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Characterization of Marshall-Olkin-G family of distributions by truncated moments

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Abstract

In this paper, a characterization of the Marshall-Olkin-G family of distribution (MO-G) [A. W. Marshall, I. Olkin, Biometrika, **84** (1997), 641–652] by left and right truncated moments based on a certain continuous function of a random variable is discussed under some necessary condition. We provide characterization of Marshall-Olkin Nadarajah-Haghighi distribution (MONH) [A. J. Lemonte, G. M. Cordeiro, G. Moreno-Arenas, Statistics, **50** (2016), 312–337] and Marshall-Olkin generalized Erlang-truncated exponential distribution (MOGETE) [I. E. Okorie, A. C. Akpanta, J. Ohakwe, Cogent Math., **4** (2017), 19 pages] for illustration.

Keywords: MO-G family, MONH model, MOGETE model, truncated Moments, failure rate, reverse failure rate, characterization of distribution.

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1. Introduction

Characterizations of probability models are a very important aspect in distribution theory. Characterizing of a probability distribution, consequently, state a characteristic that it is the only distribution that satisfied a specified condition. A characterization of probability models played a vital role in statistical studies in various fields of sciences and applied sciences. A distribution characterizes a variable when the conditions of the distribution are similar to those of the variable. Many researchers have investigated the characterizations of absolutely continuous probability distributions over the years. For instance, the characterization of distributions by truncated moments was studied by [11]. [4] discussed various techniques of characterizations of probability distributions. [6] characterized distributions by the moments of residual life. [19] investigated the characterizations of distributions by conditional expectations. Characterizations through mean residual life and failure rates were studied by [17]. Recently, [8] Characterized various distributions based on infinite divisibility. [1] provide a characterization of Lindley distribution (L) based on left and right truncated moments. [14] characterized half logistic Poisson distribution (HLP)

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[15] based on left and right truncated moments. [10] characterized Lindley distribution (L) based on truncated moments of order statistics.

MO-G distribution has received significant attention from many authors in recent years. The most recent family of MO-G include the Marshall-Olkin Kappa [9] (MO-Kp) and the Marshall-Olkin length-biased exponential (MO-LBE) [20] distributions among others. The characterization of Marshall-Olkin Power Log-Normal (MOPLN) [5] and Marshall-Olkin Extended Burr Type XII (MOEBXII) [2] based on truncated moments and hazard rate functions were considered by [7].

[1] characterized Lindley distribution by taking the conditional expectation based on a function of a non-negative random variable X^n, $n \in \mathbb{N}$.

In this paper, we follow similar procedure to [1, 14] using a different certain continuous function of a non-negative random variable under some necessary condition to characterizes the Marshal-Olkin-G family of distribution by left and right truncated expectations.

Some general results are considered to characterized the Marshal-Olkin-G family in Section 2. Two particular cases using the MONH and MOGETE distributions are studied in Section 3. Conclusion is presented in Section 4.

2. Characterization of MO-G via left and right truncated moments

In this section, a characterization of MO-G based on left and right truncated moments of a certain continuous function of a random variable is discussed under some necessary condition.

The probability density function (pdf) $f(x)$, cumulative distribution function (cdf) $F(x)$, failure rate function (hrf) $h(x)$, reverse failure rate function (rhrf) $r(x)$, and survival function $s(x)$ (sf) of MO-G family are given by

$$F(x) = \frac{G(x;\xi)}{1-(1-\beta)\bar{G}(x;\xi)}, \tag{2.1}$$

$$f(x) = \frac{\beta g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]^2},$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{g(x;\xi)}{\bar{G}(x,\xi)[1-(1-\beta)\bar{G}(x;\xi)]},$$

$$r(x) = \frac{f(x)}{F(x)} = \frac{\beta g(x;\xi)}{G(x,\xi)[1-(1-\beta)\bar{G}(x;\xi)]},$$

$$s(x) = \frac{\beta\bar{G}(x;\xi)}{G(x;\xi)+\beta\bar{G}(x;\xi)}, \tag{2.2}$$

respectively, where $\beta > 0$, ξ is a parameter vector, and $G(x;\xi)$ and $g(x;\xi)$ are any valid baseline cdf and pdf, respectively.

Interpretation 2.1. *Let (X, Z) be a random vector with joint density function $f(x, z)$ defined on $\mathbb{R}^2$. Suppose that the conditional cumulative distribution of X given $Z = z$ is $\boldsymbol{T}(x|z)$ and $Z \sim \boldsymbol{c}(z)$. Then the following defines the unconditional survival function of X,*

$$s(x) = \int \bar{\boldsymbol{T}}(x|z)\boldsymbol{c}(z)dz.$$

The survival function $s(x)$ is obtained by compounding the survival function $\bar{\boldsymbol{T}}(x|z) = 1-\boldsymbol{T}(x|z)$ and the density of $\boldsymbol{c}(z)$. Suppose that the survival function

$$\bar{\boldsymbol{T}}(x|z) = e^{-z\frac{G(x;\xi)}{\beta\bar{G}(x;\xi)}},$$

where $\beta > 0$, ξ a parameter vector, $G(x;\xi)$ is any valid cumulative distribution, $\bar{G}(x;\xi) = 1-G(x;\xi)$, and Z assumed to have exponential distribution with mean 1, then X has survival function in (2.2).

Proof. For all $x, z, \beta > 0$, the survival function is given as

$$s(x) = \int \bar{\mathbf{T}}(x|z)\mathbf{c}(z)dz = \int_0^\infty e^{-z\frac{G(x)}{\beta\bar{G}(x)}} e^{-z}dz = \int_0^\infty e^{-z\left[\frac{G(x)}{\beta\bar{G}(x)}+1\right]}dz = \left[\frac{G(x)}{\beta\bar{G}(x)}+1\right]^{-1} = \frac{\beta\bar{G}(x)}{G(x)+\beta\bar{G}(x)}.$$

□

2.1. Characterization of MO-G family by left truncated moments

Here, we provide an important lemma as a tool for the characterization of the MO-G by left truncated moments.

Lemma 2.2. *Suppose that the random variable X has an absolutely continuous c.d.f* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$ *for all* $x > 0$, *with density function* $f(x) = F'(x)$ *and failure rate* $h(x) = f(x)/[1 - F(x)]$. *Let* $q(x)$ *be a continuous function in* $x > 0$ *and* $E[q(X)] < \infty$. *If* $E[q(X)|X \geqslant x] = \tau(x)h(x), x > 0$, *where* $\tau(x)$ *is a differentiable function in* $x > 0$, *then* $f(x) = K\exp\left[-\int_0^x \frac{q(y)+\tau'(y)}{\tau(y)}dy\right]$, $x > 0$, *where* $K > 0$ *is a normalizing constant.*

Proof. Since, $E[q(X)|X \geqslant x] = \frac{1}{1-F(x)}\int_x^\infty q(y)f(y)dy$, it follows that

$$\int_x^\infty q(y)f(y)dy = \tau(x)f(x), \tag{2.3}$$

differentiating both side of (2.3) we get

$$-q(x)f(x) = \tau(x)f'(x) + \tau'(x)f(x),$$

this implies

$$f'(x) + \left(\frac{q(x)+\tau'(x)}{\tau(x)}\right)f(x) = 0, \tag{2.4}$$

which is first order linear differential equation w.r.t $f(x)$. From the general solution of (2.4), we have

$$f(x) = K\exp\left[-\int_0^x \frac{q(y)+\tau'(y)}{\tau(y)}dy\right], \quad x > 0,$$

where K is normalizing constant. □

Now, we provide the characterization of MO-G based on Lemma 2.2.

Theorem 2.3. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$ $\forall x > 0$, *with density function* $f(x) = F'(x)$ *and failure rate* $h(x) = f(x)/[1 - F(x)]$. *Assume that*

$$E\left[1-(1-\beta)\bar{G}(X;\xi)\right] < \infty$$

for $\beta > 0, \xi \in \mathbb{R}$, *then X has the MO-G distribution if and only if*

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \geqslant x\right] = h(x)\tau(x), \quad x > 0,$$

where $\tau(x) = \frac{[1-(1-\beta)\bar{G}(x;\xi)]^2\ln[1-(1-\beta)\bar{G}(x;\xi)]}{(1-\beta)g(x,\xi)}$, *provided* $\lim_{x\to 0} g(x;\xi) \neq 0$ *and exist in the parameter region.* $G(x;\xi)$ *and* $g(x;\xi)$ *are any valid baseline cdf and pdf respectively.*

Proof. For necessity, suppose that $\lim_{\to 0} g(x;\xi) \neq 0$ and exist for $\xi \in \mathbb{R}$. For sufficiently, let $q(x) = 1-(1-\beta)\bar{G}(x;\xi)$, then

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \geqslant x\right] = \frac{h(x)}{f(x)}\int_x^\infty \frac{\beta g(y;\xi)}{1-(1-\beta)\bar{G}(y;\xi)}dy,$$

by letting $t = 1-(1-\beta)\bar{G}(x;\xi)$ and after some algebra we have

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \geqslant x\right] = h(x)\tau(x),$$

where $\tau(x) = \frac{[1-(1-\beta)\bar{G}(x;\xi)]^2 \ln[1-(1-\beta)\bar{G}(x;\xi)]}{(\beta-1)g(x;\xi)}$. Now, we obtain $\frac{q(x)}{\tau(x)}$ and $\frac{\tau'(x)}{\tau(x)}$ as

$$\frac{q(x)}{\tau(x)} = \frac{(\beta-1)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]\ln[1-(1-\beta)\bar{G}(x;\xi)]}$$

and

$$\frac{\tau'(x)}{\tau(x)} = \frac{2(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]} + \frac{(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]\ln[1-(1-\beta)\bar{G}(x;\xi)]} - \frac{g'(x;\xi)}{g(x;\xi)}.$$

This implies

$$\frac{\tau'(x)+q(x)}{\tau(x)} = \frac{2(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]} - \frac{g'(x;\xi)}{g(x;\xi)},$$

therefore

$$\int_0^x \frac{\tau'(y)+q(y)}{\tau(y)}dy = \ln[1-(1-\beta)\bar{G}(x;\xi)]^2 - \ln\beta^2 - \ln g(x;\xi) + \ln g(0),$$

hence

$$f(x) = K\exp\left[-\int_0^x \frac{\tau'(y)+q(y)}{\tau(y)}dy\right] = \frac{K\beta^2 g(x;\xi)g^{-1}(0)}{[1-(1-\beta)\bar{G}(x;\xi)]^2},$$

thus $K = g(0)\beta^{-1}$. □

2.2. Characterization of MO-G family by right truncated moments

Now, we provide a supportive lemma for the characterization of the MO-G based on right truncated moments.

Lemma 2.4. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$ $\forall x > 0$, *with density function* $f(x) = F'(x)$ *and reverse failure rate* $r(x) = f(x)/F(x)$. *Let* $q(x)$ *be a continuous function in* $x > 0$ *and* $E[q(X)] < \infty$. *If* $E[q(X)|X \leqslant x] = w(x)r(x), x > 0$, *where* $w(x)$ *is a differentiable function in* $x > 0$, *then*

$$f(x) = K\exp\left[-\int_0^x \frac{w'(y)-q(y)}{w(y)}dy\right], \quad x > 0,$$

where $K > 0$ *is a normalizing constant.*

Proof. We start by

$$E[q(X)|X \leqslant x] = \frac{1}{F(x)}\int_0^x q(y)f(y)dy,$$

we have that

$$\int_0^x q(y)f(y)dy = w(x)f(x). \tag{2.5}$$

Differentiating both side of (2.5) we get

$$q(x)f(x) = w(x)f'(x) + w'(x)f(x),$$

this implies

$$f'(x) + \left(\frac{w'(x)-q(x)}{w(x)}\right)f(x) = 0, \tag{2.6}$$

which is the first order homogeneous linear differential equation w.r.t $f(x)$. From the general solution of (2.6) we get

$$f(x) = K\exp\left[-\int_0^x \frac{w'(y)-q(y)}{w(y)}dy\right], \quad x > 0,$$

where K is normalizing constant. □

Theorem 2.5. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$ $\forall x > 0$, *with density function* $f(x) = F'(x)$ *and reverse failure rate* $r(x) = f(x)/F(x)$. *Assume that*

$$E\left[1-(1-\beta)\bar{G}(X;\xi)\right] < \infty$$

for $\beta > 0, \xi \in \mathbb{R}$, *then, X has the MO-G family of distribution if and only if*

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \leqslant x\right] = r(x)w(x), \quad x > 0,$$

where $w(x) = \frac{[1-(1-\beta)\bar{G}(x;\xi)]^2[\ln[1-(1-\beta)\bar{G}(x;\xi)]-\ln\beta]}{(1-\beta)g(x,\xi)}$ *provided* $\lim_{\to 0} g(x;\xi) \neq 0$ *and exist in the parameter region.* $G(x;\xi)$ *and* $g(x;\xi)$ *are any valid baseline cdf and pdf respectively.*

Proof. For necessity, suppose that $\lim_{x\to 0} g(x;\xi) \neq 0$ and exist in the parameter region. For sufficiently, let $q(x) = 1-(1-\beta)\bar{G}(x;\xi)$, then

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \leqslant x\right] = \frac{r(x)}{f(x)}\int_0^x \frac{\beta g(y;\xi)}{1-(1-\beta)\bar{G}(y;\xi)}dy,$$

by letting $t = 1-(1-\beta)\bar{G}(x;\xi)$ and after some algebraic manipulations we have

$$E\left[1-(1-\beta)\bar{G}(X;\xi)|X \leqslant x\right] = r(x)w(x),$$

where $w(x) = \frac{[1-(1-\beta)\bar{G}(x;\xi)]^2[\ln[1-(1-\beta)\bar{G}(x;\xi)]-\ln\beta]}{(1-\beta)g(x,\xi)}$. Now, we obtained $\frac{q(x)}{w(x)}$ and $\frac{w'(x)}{w(x)}$ as

$$\frac{q(x)}{w(x)} = \frac{(1-\beta)g(x,\xi)}{[1-(1-\beta)\bar{G}(x;\xi)][\ln[1-(1-\beta)\bar{G}(x;\xi)]-\ln\beta]}$$

and

$$\frac{w'(x)}{w(x)} = \frac{2(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]} + \frac{(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)][\ln[1-(1-\beta)\bar{G}(x;\xi)]-\ln\beta]} - \frac{g'(x;\xi)}{g(x;\xi)}.$$

Thus

$$\frac{w'(x)-q(x)}{w(x)} = \frac{2(1-\beta)g(x;\xi)}{[1-(1-\beta)\bar{G}(x;\xi)]} - \frac{g'(x;\xi)}{g(x;\xi)},$$

therefore

$$\int_0^x \frac{w'(y)-q(y)}{w(y)}dy = \ln[1-(1-\beta)\bar{G}(x;\xi)]^2 - \ln\beta^2 - \ln g(x;\xi) + \ln g(0).$$

Hence

$$f(x) = K\exp\left[-\int_0^x \frac{\tau'(y)-q(y)}{\tau(y)}dy\right] = \frac{K\beta^2 g(x;\xi)g^{-1}(0)}{[1-(1-\beta)\bar{G}(x;\xi)]^2},$$

and the normalizing constant K, must be $K = g(0)\beta^{-1}$. □

3. Characterization of MONH and MOGETE distributions based on left and right truncated moments

In this section, we consider two particular cases of MO-G family. A characterization of MONH and MOGETE is presented based on the results in Section 2.

3.1. Characterization of MONH distribution based on left and right truncated moments

The following gives an example of characterization of MO-G based on a particular distribution called Marshall-Olkin Nadarajah-Haghighi (MONH). The MONH distribution is obtained by considering the baseline distribution $G(x;\xi)$ in (2.1) to be Nadarajah-Haghighi exponential type of distribution (NH) [16]. The NH has cdf and pdf as $G_{NH}(x) = 1-e^{[1-(1+\lambda x)^{\alpha}]}$ and $g_{NH}(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}$, respectively, for $x, \alpha, \lambda > 0$. Therefore, the cdf, pdf, hrf, and rhrf of the MONH distribution are:

$$F_{MONH}(x) = \frac{1 - e^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}}},$$
$$f_{MONH}(x) = \frac{\alpha\beta\lambda(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha})]}]^2},$$
$$h_{MONH}(x) = \frac{f_{MONH}(x)}{1-F_{MONH}(x)} = \frac{\alpha\lambda(1+\lambda x)^{\alpha-1}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}},$$
$$r_{MONH}(x) = \frac{f_{MONH}(x)}{F_{MONH}(x)} = \frac{\alpha\beta\lambda(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-e^{[1-(1+\lambda x)^{\alpha}]}][1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]},$$

respectively, where $x, \alpha, \lambda, \beta > 0$. The following Proposition 3.1 discusses the characterization of MONH base on left truncated moment while Proposition 3.2 provides the characterization of MONH base on right truncated moment.

Proposition 3.1. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0\ \forall x > 0$, *with density function* $f(x) = F'(x)$ *and failure rate* $h(x) = f(x)/[1 - F(x)]$. *Assume that* $E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}\right] < \infty$ *for* $\alpha, \beta, \lambda > 0$, *then X has the MONH distribution if and only if*

$$E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \geqslant x\right] = h(x)\tau(x), \quad x > 0,$$

where,

$$\tau(x) = \frac{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2 \ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]}{\alpha\lambda(\beta-1)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}.$$

Proof. For necessity, $\lim_{x\to 0} g_{NH}(x) = \alpha\lambda, \forall \alpha, \lambda$. For sufficiently, let $q(x) = 1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}$, then

$$\begin{aligned} E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \geqslant x\right] &= \frac{h(x)}{f(x)}\int_x^{\infty}(1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]})f(y)dy \\ &= \frac{h(x)[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2}{(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}\int_x^{\infty}\frac{(1+\lambda y)^{\alpha-1}e^{[1-(1+\lambda y)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]}}dy. \end{aligned}$$

Letting $u = 1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]}$, then by some algebraic manipulations we obtained

$$E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \geqslant x\right] = \frac{h(x)[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2 \ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]}{\alpha\lambda(\beta-1)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}.$$

Thus

$$\tau(x) = \frac{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2 \ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]}{\alpha\lambda(\beta-1)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}.$$

We compute $\frac{q(x)}{\tau(x)}$ and $\frac{\tau'(x)}{\tau(x)}$ as

$$\frac{q(x)}{\tau(x)} = \frac{\alpha\lambda(\beta-1)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]}$$

and

$$\begin{aligned} \frac{\tau'(x)}{\tau(x)} = &\frac{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]} \\ &+ \frac{2\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}} - \frac{(\alpha-1)\lambda}{1+\lambda x} + \alpha\lambda(1+\lambda x)^{\alpha-1}, \end{aligned}$$

respectively, therefore

$$\frac{\tau'(x)+q(x)}{\tau(x)} = \frac{2\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}} - \frac{(\alpha-1)\lambda}{1+\lambda x} + \alpha\lambda(1+\lambda x)^{\alpha-1}$$

and

$$\int_0^y \frac{\tau'(y)+q(y)}{\tau(y)}dy = 2\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}] - 2\ln\beta - \ln(1+\lambda x)^{\alpha-1} + [(1+\lambda x)^{\alpha}-1].$$

Hence, we have

$$f(x) = K\exp\left[-\int_0^y \frac{\tau'(y)+q(y)}{\tau(y)}dy\right] = \frac{K\beta^2(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2},$$

where the normalizing constant is $K = \alpha\lambda\beta^{-1}$. □

Proposition 3.2. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$, $\forall x > 0$, *with density function* $f(x) = F'(x)$ *and reverse failure rate* $r(x) = f(x)/F(x)$. *Assume that* $E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}\right] < \infty$ *for* $\alpha, \beta, \lambda > 0$, *then, X has the MONH family of distribution if and only if* $E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \leqslant x\right] = r(x)w(x)$, $x > 0$, *where,*

$$w(x) = \frac{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2[\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}] - \ln\beta]}{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}.$$

Proof. For necessity, $\lim_{x\to 0} g_{NH}(x) = \alpha\lambda$, $\forall\alpha,\lambda$. For sufficiently, let $q(x) = 1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}$, we have that

$$\begin{aligned} E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \leqslant x\right] &= \frac{r(x)}{f(x)}\int_0^x (1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]})f(y)dy \\ &= \frac{r(x)[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2}{(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}\int_0^x \frac{(1+\lambda y)^{\alpha-1}e^{[1-(1+\lambda y)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]}}dy. \end{aligned}$$

Letting $u = 1-(1-\beta)e^{[1-(1+\lambda y)^{\alpha}]}$ and after some algebra we get

$$E\left[1-(1-\beta)e^{[1-(1+\lambda X)^{\alpha}]}|X \leqslant x\right] = r(x)w(x),$$

where

$$w(x) = \frac{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2[\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}] - \ln\beta]}{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}.$$

Then we obtain $\frac{q(x)}{w(x)}$ and $\frac{w'(x)}{w(x)}$ as

$$\frac{q(x)}{w(x)} = \frac{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}][\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}] - \ln\beta]}$$

and

$$\begin{aligned} \frac{w'(x)}{w(x)} &= \frac{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}][\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}] - \ln\beta]} \\ &\quad + \frac{2\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}} - \frac{(\alpha-1)\lambda}{1+\lambda x} + \alpha\lambda(1+\lambda x)^{\alpha-1}, \end{aligned}$$

respectively, thus

$$\frac{w'(x)-q(x)}{w(x)}=\frac{2\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}}-\frac{(\alpha-1)\lambda}{1+\lambda x}+\alpha\lambda(1+\lambda x)^{\alpha-1}$$

and

$$\int_0^y\frac{w'(y)-q(y)}{w(y)}dy=\ln[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2-\ln\beta^2-\ln(1+\lambda x)^{\alpha-1}+[(1+\lambda x)^{\alpha}-1].$$

Finally we have

$$f(x)=K\exp\left[-\int_0^y\frac{w'(y)-q(y)}{w(y)}dy\right]=\frac{K\beta^2(1+\lambda x)^{\alpha-1}e^{[1-(1+\lambda x)^{\alpha}]}}{[1-(1-\beta)e^{[1-(1+\lambda x)^{\alpha}]}]^2},$$

and the normalizing constant $K=\alpha\lambda\beta^{-1}$. □

Note: If $\alpha=1$ we obtained the characterization of Marshal-Olkin exponential distribution (MOE) [13].

3.2. *Characterization of MOGETE distribution based on left and right truncated moments*

The following is the characterization of Marshall-Olkin generalized Erlang-truncated exponential distribution (MOGETE) [18]. The MOGETE distribution is obtained by considering the baseline distribution $G(x;\xi)$ in (2.1) to be the Erlang- truncated exponential distribution (ETE) [3]. The ETE has cdf and pdf as $G_{ETE}(x)=1-e^{-\alpha(1-e^{-\theta})x}$ and $g_{ETE}(x)=\alpha(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}$, respectively, for $x,\alpha,\theta>0$. Therefore, the cdf, pdf, hrf, and rhrf of the MOGETE distribution are:

$$\begin{aligned}
F_{MOGETE}(x)&=\frac{1-e^{-\alpha(1-e^{-\theta})x}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}},\\
f_{MOGETE}(x)&=\frac{\alpha\beta(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]^2},\\
h_{MOGETE}(x)&=\frac{f_{MOGETE}(x)}{1-F_{MOGETE}(x)}=\frac{\alpha(1-e^{-\theta})}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}},\\
r_{MOGETE}(x)&=\frac{f_{MOGETE}(x)}{F_{MOGETE}(x)}=\frac{\alpha\beta(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{[1-e^{-\alpha(1-e^{-\theta})x}][1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]},
\end{aligned}$$

respectively, where $x,\alpha,\beta,\theta>0$. The following Proposition 3.3 discusses the characterization of MOGETE by left truncated moment and Proposition 3.4 is the characterization of MOGETE by right truncated moment based on certain continuous function.

Proposition 3.3. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0)=0$, $F(x)>0$, $\forall x>0$, *with density function* $f(x)=F'(x)$ *and failure rate* $h(x)=f(x)/[1-F(x)]$. *Assume that* $E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}\right]<\infty$ *for* $\alpha,\beta,\theta>0$, *then X has the MOGETE distribution if and only if* $E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}|X>x\right]=\frac{h(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]}{\alpha(\beta-1)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}$, $x>0$.

Proof. For necessity, $\lim_{x\to0}g_{ETE}(x)=\alpha(1-e^{-\theta})$. For sufficiently, let $q(x)=1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}$, then

$$\begin{aligned}
E[q(X)|X>x]&=\frac{h(x)}{f(x)}\int_x^{\infty}(1-(1-\beta)e^{-\alpha(1-e^{-\theta})y})f(y)dy\\
&=\frac{h(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2}{e^{-\alpha(1-e^{-\theta})x}}\int_x^{\infty}\frac{e^{-\alpha(1-e^{-\theta})y}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})y}}dy.
\end{aligned}$$

By letting $u = 1-(1-\beta)e^{-\alpha(1-e^{-\theta})y}$ and some algebra we get,

$$E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}|X>x\right] = \frac{h(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2 \ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]}{\alpha(\beta-1)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}},$$

thus

$$\tau(x) = \frac{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2 \ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]}{\alpha(\beta-1)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}.$$

In similar way, we get $\frac{q(x)}{\tau(x)}$ and $\frac{\tau'(x)}{\tau(x)}$ as

$$\frac{q(x)}{\tau(x)} = \frac{\alpha(\beta-1)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]}$$

and

$$\begin{aligned}\frac{\tau'(x)}{\tau(x)} &= \frac{\alpha(1-\beta)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]} \\ &\quad + \frac{2\alpha(1-e^{-\theta})(1-\beta)e^{-\alpha(1-e^{-\theta})x}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}} + \alpha(1-e^{-\theta}),\end{aligned}$$

respectively. Next, we have that $\frac{q(x)+\tau'(x)}{\tau(x)} = \frac{2\alpha(1-e^{-\theta})(1-\beta)e^{-\alpha(1-e^{-\theta})x}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}} + \alpha(1-e^{-\theta})$ and

$$\int_0^x \frac{q(y)+\tau'(y)}{\tau(y)}dy = 2\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}] - 2\ln\beta + \alpha(1-e^{-\theta})x,$$

therefore we have

$$f(x) = K\exp\left[-\int_0^x \frac{q(y)+\tau'(y)}{\tau(y)}dy\right] = \frac{K\beta^2 e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2}.$$

Hence the normalizing constant $K = \frac{\alpha(1-e^{-\theta})}{\beta}$. □

Proposition 3.4. *Suppose that the random variable X has an absolutely continuous cdf* $F(x)$ *with* $F(0) = 0$, $F(x) > 0$, $\forall x > 0$, *with density function* $f(x) = F'(x)$ *and reverse failure rate* $r(x) = f(x)/F(x)$. *Assume that* $E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}\right] < \infty$ *for* $\alpha, \beta, \theta > 0$, *then X has the MOGETE family of distribution if and only if* $E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}|X \leqslant x\right] = \frac{r(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2[\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]-\ln\beta]}{\alpha(1-\beta)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}$, $x > 0$.

Proof. For necessity, $\lim_{x\to 0} g_{ETE}(x) = \alpha(1-e^{-\theta})$. For sufficiently, let $q(x) = 1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}$, then

$$\begin{aligned}E[q(X)|X \leqslant x] &= \frac{r(x)}{f(x)}\int_0^x (1-(1-\beta)e^{-\alpha(1-e^{-\theta})y})f(y)dy \\ &= \frac{r(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2}{e^{-\alpha(1-e^{-\theta})x}}\int_0^x \frac{e^{-\alpha(1-e^{-\theta})y}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})y}}dy.\end{aligned}$$

By setting $u = 1-(1-\beta)e^{-\alpha(1-e^{-\theta})y}$ and after some algebraic manipulations we have

$$E\left[1-(1-\beta)e^{-\alpha(1-e^{-\theta})X}|X \leqslant x\right] = \frac{r(x)(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2[\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}]-\ln\beta]}{\alpha(1-\beta)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}},$$

thus

$$w(x) = \frac{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2[\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}] - \ln\beta]}{\alpha(\beta-1)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}.$$

Therefore, we have $\frac{q(x)}{w(x)}$ and $\frac{w'(x)}{w(x)}$ as

$$\frac{q(x)}{w(x)} = \frac{\alpha(1-\beta)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})[\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}] - \ln\beta}$$

and

$$\frac{w'(x)}{w(x)} = \frac{\alpha(1-\beta)(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})[\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}] - \ln\beta]} + \frac{2\alpha(1-e^{-\theta})(1-\beta)e^{-\alpha(1-e^{-\theta})x}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}} + \alpha(1-e^{-\theta}),$$

respectively. Next, we have that $\frac{w'(x)-q(x)}{w(x)} = \frac{2\alpha(1-e^{-\theta})(1-\beta)e^{-\alpha(1-e^{-\theta})x}}{1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}} + \alpha(1-e^{-\theta})$ and

$$\int_0^x \frac{w'(y)-q(y)}{w(y)}dy = 2\ln[1-(1-\beta)e^{-\alpha(1-e^{-\theta})x}] - 2\ln\beta + \alpha(1-e^{-\theta})x,$$

therefore

$$f(x) = K\exp\left[-\int_0^x \frac{w'(y)-q(y)}{w(y)}dy\right] = \frac{K\beta^2 e^{-\alpha(1-e^{-\theta})x}}{(1-(1-\beta)e^{-\alpha(1-e^{-\theta})x})^2},$$

and $K = \frac{\alpha(1-e^{-\theta})}{\beta}$. □

4. Conclusion

In this work, the characterization of the MO-G family of distribution by left and right truncated moments based on some certain continuous function of a non-negative random variable is provided. The characterizations of two families of MO-G are discussed, in particular, the MONH and MOGETE distributions.

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