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Interval Interpolation by Newton's Divided Differences

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Abstract

In this paper, we present applied of interval algebra operation in interpolation, when the support points are intervals. We compute interpolation polynomial that coefficients are interval. This polynomial named inters polar polynomial. We compute interpolation polynomial by Newton`s divided difference formula.

Keywords: Interval value, Interpolation polynomial, Interval interpolation, Newton`s Divided differences.

1. Introduction

Interval analysis play an important role in many fields, such as Fuzzy Theory, Statistics and Probability, Approximation theory, Game Theory and Computer Science; for examples see[1-10]. In this paper, we first present the real interval and their properties. Then we present Interval interpolation.

We consider the following definitions and theorems in [1-6].

1.1 Definition: A real interval, or just an interval [x], is a nonempty closed and Bounded subset of the real numbers $\mathbb{R}$

$[x] := [\underline{x}, \overline{x}] := \{ x\in\mathbb{R} \mid \underline{x}\leq x\leq\overline{x} \}$, (1.1.1)

where $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of the interval [x], respectively. Then $\underline{x}\leq \overline{x}$, $\underline{x}$ and $\overline{x}$ called Infimum and Suprimum.

The set of all intervals is denoted by $\mathbb{IR}$:

$\mathbb{IR} := \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x}\in\mathbb{R}, \underline{x}\leq\overline{x}\}$. (1.1.2)

1.2 Definition: An open interval is denoted by:

$]\underline{x}, \overline{x}[:= \{x \in \mathbb{R} \mid \underline{x} < x < \overline{x} \}$. (1.1.3)

1. 3 Definition: If X is bounded and nonempty subset of $\mathbb{R}$:

$\lozenge X = [\inf X, \sup X]$. (1.1.4)

◊ X is hull of X , and hull coincidence involves X**.**

1.2. Interval algebra operator

1.2.1 Definition: If x and y are real intervals then specific equations for interval operations are :

$[x] + [y] = [\underline{x}+\underline{y}, \overline{x}+\overline{y}],$ (1.2.1)

$[x] - [y] = [\underline{x}-\overline{y}, \overline{x}-\underline{y}],$ (1.2.2)

$[x] \cdot [y] = [\min(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}), Max\ (\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy})],$ (1.2.3)

$[x]/[y] = [x] \cdot [1/\overline{y}, 1/\underline{y}]$ *if* $0 \notin [y],$ (1.2.4)

$1/[y] = [1/\overline{y}, 1/\underline{y}]$ *if* $0 \notin [y].$ (1.2.5)

And when $0 \in [y]$, Hansen has defined a set of extended rules for interval division:

$$[x]/[y]=\begin{cases}[\overline{x}/\underline{y}, \infty) & if\ \overline{x} \le 0 \quad and \quad \overline{y}=0\\ (-\infty, \overline{x}/\overline{y}] \cup [\overline{x}/\underline{y}, \infty) & if\ \overline{x} \le 0 \quad and \quad \underline{y}<0<\overline{y}\\ (-\infty, \overline{x}/\overline{y}] & if\ \overline{x} \le 0 \quad and \quad \underline{y}=0\\ (-\infty, \infty) & if\ \underline{x}<0<\overline{x}\\ (-\infty, \underline{x}/\underline{y}] & if\ \underline{x} \ge 0 \quad and \quad \overline{y}=0\\ (-\infty, \underline{x}/\underline{y}] \cup [\underline{x}/\underline{y}, \infty) & if\ \underline{x} \ge 0 \quad and \quad \underline{y}<0<\overline{y}\\ [\underline{x}/\overline{y}, \infty) & if\ \overline{x} \ge 0 \quad and \quad \underline{y}=0\end{cases} \quad (1.2.6)$$

The result of an interval operator is also an interval (except for the special case of division by an Interval containing zero).

1.3. Property of interval algebra operator

We consider some property of real number algebra for interval.

If x, y and z are real interval the removable property is:

$x + y = y + x$ (1.3.1)

Associative property is:

$(x + y) + z = x + (y + z)$ (1.3.2)

$(xy)z = x(yz)$ (1.3.3)

Zero and unity in $\mathbb{IR}$:

$x + 0 = 0 + x = x$ (1.3.4)

$x * 1 = 1 * x = x$ (1.3.5)

the sub-distributive law:

$x\,(y \pm z) \subseteq xy \pm xz$ (1.3.6)

$(x \pm y)z \subseteq xz \pm yz$ (1.3.7)

And:

$x - y \subseteq (x + z) - (y + z)$ (1.3.8)

$x/y \subseteq (xz)/(yz)$ (1.3.9)

1.4. Interval functions

Another advantage offered by interval mathematics is the ability to compute guaranteed bounds on the range functions defined over interval domains. Therefore, we can compute bounds on the output of a function with uncertain arguments.

Given a real function f of real variables x = $(x_1, x_2, \cdots, x_n)^T$, which belong to the intervals [x] =$([x_1], [x_2], \cdots, [x_n])^T$, the ideal interval extension of f would be a function that provides the exact range of f in the domain $([x_1], [x_2], \cdots [x_n])^T$ [1]**.**

1.5. Exact range

The exact range of f: D ⊆ $\mathbb{R}^n \to \mathbb{R}$ on [x] ⊆ D is denoted by:

f([x]) := {f (x)/x ∈ [x]}. (1.5.1)

An interval function is an interval value that depends on one or several interval variables.

Consider f as a real function of the real variables $x_1, x_2, \ldots, x_n$ and F as an interval function of the interval variables $[x_1], [x_2], \ldots, [x_n]$.

1.5.1 Definition*: The interval function* F *is an interval extension of f if*

F(x) = f(x), x ∈ D. (1.5.2)

Therefore, if the arguments of F are degenerate intervals, then the result of computing F(x) must be a degenerate interval equal to f(x). This definition assumes that the interval arithmetic is exact. In practice, there are rounding errors, and the result of computing F is an interval that contains f(x)

f(x) ∈ F([x]). (1.5.3)

To compute the range of the function f, it is not enough to have an interval extension F.

Moreover, F must be an inclusion function and must be inclusion monotonic.

1.5.2 Definition: An interval function is inclusion monotonic if $[x_i] \subseteq [y_i]$ (i = 1, 2· · · , n) implies

$F([x_1], [x_2], \cdots, [x_n]) \subseteq F([y_1], [y_2], \cdots, [y_n])$.(1.5.4)

1..1 Theorem: If F([x]) is an inclusion monotonic interval extension of a real function f(x), then

f([x]) ⊆F([x]);

that is, the interval extension $F([x_1], [x_2], \cdots, [x_n])$ contains the range of values of

$f(x_1, x_2, \cdots, x_n)$ for all $x_i \in [x_i]$ (i= 1, 2, · · · , n).

Proof: (see [5]).

2. Interval interpolation

If $x_0, x_1, \cdots x_n \in \mathbb{IR}$, We will compute $F([x])$, that $[x]$ is an interval between $x_0, x_1, \cdots x_n \in \mathbb{IR}$. We find
$p_n([x]) = A_0 + (x - x_0)A_1 + (x - x_0)(x - x_1)A_2 + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})A_n$ that $A_i \in \mathbb{IR}$. The table of Newton`s divided differences is:

x_i	F_i	1	2	…	n
x_0	F_0				
x_1	F_1	$F[x_0, x_1]$	$F[x_0, x_1, x_2]$		
x_2	F_2	$F[x_1, x_2]$	$\vdots$		$F[x_0, x_1, \cdots x_n]$
$\vdots$	$\vdots$	$\vdots$			
x_n	F_n	$F[x_{n-1}, x_n]$	$F[x_{n-2}, x_{n-1}, x_n]$		

From first order divided differences we have:

$$F[x, x_0] = \frac{F_0 - F([x])}{x_0 - x} \Rightarrow F_0 - F([x]) = (x_0 - x)F[x, x_0]$$
$$\Longrightarrow F([x]) = F_0 + (x - x_0)F[x, x_0]. \qquad (2.1)$$

And from second order divided differences we have:

$$F[x, x_0, x_1] = \frac{F[x_0, x_1] - F[x, x_0]}{x_1 - x}$$
$$\Longrightarrow F[x_0, x_1] - F[x, x_0] = (x_1 - x)F[x, x_0, x_1]$$
$$F[x_1, x_0] = F[x, x_0] + (x - x_1)F[x, x_0, x_1]. \qquad (2.2)$$

And in general we have:

$$F(x) = F_0 + (x - x_0)F[x_0, x_1] + (x - x_0)(x - x_1)F[x_0, x_1, x_2] + \cdots$$
$$+(x - x_0)(x - x_1)\cdots(x - x_{n-1})F[x_0, x_1, \cdots, x_n] + (x - x_0)(x - x_1)\cdots(x - x_n)\, F[x_0, x_1, \cdots, x_{n+1}]. \quad (2.3)$$

From (2.3) *the* interpolation polynomial is:

$$p_n([x]) = F_0 + (x - x_0)F[x_0, x_1] + (x - x_0)(x - x_1)F[x_0, x_1, x_2] + \cdots$$
$$+(x - x_0)(x - x_1)\cdots(x - x_{n-1})F[x_0, x_1, \cdots, x_n]. \qquad (2.4)$$

And interpolation polynomial error is:

$$F(x) - p_n([x]) = R_n([x]) = (x - x_0)(x - x_1)\cdots(x - x_n)\, F[x_0, x_1, \cdots, x_{n+1}]. \quad (2.5)$$

Note: The maximum degree of divided differences Newton`s interpolation polynomial is n.

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3. Examples

3.1 Example: Consider the function f(x) = x · x, with [x] = [−1, 2]. It is easily seen that

f([x]) = f([−1, 2])= [0, 4]. On the other hand F([x]) = F([−1, 2]) = [x] · [x] = [−1, 2] · [−1, 2]

= [−2, 4].

Hence the range obtained by computing the interval extension F([x])is overestimating the exact range of f into [x]. A real-valued function may be defined by several equivalent arithmetic expressions. Mathematical equivalent expressions do not necessarily yield equivalent interval extensions.

The following example illustrate this point.

3.2 Example: Consider the function $f(x) = x^2 - 2x + 1$= x(x–2) + 1 = $(x-1)^2$.

Three possible interval extension functions are:

F_1([x]) = $[x]^2$− 2[x] + 1,

F_2 ([x]) = [x]([x] − 2) + 1,

F_3 ([x]) = $([x] - 1)^2$.

If we let [x] = [1, 2], then

F_1 ([1, 2]) = [1, 2]2 − 2[1, 2] + 1 = [−2, 3],

F_2 ([1, 2]) = [1, 2]([1, 2] −2) + 1 = [−1, 1],

F_3 ([1, 2]) = $([1,2] - 1)^2$= [0, 1].

Three mathematical equivalent expressions yield different answers. The true range of f(x) over

x∈[1, 2] is [0, 1].

3.3 Example: If real function is f(x) = x .x on $x_0 = [1,2], x_1 = [3,5], x_2 = [4,6]$ points interval

contents are:

$$F[x_0] = [1,2] * [1,2] = [1,4]$$

$$F[x_1] = [3,5] * [3,5] = [9,16]$$

$$F[x_2] = [4,6] * [4,6] = [16,36]$$

And divided difference table is:

x_i	$F(x_i)$	1	2

[1, 2]	[1, 4]	$\frac{[9,25]-[1,4]}{[3,5]-[1,2]}=[\frac{5}{4},24]$	
[3, 5]	[9, 25]		$\frac{[-27,9]-\left[\frac{5}{4},24\right]}{[4,6]-[1,2]}=\left[\frac{-51}{5},\frac{31}{8}\right]$
[4, 6]	[16, 36]	$\frac{[16,36]-[9,25]}{[4,6]-[3,5]}=[-27,9]$	

Interpolation polynomial is:

$$p_2([x]) = F_0 + (x - x_0)F[x_0, x_1] + (x - x_0)(x - x_1)F[x_0, x_1, x_2]$$

And:

$$p_2([x]) = [1,4] + (x - [1,2])\left[\frac{5}{4}, 24\right] + (x - [1,2])(x - [3,5])\left[\frac{-51}{5}, \frac{31}{8}\right]$$

$$= \left[-\frac{51}{5}, \frac{31}{8}\right]x^2 + \left[-\frac{207}{8}, \frac{477}{5}\right]x + \left[-149, \frac{83}{2}\right]$$

4. Conclusion

We present applied of interval algebra operation in interpolation, when the support points are intervals. We compute interpolation polynomial with Newton`s divided differences. Numerical results show that the method is working well.

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