A	N/A	N/A

Proposition 3.11. *For $a \leqslant -2$, there are seven straight lines $y + 10x = 23 + j$ for $j = 1, 2, \dots, 7$ such that the seven points $(a, 24 - 10a)$, $(a, 25 - 10a), \dots, (a, 30 - 10a)$ lie on these lines for $j = 1, 2, \dots, 7$, respectively which give the respective GH codewords* 0101000011, 0100100011, 0000010011, 0010010011, 1001010011, 1000001011, *and* 1010001011.

Proposition 3.12. *For $a \leqslant -2$, there are seven straight lines $y + 11x = 26 + j$ for $j = 1, 2, \dots, 7$ such that the seven points $(a, 27 - 11a)$, $(a, 28 - 11a), \dots, (a, 33 - 11a)$ lie on these lines for $j = 1, 2, \dots, 7$, respectively which give the respective GH codewords* 0100010011, 0001010011, 0000001011, 0010001011, 1001001011, 1000101011, *and* 1010101011.

Proposition 3.13. *For $a \leqslant -2$, there are seven straight lines $y + 12x = 28 + j$ for $j = 1, 2, \dots, 7$ such that the seven points $(a, 29 - 12a)$, $(a, 30 - 12a), \dots, (a, 35 - 12a)$ lie on these lines for $j = 1, 2, \dots, 7$, respectively which give the respective GH codewords* 0101010011, 0100001011, 0001001011, 0000101011, 0010101011, 10000000011, *and* 10100000011.

Proposition 3.14. *For $a \leqslant -2$, there are seven straight lines $y + 13x = 31 + j$ for $j = 1, 2, \dots, 7$ such that the seven points $(a, 32 - 13a)$, $(a, 33 - 13a), \dots, (a, 38 - 13a)$ lie on these lines for $j = 1, 2, \dots, 7$, respectively which give the respective GH codewords* 0101001011, 0100101011, 00000000011, 00100000011, 10010000011, 10001000011, *and* 10101000011.

Thus we can determine the GH codeword of (a, n), $a \leqslant -2$, $n \leqslant 38 - 13a$, if it exists from the above mentioned properties.

Definition 3.15. The straight line $y + mx = c$ is called GH code straight line if all the integral points (a, n) for $a \leqslant -2$, $n \geqslant 1$ on this straight line have GH codeword. Otherwise it is called Non-GH code straight line.

Without any loss of generality we assume that m and c both are non-negative integers.

Note 3.16. The point (a, n) satisfying more than one GH code straight line, does not have unique GH codeword.

Theorem 3.17. *The point* (a, n) *gives the GH codeword* $a_1a_2 \dots a_l1$ *if and only if the point* (a, n) *satisfies the straight line* $y + \{\sum_{k=1}^{l} a_k F(k-4)\}x = \sum_{k=1}^{l} a_k F(k-2)$ *where* $a_k \in \{0, 1\}$ *and the string* $a_1a_2 \dots a_l$ *does not contain any consecutive* 1*'s.*

Proof. Let (a, n) gives the codeword $a_1a_2 \dots a_l1$. Then $n = \sum_{k=1}^{l} a_k VF_a(k) = \sum_{k=1}^{l} a_k F(k-2) - a\sum_{k=1}^{l} a_k F(k-4)$, by Theorem 2.1. The point $(a, \sum_{k=1}^{l} a_k F(k-2) - a\sum_{k=1}^{l} a_k F(k-4))$ lies on the straight line $y + \{\sum_{k=1}^{l} a_k F(k-4)\}x = \sum_{k=1}^{l} a_k F(k-2)$. Hence, the point (a, n) lies on the straight line $y + \{\sum_{k=1}^{l} a_k F(k-4)\}x = \sum_{k=1}^{l} a_k F(k-2)$.

Conversely, let the point (a, n) lies on the straight line $y + \{\sum_{k=1}^{l} a_k F(k-4)\}x = \sum_{k=1}^{l} a_k F(k-2)$. Therefore, for the point (a, n), we have $n = \sum_{k=1}^{l} a_k F(k-2) - a\sum_{k=1}^{l} a_k F(k-4) = \sum_{k=1}^{l} a_k VF_a(k)$, by Theorem 2.1. Hence for $a \leqslant -2$, the GH codeword of (a, n) is $a_1a_2 \dots a_l1$ where $a_k \in \{0, 1\}$ and the string $a_1a_2 \dots a_l$ does not contain any consecutive 1's. □

Corollary 3.18. $(a, VF_a(k))$ *lies on the straight line* $y + F(k-4)x = F(k-2)$ *whose gradient is the Fibonacci number* $-F(k-4)$.

4. Determination of GH codeword for the point (a, n)

4.1. Method 1

In this section, we propose an algorithm which determines the GH codeword of a point (a, n) where $a \leqslant -2$ and $n > 38 - 13a$, if it exists.

Algorithm 4.1.

Step 1. Take $r = 1$, $m = n$.

Step 2. If $m = VF_a(i_r)$, then GH codeword of n always exists and is $\underbrace{00\dots0}_{i_r-1 \text{ times}}11$ and then go to Step 12.

Step 3. If $m - a = VF_a(i_r)$, then GH codeword of n always exists and is $1\underbrace{00\dots0}_{i_r-2 \text{ times}}11$ and then go to Step 12.

Step 4. If $m - a - 1 = VF_a(i_r)$, then GH codeword of n always exists and is $101\underbrace{00\dots0}_{i_r-4 \text{ times}}11$ and then go to Step 12.

Step 5. Let $m_r = VF_a(i_r)$ be the greatest integer less than m.

Step 6. If $(m - m_r) > 38 - 13a$, then consider $m = m - m_r$, $r = r + 1$ and go to Step 5.

Step 7. If $m - m_r$ satisfies at least one straight line given in the Properties 3.1-3.14, then the GH codeword of (a, n) exists and go to Step 8. Otherwise go to Step 9.

Step 8. The GH codeword corresponding to (a, n) is the GH codeword of $(a, m - m_r)$ by deleting the last 1 then put 1 at the positions $i_r, i_{r-1}, \dots, i_1$ and all other positions are zero and then put 1 at the last position and go to Step 12.

Step 9. Let $m_{r+1} = VF_a(i_{r+1})$, for some $k = i_{r+1}$, be the greatest number but not greater than m_r.

Step 10. Let $m - m_{r+1} \geqslant m_{r+1}$, then the GH codeword of (a, n) does not exist and go to Step 12.

Step 11. Otherwise $r = r + 1$ and go to Step 5.

Step 12. End.

Example 4.2. Determination of the GH codeword of $(-6, 467)$ if it exists.

It is clear that $467 \notin VF_{-6}(k)$. By Step 5, $m_1 = 293 = VF_{-6}(12)$ is the greatest integer less than 467 $(= m)$. Now, $m - m_1 = 467 - 293 = 174 > 38 - 13(-6)$. By Step 6, $m = 174$ and by Step 5, $m_2 = 112 = VF_{-6}(10)$ is the greatest integer less than 174. Now, $m - m_2 = 174 - 112 = 62 < 38 - 13(-6)$ and $(-6, 62)$ satisfies the straight line $y + 7x = 15 + 5$ (Property 3.8).

Therefore the GH codeword of $(-6, 62)$ is 100101011. Hence by Step 8, the GH codeword of $(-6, 467)$ is 1001010101011.

Example 4.3. Determination of the GH codeword of $(-5, 683)$ if it exists.

It is obvious that $683 \notin VF_{-5}(k)$. By Step 5, $m_1 = 678 = VF_{-5}(14)$ is the greatest number less than $683(= m)$. We see that $m - m_1 = 683 - 678 = 5 < 38 - 13(-5)$ and $5 < 678$. But $(-5, 5)$ does not satisfy any one of the straight lines given in the Properties 3.1-3.14. So, by Step 9, we take $m_1 = 419 = VF_{-5}(13)$. Now, $m - m_1 = 683 - 419 = 264 > 38 - 13(-5)$. By Step 6, $m = 264$ and by Step 5, $m_2 = 259 = VF_{-5}(12)$ is the greatest number less than 264. Again, $m - m_2 = 264 - 259 = 5 < 38 - 13(-5)$ and $5 < 259$. So according to Step 10, we take $m_2 = 160 = VF_{-5}(11)$. Now, $m - m_2 = 264 - 160 = 104 > 38 - 13(-5)$. By Step 6, $m = 104$ and by Step 5, $m_3 = 99 = VF_{-5}(10)$ is the greatest number less than 104. Check that $m - m_3 = 104 - 99 = 5 < 38 - 13(-5)$ and $5 < 99$. So, by Step 9, we take $m_3 = 61 = VF_{-5}(9)$. Now, $m - m_3 = 104 - 61 = 43 > 38 - 13(-5)$ and $43 < 61$. But $(-5, 43)$ does not satisfy any one of the straight lines given in the Properties 3.1-3.14. By Step 10, we take $m_3 = 38 = VF_{-5}(8)$. $m - m_3 = 104 - 38 = 66 < 38 - 13(-5)$ and $66 > 38$. By Step 10, the GH codeword of $(-5, 683)$ does not exist.

4.2. Determination of c depending on $m(\geqslant 2)$ of the GH code straight line $y + mx = c + j$, $1 \leqslant j \leqslant 7$

Let (a, n) be any point on the GH code straight line $y + mx = c + 1$, $m \geqslant 2$ and the GH codeword of (a, n) be $a_1 a_2 \ldots a_l 1$ where $a_k \in \{0, 1\}$ and the string $a_1 a_2 \ldots a_l$ does not contain any consecutive 1's. From the Properties 3.1-3.14 and using Method 1, for the GH code straight line $y + mx = c + 1$ we have always $a_1 = 0, a_2 = 1, a_3 = 0, a_5 = 0$. Now $n = \sum_{k=1}^{l} a_k VF_a(k) = \sum_{k=1}^{l} a_k F(k-2) - a \sum_{k=1}^{l} a_k F(k-4)$, by Theorem 2.1. Again by Theorem 3.17, $m = \sum_{k=1}^{l} a_k F(k-4) = a_1 F(-3) + a_2 F(-2) + a_3 F(-1) + a_4 F(0) + \sum_{k=5}^{l} a_k F(k-4) = 1 + a_4 + \sum_{k=5}^{l} a_k F(k-4)$, since $F(-3) = -1$, $F(-2) = 1$, $F(-1) = 0$ and $F(0) = 1$. Therefore,

$$m - 1 - a_4 = \sum_{k=5}^{l} a_k F(k-4), \tag{4.1}$$

where $a_5 a_6 \ldots a_l$ does not contain any consecutive 1's. Thus $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m - 1 - a_4$. Now, a_4 is either 0 or 1. Hence, $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m - 2$ if and only if $a_4 = 1$ and $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m - 1$ if and only if $a_4 = 0$.

Again, $a_5 = 0$ always for the GH code straight line $y + mx = c + 1$ and $m - 2$, $m - 1$ are consecutive positive integers.

There are three possible cases $0, 1$; $1, 0$; and $0, 0$ for the coefficients of $F(1)$ in the Zeckendorf's representation of $m - 2$ and $m - 1$, respectively. The case $1, 1$ is not possible, by Property 1.1.

Case 1: For any two consecutive positive integers $m - 2$ and $m - 1$, we consider the case that the coefficients of $F(1)$ in the Zeckendorf's representation of $m - 2$ is 0 and $m - 1$ is 1. But in the equation (4.1) the coefficient of $F(1)$ for $m-1$ cannot be equal to 1 since $a_5=0$. Hence $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-2$.

Case 2: For any two consecutive positive integers $m-2$ and $m-1$, we consider the case that the coefficient of $F(1)$ in the Zeckendorf's representation of $m-2$ is 1 and $m-1$ is 0. But in the equation (4.1) the coefficient of $F(1)$ for $m-2$ cannot be equal to 1 since $a_5=0$. Hence $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-1$.

Case 3: For any two consecutive positive integers $m-2$ and $m-1$, we consider the case that the coefficient of $F(1)$ in the Zeckendorf's representation of $m-2$ is 0 and $m-1$ is 0.

Then the straight lines $y+4x=7+1$, $y+9x=20+1$ and $y+12x=28+1$ for $m=4,9,12$ satisfy this case, i.e., the coefficient of $F(1)$ in the Zeckendorf's representation of $m-2$ is 0 as well as $m-1$ is 0. For the straight line $y+4x=7+1$, we have $a_1=0, a_2=1, a_3=0, a_4=1, a_5=0, a_6=1$ (by the Property 3.5) and $a_5F(1)+a_6F(2)$ is the Zeckendorf's representation of $4-2=2$. For the straight line $y+9x=20+1$, we have $a_1=0, a_2=1, a_3=0, a_4=1, a_5=0, a_6=1, a_7=0, a_8=1$ (by the Property 3.10) and $a_5F(1)+a_6F(2)+a_7F(3)+a_8F(4)$ is the Zeckendorf's representation of $9-2=7$. For the straight line $y+12x=28+1$, we have $a_1=0, a_2=1, a_3=0, a_4=1, a_5=0, a_6=1, a_7=0, a_8=0, a_9=1$ (by the Property 3.13) and $a_5F(1)+a_6F(2)+a_7F(3)+a_8F(4)+a_9F(5)$ is the Zeckendorf's representation of $12-2=10$. Hence in this case for $m \leqslant 13$, $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-2$.

Let $m \geqslant 14$. Then we have either $m-2=\sum_{k=5}^{l} a_k F(k-4)$ or $m-1=\sum_{k=5}^{l} a_k F(k-4)$. It is obvious that we have (i) $a_5F(1)+a_6F(2)=2$ or (ii) $a_5F(1)+a_6F(2)+a_7F(3)+a_8F(4)=7$ or (iii) $a_5F(1)+a_6F(2)+a_7F(3)+a_8F(4)+a_9F(5)=10$ or (i) and (ii) or (i) and (iii). Thus we have always a_6, the coefficient of $F(2)$ is 1. Hence by using Property 1.2, $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-2$.

Therefore, we obtain the positive integer $m-1$ for the case 2, otherwise we obtain the positive integer $m-2$.

Now from Theorem 3.17, we have

$$c+1=1+a_4F(2)+\sum_{k=5}^{l} a_k F(k-2).$$

Therefore,

$$c=2a_4+\sum_{k=5}^{l} a_k F(k-2), \text{ since } F(2)=2,$$

where $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-2$ or $m-1$ according as $a_4=1$ or $a_4=0$. Hence, $c=\sum_{k=5}^{l} a_k F(k-2)+2$ if $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-2$ or $c=\sum_{k=5}^{l} a_k F(k-2)$ if $\sum_{k=5}^{l} a_k F(k-4)$ is the Zeckendorf's representation of $m-1$.

Thus we arrive at the following theorem.

Theorem 4.4. *The GH codeword exists for* (a,n) *if and only if* (a,n) *satisfies at least one of the straight lines* $y+mx=c+j$, *where* m, c, j *are non-negative integers and*

Case 1: $m=0$, $c=0$ *and* $j=1,2,3,4$;

Case 2: $m=1$, $c=0$ *and* $j=1,2,3,4,5,6$;

Case 3: $m>1$. *If the coefficient of* $F(1)$ *in the Zeckendorf's representation of* $m-2$ *is 0, then* $c=\sum_{k=1}^{l} a_k F(k+2)+2$, *where* $\sum_{k=1}^{l} a_k F(k)$ *is the Zeckendorf's representation of* $m-2$, *otherwise* $c=\sum_{k=1}^{l} a_k F(k+2)$, *where* $\sum_{k=1}^{l} a_k F(k)$ *is the Zeckendorf's representation of* $m-1$ *and for* $m=F(4)+F(1)$, $F(6)+F(1)$, $F(6)+F(4)+F(1)$, $F(7)+F(4)+F(1)$, $F(8)+F(1)$, $F(8)+F(4)+F(1)$, $F(8)+F(6)+F(1)$, $F(8)+F(6)+F(4)+F(1)$, $F(9)+F(4)+F(1)$, $F(9)+F(6)+F(1)$, $F(9)+F(6)+F(4)+F(1)$, $F(9)+F(7)+F(4)+F(1)$, $F(10)+F(1)+\cdots$, $j=1,2,3,4,5,6$, *otherwise* $j=1,2,3,4,5,6,7$.

Proof.

Case 1: It follows from Property 3.1.

Case 2: It follows from Property 3.2.

Case 3: It follows from 4.2 that if the coefficient of $F(1)$ in the Zeckendorf's representation of $m-2$ is 0, then $c = \sum_{k=1}^{l} a_k F(k+2) + 2$ where $\sum_{k=1}^{l} a_k F(k)$ is the Zeckendorf's representation of $m-2$ otherwise $c = \sum_{k=1}^{l} a_k F(k+2)$ where $\sum_{k=1}^{l} a_k F(k)$ is the Zeckendorf's representation of $m-1$.

Property 3.1 gives for $m = 0$, $j = 1,2,3,4$. Property 3.2 gives for $m = 1 = F(1)$, $j = 1,2,3,4,5,6$. Properties 3.3-3.14 give for $m = 6 = F(4) + F(1)$, $j = 1,2,3,4,5,6$ and for $m = 2,3,4,5,7,\ldots,13$, $j = 1,2,3,4,5,6,7$. Also from Theorem 3.17 we have $(a, VF_a(k))$ lies on the straight line $y + F(k-4)x = F(k-2)$.

Hence by repeatedly applying Method 1 we have, for $m = F(4)+F(1)$, $F(6)+F(1)$, $F(6)+F(4)+F(1)$, $F(7)+F(4)+F(1)$, $F(8)+F(1)$, $F(8)+F(4)+F(1)$, $F(8)+F(6)+F(1)$, $F(8)+F(6)+F(4)+F(1)$, $F(9)+F(4)+F(1)$, $F(9)+F(6)+F(1)$, $F(9)+F(6)+F(4)+F(1)$, $F(9)+F(7)+F(4)+F(1)$, $F(10)+F(1)+\cdots$, $j = 1,2,3,4,5,6$, otherwise $j = 1,2,3,4,5,6,7$. □

The GH codewords for $0 \leqslant m \leqslant 14$ are given in Table 5.

The GH code straight lines for $m = 0,1,2,3,4,5,6,7,8$ are given in Figure 1.

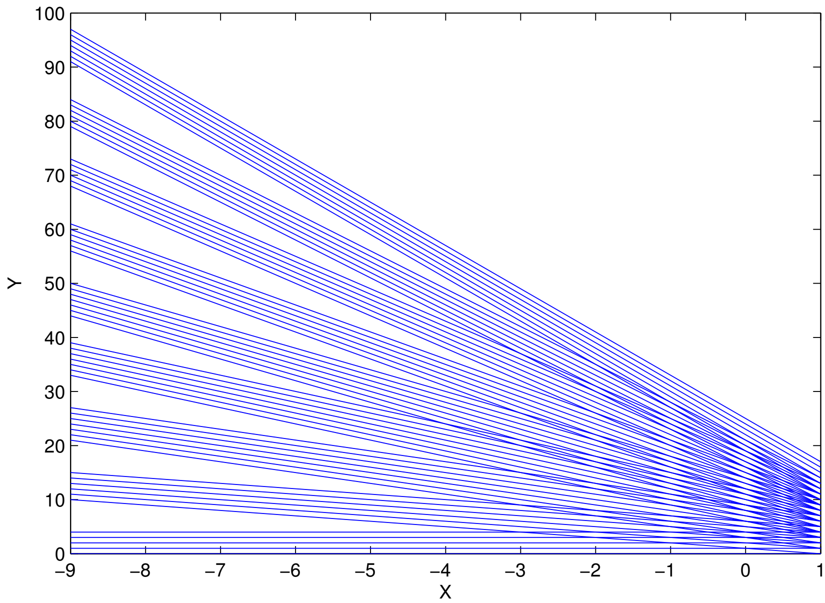

Figure 1: GH code straight lines.

Corollary 4.5. *The GH codeword of* (a,n) *is the codeword corresponding to the GH code straight line* $y + mx = c + j$ *satisfied by* (a,n).

Corollary 4.6. *The GH codeword of* (a,n) *is not unique if* (a,n) *satisfies more than one GH code straight line.*

Theorem 4.7. *GH codeword of* (a,n) *is not unique for* $-4 \leqslant a \leqslant -2$.

Proof. GH codeword of (a,n) is not unique if (a,n) lies on at least two GH code straight lines. If (a,n) lies on the GH code straight lines $y + m_1 x = c_1 + j_1$ and $y + m_2 x = c_2 + j_2$, where $m_1 < m_2$, then $a = \frac{(c_2+j_2)-(c_1+j_1)}{m_2-m_1}$. Therefore,

$$c_2 + j_2 < c_1 + j_1, \tag{4.2}$$

since $a \leqslant -2$ and $m_1 < m_2$.

Table 5: The GH codewords for $0 \leqslant m \leqslant 14$.

m	c(m)	j	GH code	m	c(m)	j	GH code
0	0	1	0011	8	18	1	010001011
		2	10011			2	000101011
		3	100011			3	0000000011
		4	101011			4	0010000011
1	0	1	011			5	1001000011
		2	00011			6	1000100011
		3	000011			7	1010100011
		4	001011	9	20	1	010101011
		5	1000011			2	010000011
		6	1010011			3	0001000011
2	2	1	01011			4	0000100011
		2	010011			5	0010100011
		3	0000011			6	1000010011
		4	0010011			7	1010010011
		5	1001011	10	23	1	0101000011
		6	10000011			2	0100100011
		7	10100011			3	0000010011
3	5	1	0100011			4	0010010011
		2	0001011			5	1001010011
		3	00000011			6	1000001011
		4	00100011			7	1010001011
		5	10010011	11	26	1	0100010011
		6	10001011			2	0001010011
		7	10101011			3	0000001011
4	7	1	0101011			4	0010001011
		2	01000011			5	1001001011
		3	00010011			6	1000101011
		4	00001011			7	1010101011
		5	00101011	12	28	1	0101010011
		6	100000011			2	0100001011
		7	101000011			3	0001001011
5	10	1	01010011			4	0000101011
		2	01001011			5	0010101011
		3	000000011			6	10000000011
		4	00100011			7	10100000011
		5	100100011	13	31	1	0101001011
		6	100010011			2	0100101011
		7	101010011			3	00000000011
6	13	1	010000011			4	00100000011
		2	000100011			5	10010000011
		3	000010011			6	10001000011
		4	001010011			7	10101000011
		5	100001011	14	34	1	01000000011
		6	101001011			2	00011000011
7	15	1	010100011			3	00001000011
		2	010010011			4	00101000011
		3	000001011			5	100001000011
		4	001001011			6	10100100011
		5	100101011				
		6	1000000011				
		7	1010000011				

We have from the equations of the GH code straight lines, for $a \leqslant -2$, (4.2) is possible only when m_1 and m_2 are consecutive, i.e., $m_2 - m_1 = 1$, otherwise $a > -2$. Therefore, $a = (c_2 + j_2) - (c_1 + j_1) \leqslant -2$. Then $j_2 - j_1 \leqslant -4$ or -5, since $c_2 - c_1 = 2$ or 3, by Theorem 4.4. So $j_2 - j_1 = -4, -5$ or -6, since $1 \leqslant j_1, j_2 \leqslant 7$. Hence $a = -2, -3, -4$.

Thus the GH codeword of (a, n) is not unique for $-4 \leqslant a \leqslant -2$. □

Theorem 4.8. *$(a, n), a \leqslant -2$ satisfies maximum two GH code straight lines.*

Proof. We know that, the gradients of two intersecting GH code straight lines for $-4 \leqslant a \leqslant -2$ must be consecutive. Hence, $(a, n), a \leqslant -2$ satisfies maximum two GH code straight lines. □

Corollary 4.9. *The maximum GH codewords of (a, n) are two if the codeword of (a, n) exists.*

Note 4.10. We have one alternative codewords of the green colored codewords of (a, n)'s in Tables 1 and 2.

For an example, $(-3, 9)$ has two codewords: 1010011 and 01011.

Corollary 4.11. *The GH codeword of (a, n) if exists, is unique for $a \leqslant -5$.*

4.3. *Method 2*

Algorithm 4.12.

Step 1. Check whether (a, n) satisfies at least one of the GH code straight line $y + mx = c + j$ or not. If it satisfies, the GH codeword of (a, n) exists otherwise not.

Step 2. Write down the GH code straight line.

Step 3. Find the point on that straight line when $a = -5$ (we can choose any value of $a \leqslant -5$). Let the point be $(-5, n_1)$.

Step 4. Determine the GH codeword of $(-5, n_1)$ by using Algorithm 4.1. The GH codeword of $(-5, n_1)$ is the GH codeword of (a, n).

Example 4.13. Determine the GH codeword of the point $(-600, 4560)$, if it exits.

Let $(-600, 4560)$ satisfies the straight line $y + mx = c + j$. Then the maximum value of m is $[\frac{4560}{600}] = 7$. Therefore $c = F(6) + 2 = 13 + 2 = 15$, since $m - 2 = 5 = F(4)$. From the equation we have, $j = 345 > 7$. Therefore $(-600, 4560)$ does not satisfy any GH code straight line and hence GH codeword of $(-600, 4560)$ does not exist.

Example 4.14. Determine the GH codeword of the point $(-1745, 15728)$, if it exits.

Let $(-1745, 15728)$ satisfies the straight line $y + mx = c + j$. Then the maximum value of m is $[\frac{15728}{1745}] = 9$. Therefore $c = F(4) + F(6) + 2 = 5 + 13 + 2 = 20$ since $m - 2 = 7 = F(2) + F(4)$. So $j = 3 < 7$. Therefore $(-1745, 15728)$ satisfies the GH code straight line $y + 9x = 20 + 3$ and hence GH codeword of $(-1745, 15728)$ exists. Now the GH code straight line $y + 9x = 20 + 3$ gives $y = 68$ when $x = -5$. Thus the GH codeword of $(-1745, 15728)$ is 0001000011 since Table 2 gives the GH codeword of $(-5, 68)$ which is 0001000011.

5. Conclusion

For the sequence $VF_{-5}(k)$, Thomas showed that it is not possible to write the Zeckendorf's representation for integers 5 and 12. Thereafter, Basu and Prasad improved the availability of GH code upto positive integer 100 for $-20 \leqslant a \leqslant -2$. But there was no general method to determine the GH codeword for any integer $a \leqslant -2$ and any positive integer n. In this paper, we have introduced the idea of GH code straight line. Then we proposed a couple of algorithms to determine the existence of GH codewords. If GH codeword exists, then it is easy to find the codeword for a given parameter (a, n), using algorithms mentioned in this paper.

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