Journal of mathematics and computer science 13 (2014), 257-280

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Mild Solution to Fractional Boundary Value Problem with Nonlinear Boundary Conditions

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Article history:
Received August 2014
Accepted September 2014
Available online October 2014

Abstract

In this paper, we consider a system of boundary value problems for fractional differential equation given by

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) = Au(t) + f\big(t, u(t)\big) + \int_0^t q(t-s)g\big(s, u(s)\big)ds, t\epsilon I = [0,T], q\epsilon(0,T), t \neq t_k \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \ \ k = 1,2,\dots,m, \\ u(0) = u_0 \epsilon X, \quad u'(0) = u_1 \epsilon X, \end{cases}$$

where ${}^cD_{0+}^{\alpha}$ is Caputo's fractional derivative of order α, $A: D(A) \subset X \to X$ is a sectorial operator of type (M, θ, α, μ) on a Banach space X, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T, I_k, J_k: \mathbb{R} \to \mathbb{R}$, $\Delta x(t_k) \coloneqq x(t_k^+) - x(t_k^-), x(t_k^-) = \lim_{h\to 0^-} x(t_k + h), x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $I_k, J_k \epsilon C(X, X)$ $(k = 1,2,\dots,m)$ are bounded function, the functions $f, g: I \times X \to X$ are given operators satisfying some assumptions and $q: I \to X$ is a integrable function on I. Several existence results of mild solutions are obtained.

Keywords: Caputo's fractional derivative, Sectorial operator, Mild solution, Analytic solution operators.

1. Introduction

Fractional calculus is a field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have found numerous applications in various fields of physics and engineering [7, 14].

Impulsive fractional differential equations have attracted a considerable interest both in mathematics and applications since Agarwal and Benchohra published the first paper on this topic [1] in 2008; see for example [2, 3, 4, 10, 16].

Our purpose in this paper is to show the existence of at least one mild solution for the following fractional system

$$(1.1)\qquad \begin{cases} {}^{c}D_{0^+}^{\alpha}u(t) = Au(t) + f\big(t, u(t)\big) + \int_0^t q(t-s)g\big(s, u(s)\big)ds, t\epsilon I = [0,T], q\epsilon(0,T), t \neq t_k \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \ k = 1,2,\dots,m, \\ u(0) = u_0\epsilon X, \qquad u'(0) = u_1\epsilon X, \end{cases}$$

where ${}^{c}D_{0+}^{\alpha}$ is Caputo's fractional derivative of order $\alpha, A: D(A) \subset X \to X$ is a sectorial operator of type (M, θ, α, μ) on a Banach space X, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $I_k, J_k: \mathbb{R} \to \mathbb{R}$, $\Delta x(t_k) \coloneqq x(t_k^+) - x(t_k^-)$, $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $I_k, J_k \epsilon C(X, X)$ $(k = 1,2,\dots,m)$ are bounded function, the functions $f, g: I \times X \to X$ are given a satisfies some assumptions and $q: I \to X$ is a integrable function on I.

The rest of the article is organized as follows: In Section 2, we shall recall certain results from the theory of the continuous fractional calculus. In Section 3, we shall provide some conditions under which the problem (1.1) has at least one mild solution. In Section 4, by suitable conditions, we will prove that the problem (1.1) into two ways has at least one mild solution.

2. Preliminaries

In this section, we recall some definition and propositions of fractional differential equation and sectorial operators.

Let X is a Banach space with the norm $|.|$. Denote C(I,X) be the Banach space of continuous functions from I into X with the norm $\|x\| = \sup_{t\epsilon I}|x(t)|, x\epsilon C(I, X)$, and L(X) represents the Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|Q\|_{L(X)} = \sup\{|Q(x)|:\ |x| = 1\}, Q\epsilon L(X)$. We also introduce the set of functions

$$PC(I, X) = \{x: I \to X:\ x\epsilon C\big((t_k, t_{k+1}], X\big),\ k = 0,1, \dots, m \quad \text{and there exit} \quad x(t_k^-) \quad \text{and} \quad x(t_k^+),\ k = 1, \dots, m \ \text{with}\ x(t_k^-) = x(t_k)\},$$

endowed with the norm $\|x\|_{PC} = \sup_{t\epsilon I}\|x(t)\|$. It is easy to see $(PC(I, X), \|.\|)$ Banach space.

Lemma 2.1. ([11, 15]) The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Lemma 2.2. ([11, 15]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f: (0, \infty) \to \mathbb{R}$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 2.3. ([8]) Let $\alpha > 0$. Then the differential equation

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{C}$, $i = 1, \dots, n$, there $n-1 < \alpha \leq n$.

Lemma 2.4. ([8]) Let $\alpha > 0$. Then the following equality holds for $u \in L^1(0,1)$, $D_{0+}^{\alpha} u \in L^1(0,1)$;

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{C}, i = 1, \dots, n$, there $n-1 < \alpha \leq n$.

Definition 2.5. For the function $f \in C_{\alpha,-1}^m$ and $m \in \mathbb{N}^+$, the fractional derivative of order $\alpha > 0$ of f in the Caputo sence is given by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \qquad m-1 < \alpha \leq m.$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\left(D_t^\alpha u(t)\right)(\lambda) = \lambda^\alpha (Lu)(\lambda) - \sum_{j=0}^{m-1} \lambda^{\alpha-j-1}\left(D^j u\right)(0), \quad m-1 < \alpha \leq m.$$

Now, we introduce some notations about sectorial operators, analytic solution operators.

An operator A is said to be sectorial, if there are $\omega \in \mathbb{R}$, $\theta \in \left[\frac{\pi}{2}, \pi\right]$ and $M > 0$ such that the following two conditions are satisfied:

$$\begin{cases} (1)\ \rho(A) \subset \sum_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ (2)\ \|R(\lambda, A)\|_{L(X)} \leq \dfrac{M}{|\lambda - \omega|}, \lambda \in \sum_{\theta,\omega}. \end{cases}$$

Consider the following Cauchy problem for the Caputo fractional derivative evolution equation of order α ($m-1 < \alpha \leq m, m > 0$ is an integer):

(2.2) $$D_t^\alpha u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, \quad k = 1,2,\dots,m-1,$$

where A is a sectorial operator. The solution operators $S_\alpha(t)$ of (2.2) is defined by (see [3])

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda,$$

where γ is a suitable path lying on $\sum_{\theta,\omega}$.

An operator A is said to belong to $\varrho^\alpha(X; M, \omega)$ or $\varrho^\alpha(M, \omega)$, if problem (2.2) has a solution operator $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \leq Me^{\omega t}, t \geq 0$. Denote $\varrho^\alpha(\omega) := \cup\{\varrho^\alpha(M,\omega) : M \geq 1\}$ and $\varrho^\alpha := \cup\{\varrho^\alpha(\omega) \geq 0\}$.

Definition 2.6. ([3]) A solution operator $S_\alpha(t)$ of (2.2) is called analytic, if $S_\alpha(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta_0\}$ for some $\theta_0 \in \left(0, \frac{\pi}{2}\right]$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0), if for each $\theta < \theta_0$ and $\omega > \omega_0$, there is an $M = M(\theta, \omega)$ such that $\|S_\alpha(t)\| \leq Me^{\omega t}, t \in \sum_\theta := \{t \in \mathbb{C} \setminus \{0\} : |\arg t < \theta|\}$. Denote $\mathfrak{A}^\alpha(\theta_0, \omega_0) := \{\mathfrak{A} \in \varrho^\alpha : \mathfrak{A}$ generates analytic solution operators $S_\alpha(t)$ of type $(\theta_0, \omega_0)\}$.

Lemma 2.7. ([3,12]) Let $\alpha \in (0,2)$. A linear closed densely defined operator A belongs to $\mathfrak{A}^{\alpha}(\theta_0, \omega_0)$ iff $\lambda^{\alpha} \in \rho(A)$ for each $\lambda \in \sum_{\theta_0+\frac{\pi}{2}}$, and for any $\omega > \omega_0$, $\theta < \theta_0$, there is a constant $C = C(\theta, \omega)$ such that

$$\|\lambda^{\alpha-1} R(\lambda^{\alpha}, A)\| \leq \frac{C}{|\lambda - \omega|}, \text{for } \lambda \in \sum_{\theta+\frac{\pi}{2}}(\omega).$$

Definition 2.8. Let $A: D \subseteq X \to X$ be a closed linear operator. A is said to be sectorial operator of type (M, θ, α, μ), if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ and $\mu \in \mathbb{R}$ such that the α −resolvent of A exist outside the sector

$$\mu + S_{\theta} = \{\mu + \lambda^{\alpha} : \lambda \in \mathbb{C}, |\arg(-\lambda^{\alpha})| < \theta\}$$

and

$$\|(\lambda^{\alpha} I - A)^{-1}\| \leq \frac{M}{\lambda^{\alpha} - \mu}, \quad \lambda^{\alpha} \notin \mu + S_{\theta}.$$

3. Definition of mild solutions

Firstly, we consider the following fractional Cauchy problem

$$(3.3) \qquad \begin{cases} D_t^{\alpha} u(t) = Au(t) + f(t), \ t \in I = [0, T], \ 1 < \alpha < 2, \\ u(0) = u_0 \in X, \quad u'(0) = u_1 \in X \end{cases}$$

where f is an abstract function defined on $[0, \infty)$ and with values in X, A is a sectorial operator.

Theorem 3.1. ([16]) Let A be a sectorial operator of type (M, θ, α, μ). If f satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$, then the unique solution of the Cauchy problem (3.3) is given by

$$u(t) = S_{\alpha}(t)u_0 + K_{\alpha}(t)u_1 + \int_0^t T_{\alpha}(t-s)\, f(s)ds$$

where

$$S_{\alpha}(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^{\alpha}, A) d\lambda,$$

$$K_{\alpha}(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-2} R(\lambda^{\alpha}, A) d\lambda,$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda,$$

with c being a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in c$.

Remark 3.2. If A is a sectorial operator of type (M, θ, α, μ), then it is not difficult to see that $T_\alpha(t)$, $K_\alpha(t)$ and $S_\alpha(t)$ are well definitions. (see [9])

Theorem 3.3. ([13]) Let A be a sectorial operator of type (M, θ, α, μ). If f satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$, then the unique solution of the Cauchy problem

$$\begin{cases} {}^c D_{0^+}^{\alpha} u(t) = Au(t) + f\big(t, u(t)\big) + \displaystyle\int_0^t q(t-s)\, g\big(s, u(s)\big) ds, \qquad t \in I = [0, T] \\ u(0) = u_0 \in X, \quad u'(0) = u_1 \in X, \end{cases}$$

is given by

$$u(t) = S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-s)\Big[f\big(s, u(s)\big) + \int_0^s q(s-\tau) g\big(\tau, u(\tau)\big) d\tau\Big] ds, \tag{3.4}$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\, \lambda^{\alpha-1}\, R(\lambda^\alpha, A) d\lambda,$$

$$K_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\, \lambda^{\alpha-2}\, R(\lambda^\alpha, A) d\lambda,$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\, R(\lambda^\alpha, A) d\lambda,$$

with c being a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in c$.

Theorem 3.4. If A be a sectorial operator of type (M, θ, α, μ) and f satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$, then any solution of the Cauchy problem (1.1) is a fixed point of the operator given below

$$\Gamma u(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-s)[f(s,u(s)) \\ \qquad + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in [0\,, t_1], \\ S_\alpha(t-t_1)\left(u(t_1^-) + I_1\big(u(t_1^-)\big)\right) + K_\alpha(t-t_1)\left(u'(t_1^-) + J_1\big(u(t_1^-)\big)\right) \\ \qquad + \int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t-t_m)\left(u(t_m^-) + I_m\big(u(t_m^-)\big)\right) + K_\alpha(t-t_m)\left(u'(t_m^-) + J_m\big(u(t_m^-)\big)\right) \\ \qquad + \int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g(\tau,u(\tau))d\tau]ds, & t \in (t_m, T]. \end{cases}$$

In fact, from (3.4) it is easy to see that Theorem 3.4 holds, so the proof is omitted.

From Theorem 3.4, we can define the mild solution of system (1.1) as follows:

Definition 3.5. A function $u : I \to X$ is called a mild solution of system (1.1), if $u \in PC(I, X)$ and satisfies the following equation

$$u(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-s)[f(s,u(s)) \\ \qquad + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in [0\,, t_1], \\ S_\alpha(t-t_1)\left(u(t_1^-) + I_1\big(u(t_1^-)\big)\right) + K_\alpha(t-t_1)\left(u'(t_1^-) + J_1\big(u(t_1^-)\big)\right) \\ \qquad + \int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t-t_m)\left(u(t_m^-) + I_m\big(u(t_m^-)\big)\right) + K_\alpha(t-t_m)\left(u'(t_m^-) + J_m\big(u(t_m^-)\big)\right) \\ \qquad + \int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g(\tau,u(\tau))d\tau]ds, & t \in (t_m, T]. \end{cases}$$

4. Existence results

In this section, we present two different existence results and uniqueness of mild solutions to the system (1.1).

To study the existence of mild solutions of (1.1), we need the following know result due to Krasnoselskii.

Theorem 4.1. ([17] Krasnoselskii Theorem). Let B be a closed convex and nonempty subset of a Banach space X. Let Γ_1 and Γ_2 be two operators such that

(i) $\Gamma_1 x_1 + \Gamma_2 x_2 \in B$ whenever $x_1, x_2 \in B$;

(ii) Γ_1 is a contraction mapping;

(iii) Γ_2 is compact and continuous.

Then there exists $x \in B$ such that $x = \Gamma_1 x + \Gamma_2 x$.

Let

(4.5) $\widetilde{M}_s := \sup_{0<t<T} \|S_\alpha(t)\|_{L(X)}$, $\widetilde{M}_T := \sup_{0<t<T} \|T_\alpha(t)\|_{L(X)}$, $\widetilde{M}_K := \sup_{0<t<T} \|K_\alpha(t)\|_{L(X)}$,

where $S_\alpha(t)$, $T_\alpha(t)$ and $K_\alpha(t)$ are as in Theorem 3.1. So we have

$$\|S_\alpha(t)\|_{L(X)} \leq \widetilde{M}_s \quad , \quad \|T_\alpha(t)\|_{L(X)} \leq t^{\alpha-1}\widetilde{M}_T \, , \quad \|K_\alpha(t)\|_{L(X)} \leq t^{\alpha-2}\widetilde{M}_k.$$

Now, we consider the following assumptions:

(A_1) $f, g : I \times X \to X$ are continuous and there exists function $\mu_1, \nu_1 \in L^1(I, R^+)$ such that

$$\|f(t,x) - f(t,y)\| \leq \mu_1(t)\|x - y\|, \qquad t \in I\,, x, y \in X,$$
$$\|g(t,x) - g(t,y)\| \leq \nu_1(t)\|x - y\|\,, \qquad t \in I, x, y \in X.$$

(A_2) For each $k = 1, \dots, m$, there exist $\rho_k, \lambda_k > 0$ such that

$$\|I_k(x) - I_k(y)\| \leq \rho_k\|x - y\|, \qquad x, y \in X,$$
$$\|J_k(x) - J_k(y)\| \leq \lambda_k \|x - y\|\,, \qquad x, y \in X.$$

(A_3) $\Theta = \max_{1\leq i\leq m}\{(1+\rho_i)\widetilde{M}_s + \lambda_i \left(t - t_i\right)^{\alpha-1}\widetilde{M}_T\}$.

In what follows, we use the notation $q = \max_{t\in[0,T]} \int_0^t |q(t-s)|ds$.

Theorem 4.2. Assume that $(A_1) - (A_3)$. Then (1.1) has at least one mild solution on I.

Proof. Choose

$$r \geq \max_{1 \leq i \leq m}\{ \overline{M}(\|u(t_i^-)\| + \|I_i(u(t_i^-))\| + \|u'(t_i^-)\| + \|J_i(u(t_i^-))\| + T\|\mu_r\|_{L^\infty(I,R^+)} + qT^2\|\nu_r\|_{L^\infty(I,R^+)})\},$$

where $\overline{M} = \widetilde{M}_s = (t-s)^{\alpha-1}\widetilde{M}_T = (t-t_i)^{\alpha-2}\widetilde{M}_K$, $(i = 1, \dots, m)$ and consider $B_r = \{x \in X : \|x\| \leq r\}$. Define $\Gamma: B_r \to B_r$ by

$$\Gamma u(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-s)[f\big(s,u(s)\big) \\ \qquad + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in [0\,,t_1], \\ S_\alpha(t-t_1)\Big(u(t_1^-) + I_1\big(u(t_1^-)\big)\Big) + K_\alpha(t-t_1)\Big(u'(t_1^-) + J_1\big(u(t_1^-)\big)\Big) \\ \qquad + \int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_1,t_2], \\ \vdots \\ S_\alpha(t-t_m)\Big(u(t_m^-) + I_m\big(u(t_m^-)\big)\Big) + K_\alpha(t-t_m)\Big(u'(t_m^-) + J_m\big(u(t_m^-)\big)\Big) \\ \qquad + \int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g(\tau,u(\tau))d\tau]ds, & t \in (t_m,T]. \end{cases}$$

Set

$$\Gamma_1 x(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1, & t \in [0,t_1], \\ S_\alpha(t-t_1)\Big(u(t_1^-) + I_1\big(u(t_1^-)\big)\Big) + K_\alpha(t-t_1)\Big(u'(t_1^-) + J_1\big(u(t_1^-)\big)\Big), & t \in (t_1,t_2], \\ \vdots \\ S_\alpha(t-t_m)\Big(u(t_m^-) + I_m\big(u(t_m^-)\big)\Big) + K_\alpha(t-t_m)\Big(u'(t_m^-) + J_m\big(u(t_m^-)\big)\Big), & t \in (t_m,T] \end{cases}$$

and

$$\Gamma_2 x(t) = \begin{cases} \int_0^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in [0,t_1], \\ \int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_1,t_2], \\ \vdots \\ \int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_m,T]. \end{cases}$$

We show that Γ_1 ana Γ_2 fulfill the conditions of Theorem 4.1.

Let us observe that if $x, y \in B_r$, then $\Gamma_1 x + \Gamma_2 y \in B_r$. Indeed, in view of $(A_1) - (A_3)$ and $\left\|\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau\right\| \leq \int_0^s \|q(s-\tau)\|d\tau \int_0^s \left\|g\big(\tau,u(\tau)\big)\right\|d\tau$(see[5]), we have for any $x, y \in B_r$ and $t \in [0,t_1]$

$$\|(\Gamma_1 x)(t) + (\Gamma_2\, y)(t)\| \leq \|S_\alpha(t)\|\|u_0\| + \|K_\alpha(t)\|\|u_1\| + \|T_\alpha(t-s)\|$$

$$\times\left[\|\int_0^t f(s,u(s))ds\|+q\|\int_0^t(\int_0^s g(\tau,u(\tau))d\tau)ds\|\right]$$

which according to $(A_1)-(A_3)$ gives

$$\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le \overline{M}\bar{Z}+\overline{M}\,T\|\mu_r\|_{L^\infty(I,R^+)}+q\overline{M}T^2\|\nu_r\|_{L^\infty(I,R^+)},$$

where $\overline{Z}=(\|u_0\|+\|u_1\|)$ and $\overline{M}=\widetilde{M}_s=(t-s)^{\alpha-1}\widetilde{M}_T=t^{\alpha-2}\widetilde{M}_K$.

If $t\in(t_1,t_2)$, then

$$\begin{aligned}\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le{}& \|S_\alpha(t)\|\left[\|u(t_1^-)\|+\left\|I_1\left(u(t_1^-)\right)\right\|\right]\\ &+\|K_\alpha(t-t_1)\|\left[\left\|u'(t_1^-)\right\|+\left\|J_1\left(u(t_1^-)\right)\right\|\right]\\ &+\|T_\alpha(t-s)\|\times[\left\|\int_{t_1}^{t_2} f(s,u(s))ds\right\|\\ &+q\left\|\int_{t_1}^{t_2}(\int_0^s g(\tau,u(\tau))d\tau)ds\right\|]\end{aligned}$$

which according to $(A_1)-(A_3)$ gives

$$\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le \overline{M}\,\overline{Z}+\overline{M}\,T\|\mu_r\|_{L^\infty(I,R^+)}+q\overline{M}T^2\|\nu_r\|_{L^\infty(I,R^+)},$$

where $\overline{Z}=(\|u(t_1^-)\|+\|I_1(u(t_1^-))\|+\|u'(t_1^-)\|+\|J_1(u(t_1^-))\|)$ and $\overline{M}=\widetilde{M}_s=(t-s)^{\alpha-1}\widetilde{M}_T=(t-t_1)^{\alpha-2}\widetilde{M}_K$.

Similary, for $t\in(t_i,t_{i+1}),(i=2,\dots,m-1)$ we have

$$\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le \overline{M}\,\overline{Z}+\overline{M}\,T\left\|\mu_r\right\|_{L^\infty(I,R^+)}+q\overline{M}T^2\|\nu_r\|_{L^\infty(I,R^+)}$$

and

$$\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le \overline{M}\,\overline{Z}+\overline{M}\,T\left\|\mu_r\right\|_{L^\infty(I,R^+)}+q\overline{M}T^2\|\nu_r\|_{L^\infty(I,R^+)},\qquad t\in(t_m,T],$$

where $\overline{Z}=(\|u(t_i^-)\|+\|I_i(u(t_i^-))\|+\|u'(t_i^-)\|+\|J_i(u(t_i^-))\|)$ and $\overline{M}=\widetilde{M}_s=(t-s)^{\alpha-1}\widetilde{M}_T=(t-t_1)^{\alpha-2}\widetilde{M}_K$, $(i=2,\dots,m)$. Thus, for all $t\in[0,T]$, we have

$$\|(\Gamma_1 x)(t)+(\Gamma_2 y)(t)\| \le \max_{1\le i\le m}\left\{\overline{M}\,\overline{Z}+\overline{M}\,T\|\mu_r\|_{L^\infty(I,R^+)}+q\overline{M}T^2\|\nu_r\|_{L^\infty(I,R^+)}\right\} \le r.$$

Hence, we deduce that $\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \le r$.

Let $t \in [0, t_1]$ and $x, y \in B_r$. By (4.5), (A_1) and (A_2), we have

$$\|(\Gamma_1 x)(t) - (\Gamma_2 y)(t)\| = 0 \le \|x - y\|.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned}
\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| &\le \|S_\alpha(t - t_1)\| \|x(t_1^-) - y(t_1^-)\| \\
&\quad + \|S_\alpha(t - t_1)\| \|I_1(x(t_1^-)) - I_1(y(t_1^-))\| \\
&\quad + \|K_\alpha(t - t_1)\| \|J_1(x(t_1^-)) - J_1(y(t_1^-))\| \\
&\le \|S_\alpha(t - t_1)\| \|x - y\| + \|S_\alpha(t - t_1)\| \rho_1 \|x - y\| \\
&\quad + \|K_\alpha(t - t_1)\| \lambda_1 \|x - y\| \\
&\le \widetilde{M}_s \|x - y\| + \rho_1 \widetilde{M}_s \|x - y\| \\
&\quad + \lambda_1 (t - t_1)^{\alpha - 1} \widetilde{M}_T \|x - y\| \\
&\le \left\{ \left(1 + \rho_1\right) \widetilde{M}_S + \lambda_1 (t - t_1)^{\alpha - 1} \widetilde{M}_T \right\} \|x - y\|.
\end{aligned}$$

Similary, for $t \in (t_i, t_{i+1}]$, $(i = 2, \dots, m - 1)$ we have

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \le \left\{ \left(1 + \rho_i\right) \widetilde{M}_s + \lambda_i (t - t_i)^{\alpha - 1} \widetilde{M}_T \right\} \|x - y\|$$

and

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \le \left\{ \left(1 + \rho_m\right) \widetilde{M}_s + \lambda_m (t - t_m)^{\alpha - 1} \widetilde{M}_T \right\} \|x - y\|, \qquad t \in (t_m, T].$$

Thus, for all $t \in [0, T]$, we have

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \le \max_{1 \le i \le m} \left\{ (1 + \rho_i) \widetilde{M}_s + \lambda_i (t - t_i)^{\alpha - 1} \widetilde{M}_T \right\} \|x - y\| = \Theta \|x - y\|.$$

Hence, by Assumption (A_3), Γ_1 is a contraction mapping.

Next, we show that Γ_2 is continuous.

Let (x_n) be a sequence in B_r such that $x_n \to x$ in B_r. Then by (A_1)

$$f\left(s, x_n(s)\right) \to f\left(s, x(s)\right), \quad g\left(s, x_n(s)\right) \to g\left(s, x(s)\right) \text{ as } n \to \infty.$$

Because the functions f, g are continuous on $I \times X$. Then for every $t \in [0, t_1]$, we have

$$\|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \le \int_0^{t_1} \|T_\alpha(t-s)\|[\|f(s,x_n(s)) - f(s,x(s))\|$$
$$+q\|\int_0^s \|g(\tau,x_n(\tau)) - g(\tau,x(\tau))\|d\tau\|]ds$$
$$\le \frac{1}{\alpha} T^\alpha \widetilde{M}_T[\|f(s,x_n(s)) - f(s,x(s))\|$$
$$+qT\|g(\tau,x_n(\tau)) - g(\tau,x(\tau))\|]$$
$$\le \frac{1}{\alpha} T^\alpha \widetilde{M}_T(1+qT)\epsilon, (\epsilon > 0\,, \epsilon \to 0 (n\to\infty)). \tag{4.6}$$

Moreover, for every $t \in (t_i, t_{i+1}], (i = 2, \dots, m-1)$

$$\|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \le \frac{1}{\alpha} T^\alpha \widetilde{M}_T(1+qT)\epsilon, \quad (\epsilon > 0\,, \epsilon \to 0 (n\to\infty)) \tag{4.7}$$

and for any $t \in (t_m, T]$

$$\|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \le \frac{1}{\alpha} T^\alpha \widetilde{M}_T(1+qT)\epsilon, \quad (\epsilon > 0\,, \epsilon \to 0 (n\to\infty)). \tag{4.8}$$

By the (4.6)-(4.8), we have

$$\lim_{n\to\infty} \|(\Gamma_2 x_n) - (\Gamma_2 x)\| = 0$$

To prove the compactness of Γ_2, we shall use the Ascoli-Arzela Theorem. We prove that Γ_2 maps bounded sets into bounded sets in B_r and $\Gamma_2(B_r)$ is equicontinuous.

It suffices to prove that for any $r > 0$ there exists $\delta > 0$ such that $\|\Gamma_2 x\| \le \delta$ for each $x \in B_r$. Denote For any $k>0$ positive functions $\mu_k, \nu_k \in L^\infty([0,T], R^+)$ such that

$$\sup_{\|u\|\le k} \|f(t,u)\| \le \mu_k(t), \qquad \sup_{\|u\|\le k} \|g(t,u)\| \le \nu_k(t).$$

Then, for any $x \in B_r$, $t \in [0, t_1]$, we have

$$\|(\Gamma_2 x)(t)\| \le \int_0^{t_1} \|T_\alpha(t-s)\| \left[\|f(s,x(s))\| + q\int_0^s \|g(\tau,x(\tau))\|d\tau\right] ds$$
$$\le \frac{1}{\alpha} T^\alpha \widetilde{M}_T \left[\|\mu_r\|_{L^\infty(I,R^+)} + qT\|\nu_r\|_{L^\infty(I,R^+)}\right] < \infty.$$

Similary, we have

$$\|(\Gamma_2 x)(t)\| \le \frac{1}{\alpha} T^\alpha \widetilde{M}_T[\|\mu_r\|_{L^\infty(I,R^+)} + qT\|\nu_r\|_{L^\infty(I,R^+)}], \ \ t \in (t_i, t_i], (i = 1, \dots, m).$$

We obtain

$$\left\|(\Gamma_2 x)(t)\right\| \le \frac{1}{\alpha} T^{\alpha} \widetilde{M}_T \left[\|\mu_r\|_{L^{\infty}(I,R^+)} + qT\|\nu_r\|_{L^{\infty}(I,R^+)}\right] := \delta, \quad t \in [0,T].$$

Next, we show that $\Gamma_2(Br)$ is equicontinuous. Set

$$\mu = \max_{0\le t\le T, x\in B_r} \|f(t,x)\|, \ \ \nu = \max_{0\le t\le T, x\in B_r} \|g(t,x)\|.$$

Now, let us prove that $\Gamma_2(B_r)$ is equicontinuous. The functions $\Gamma_2(x)$, $x \in B_r$ are equicontinuous at t=0. For $s_1, s_2 \in [0,T]$, we have

$$\begin{aligned}
\left\|(\Gamma_2 x)(s_1) - (\Gamma_2 x)(s_2)\right\| \le\ & \Big\|\int_0^{s_2} T_\alpha(s_1 - s)\left[f\big(s,u(s)\big) + \int_0^s q(s-\tau) g\big(\tau,u(\tau)\big)d\tau\right]ds \\
& + \int_{s_2}^{s_1} T_\alpha(s_1 - s)\left[f\big(s,u(s)\big) + \int_0^s q(s-\tau) g\big(\tau,u(\tau)\big)d\tau\right]ds \\
& - \int_0^{s_2} T_\alpha(s_2 - s)\left[f\big(s,u(s)\big) + \int_0^s q(s-\tau) g\big(\tau,u(\tau)\big)d\tau\right]ds\Big\| \\
\le\ & \int_0^{s_2} \|T_\alpha(s_1 - s) - T_\alpha(s_2 - s)\|\Big[\left\|f\big(s,u(s)\big)\right\| \\
& + q\int_0^s \left\|g\big(\tau,u(\tau)\big)\right\|d\tau\Big]ds + \int_{s_2}^{s_1} \|T_\alpha(s_1 - s)\|\Big[\|f(s,u(s))\| \\
& + q\int_0^s \left\|g\big(\tau,u(\tau)\big)\right\|d\tau\Big]ds \\
\le\ & I_1 + I_2
\end{aligned}$$

where

$$I_1 = \int_0^{s_2} \|T_\alpha(s_1 - s) - T_\alpha(s_2 - s)\| \left[\|f(s,u(s))\| + q\int_0^s \|g(\tau,u(\tau))\|d\tau\right] ds,$$

$$I_2 = \int_{s_2}^{s_1} \|T_\alpha(s_1 - s)\| \left[\|f(s,u(s))\| + q\int_0^s \|g(\tau,u(\tau))\|d\tau\right] ds.$$

Actually, I_1, I_2 tend to 0 independently $x \in B_r$ when $s_2 \to s_1$. Indeed, let $x \in B_r$, we have

$$\begin{aligned}
I_1 &= \int_0^{s_2} \|T_\alpha(s_1 - s) - T_\alpha(s_2 - s)\| \left[\|f(s,u(s))\| + q\int_0^s \|g(\tau,u(\tau))\|d\tau\right] ds \\
&\le \int_0^{s_2} \|T_\alpha(s_1 - s) - T_\alpha(s_2 - s)\|\left[\left\|f\big(s,u(s)\big)\right\| + qT\left\|g\big(s,u(s)\big)\right\|\right]ds \\
&\le \left(\|\mu\|_{L^1(I,R^+)} + qT\|\nu\|_{L^1(I,R^+)}\right)\frac{\widetilde{M}_T(s_1^\alpha + s_2^\alpha - (s_1 - s_2)^\alpha)}{\alpha}
\end{aligned}$$

Therefore the continuity of the function $t \to \|T_\alpha(t)\|$ for $t \in (0, T]$, allow us to conclude

$\lim_{s_2 \to s_1} = 0.$

$$\begin{aligned} I_2 &= \int_{s_2}^{s_1} \|T_\alpha(s_1 - s)\| \left[\|f(s, u(s))\| + q \int_0^s \|g(\tau, u(\tau))\| d\tau\right] ds \\ &\le \int_{s_2}^{s_1} \|T_\alpha(s_1 - s)\| \left[\|f(s, u(s))\| + qT \int_0^s \|g(s, u(s))\|\right] ds \\ &\le \left(\|\mu\|_{L^1(I,R^+)} + qT\|\nu\|_{L^1(I,R^+)}\right) \frac{\widetilde{M}_T(s_2^\alpha - s_1^\alpha)}{\alpha} \end{aligned}$$

consequently $\lim_{s_2 \to s_1} = 0.$

We show that $(\Gamma_2 x)(t): x \in B_r$ relatively compact in X, for all $t \in I$.

We have proved that $(\Gamma_2 x)(t): x \in B_r$ is bounded and we have proved that $(\Gamma_2 x)(t): x \in B_r$ is equicontinuous.

In short, we have proved that $\Gamma_2(B_r)$ is relatively compact, for $t \in I, (\Gamma_2 x)(t): x \in B_r$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, Γ_2 is compact.

Thus $\Gamma = \Gamma_1 + \Gamma_2$ fulfils the assumptions of Theorem 4.1, and we conclude that (1.1) has at least one mild solution on I. ■

To obtain the uniqueness of mild solution for (1.1), we replace (A_3) by the following assumption:

$$(\widehat{A}_3)\ \ \Theta' \coloneqq \max_{1 \le i \le m} \left\{\widetilde{M}_s(\rho_i + 1) + T^{\alpha-2}\widetilde{M}_K \lambda_i + \frac{1}{\alpha} T^\alpha \widetilde{M}_T\left(\|M_1\|_{L^1(I,R^+)} + qT\|\nu_1\|_{L^1(I,R^+)}\right)\right\} < 1$$

where $q = \max_{t \in [0,T]} \int_0^t |q(t - s)| ds.$

Theorem 4.3. Assume that (A_1), (A_2) and $(\widehat{A}_3)$ hold. Then (1.1) has a unique mild solution $x \in PC(I, X)$.

Proof. Define $\Gamma: PC(I, X) \to PC(I, X)$ by

$$\Gamma u(t)=\begin{cases} S_\alpha(t)u_0+K_\alpha(t)u_1+\int_0^t T_\alpha(t-s)[f\big(s,u(s)\big) \\ \qquad +\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t\in[0,t_1], \\ S_\alpha(t-t_1)\Big(u(t_1^-)+I_1\big(u(t_1^-)\big)\Big)+K_\alpha(t-t_1)\Big(u'(t_1^-)+J_1\big(u(t_1^-)\big)\Big) \\ \qquad +\int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big)+\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t\in(t_1,t_2], \\ \vdots \\ S_\alpha(t-t_m)\Big(u(t_m^-)+I_m\big(u(t_m^-)\big)\Big)+K_\alpha(t-t_m)\Big(u'(t_m^-)+J_m\big(u(t_m^-)\big)\Big) \\ \qquad +\int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big)+\int_0^s q(s-\tau)g(\tau,u))d\tau]ds, & t\in(t_m,T]. \end{cases}$$

Note that Γ is well defined on PC(I,X). Now, take $t\in[0,t_1]$ and $x,y\in PC(I,X)$. By (4.5), (A_1) and (A_2), we have

$$\begin{aligned}\|(\Gamma x)(t)-(\Gamma y)(t)\| &\le \widetilde{M}_T\int_0^t(t-s)^{\alpha-1}\big[\mu_1(s)\,\|x(s)-y(s)\| \\ &\quad +q\int_0^s \nu_1(\tau)\,\|x(\tau)-y(\tau)\|d\tau]ds \\ &\le \widetilde{M}_T\int_0^t(t-s)^{\alpha-1}\big[\mu_1(s)\,\|x-y\|_{PC} \\ &\quad +q\,T\,\|\nu_1\|_{L^1(I,R^+)}\,\|x-y\|_{PC}]ds \\ &\le \frac{1}{\alpha}T^\alpha\widetilde{M}_T\Big(\|\mu_1\|_{L^1(I,R^+)}+qT\|\nu_1\|_{L^1(I,R^+)}\Big)\|x-y\|_{PC}.\end{aligned}$$

For $t\in(t_1,t_2]$, we have

$$\begin{aligned}\|(\Gamma x)(t)-(\Gamma y)(t)\| &\le \widetilde{M}_S\big(\|x(t_1^-)-y(t_1^-)\|+\rho_1\|x(t_1^-)-y(t_1^-)\|\big) \\ &\quad +T^{\alpha-2}\widetilde{M}_K\lambda_1\|x(t_1^-)-y(t_1^-)\| \\ &\quad +\widetilde{M}_T\int_0^t(t-s)^{\alpha-1}\big[\mu_1(s)\|x(s)-y(s)\| \\ &\quad +q\int_0^s\nu_1(\tau)\|x(\tau)-y(\tau)\|d\tau\Big]ds \\ &\le \widetilde{M}_S\big(\rho_1+1\big)\|x-y\|_{PC}+T^{\alpha-2}\widetilde{M}_K\lambda_1\|x-y\|_{PC} \\ &\quad +\frac{1}{\alpha}T^\alpha\widetilde{M}_T\Big(\|\mu_1\|_{L^1(I,R^+)}+qT\|\nu_1\|_{L^1(I,R^+)}\Big)\|x-y\|_{PC} \\ &\le [\widetilde{M}_S\big(\rho_1+1\big)+T^{\alpha-2}\widetilde{M}_K\lambda_1+\frac{1}{\alpha}T^\alpha\widetilde{M}_T\Big(\|\mu_1\|_{L^1(I,R^+)} \\ &\quad +qT\|\nu_1\|_{L^1(I,R^+)}\big)\Big]\|x-y\|_{PC}.\end{aligned}$$

Similary, we have

$$\|(\Gamma x)(t)-(\Gamma y)(t)\| \le \Big[\widetilde{M}_S(\rho_i+1)+T^{\alpha-2}\widetilde{M}_K\lambda_i+\frac{1}{\alpha}T^{\alpha}\widetilde{M}_T\Big(\|\mu_1\|_{L^1(I,R^+)} +qT\|\nu_1\|_{L^1(I,R^+)}\Big)\Big]\|x-y\|_{PC,} \qquad t\in(t_i,t_{i+1}]$$

and

$$\|(\Gamma x)(t)-(\Gamma y)(t)\| \le [\widetilde{M}_S(\rho_m+1)+T^{\alpha-2}\widetilde{M}_K\lambda_m+\frac{1}{\alpha}T^{\alpha}\widetilde{M}_T\Big(\|\mu_1\|_{L^1(I,R^+)} +qT\|\nu_1\|_{L^1(I,R^+)}\Big)]\|x-y\|_{PC} \qquad t\in(t_m,T].$$

Thus, for all $t\in[0,T]$, we have

$$\|(\Gamma x)(t)-(\Gamma y)(t)\| \le \max_{1\le i\le m}\{\widetilde{M}_S(\rho_m+1)+T^{\alpha-2}\widetilde{M}_K\lambda_m+\frac{1}{\alpha}T^{\alpha}\widetilde{M}_T(\|\mu_1\|_{L^1(I,R^+)} +qT\|\nu_1\|_{L^1(I,R^+)})\}\|x-y\|_{PC} = \Theta' \|x-y\|_{PC}.$$

Hence, by Assumption $(\widehat{A}_3)$, Γ is a contraction mapping. So, it has a unique fixed point $u_0\in PC(I,X)$ which gives the unique mild solution of (1.1). ∎

Next, we prove our second existence result. To do this, we need the following Lemma.

Lemma 4.4. ([6] Leray Schauder Alternative). Let D be a closed convex subset of a Banach space $(Z,\|.\|_z)$ and assume that $0\in D$. If $F: D\to D$ is a completely continuous map, then the set $\{x\in D: x=\lambda F(x),\ 0<\lambda<1\}$ is unbounded or the map F has a fixed point in D.

We also, consider the following assumptions:

(A_4) The functions $f,g: I\times X\to X$ are completely continuous, and there exist continuous functions $m,m': I\to[0,\infty)$ and a continuous non-decreasing function $W:[0,\infty)\to(0,\infty)$ such that

$$\|f(t,x)\|\le m(t)\,W(\|x\|), \qquad t\in I,$$
$$\|g(t,x)\|\le m'(t)\,W(\|x\|), \qquad t\in I,$$

and

$$\int_0^\infty \frac{ds}{W(s)}<\infty.$$

(A_5) The function $I_k,J_k:\mathbb{R}\to\mathbb{R}$, $k=1,\dots,m$, are completely continuous and uniformly bounded.

(A_6) The operator families $\{S_\alpha(t)\}_{t\ge0}$, $\{K_\alpha(t)\}_{t\ge0}$ and $\overline{\{T_\alpha(t)\}}_{t\ge0}$ compact, where

$\overline{T_\alpha(t)} = t^{1-\alpha} T_\alpha(t)$.

In what follows, we use the notation $N_k = \sup\{\|I_k(x)\| : x \in X\}$, $N'_k = \sup\{\|J_k(x)\| : x \in X\}$, $k = 1, \dots, m$ and $q = \max_{t\in[0,T]} \int_0^t |q(t-s)| ds$.

Theorem 4.5. Assume that conditions (A_4)-(A_6) are satisfied. If $A \in \mathfrak{A}^\alpha(\theta_0, \omega_0)$ and assume moreover,

(a) $\widetilde{M}_s < 1$

(b) $\dfrac{T^\alpha}{(1-\widetilde{M}_s)\alpha} \widetilde{M}_T \displaystyle\int_0^T \big(m(s) + qTm'(s)\big) ds < \int_c^\infty \frac{ds}{W(s)}$,

where

$$c = \max_{1\le i\le m} \left\{ \frac{\widetilde{M}_s\|x_0\| + \widetilde{M}_s N_i + T^{\alpha-2}\widetilde{M}_K\|x_0\| + T^{\alpha-2}\widetilde{M}_K N'_i}{1-\widetilde{M}_s} \right\}.$$

Then system (1.1) has at least one mild solution defined on I.

Proof. Define operator $\Gamma: PC(I,X) \to PC(I,X)$ as in Theorem 4.2 by

$$\Gamma u(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1 + \displaystyle\int_0^t T_\alpha(t-s)[f\big(s,u(s)\big) \\ \qquad + \displaystyle\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in [0, t_1], \\ S_\alpha(t-t_1)\left(u(t_1^-) + I_1\big(u(t_1^-)\big)\right) + K_\alpha(t-t_1)\left(u'(t_1^-) + J_1\big(u(t_1^-)\big)\right) \\ \qquad + \displaystyle\int_{t_1}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)d\tau]ds, & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t-t_m)\left(u(t_m^-) + I_m\big(u(t_m^-)\big)\right) + K_\alpha(t-t_m)\left(u'(t_m^-) + J_m\big(u(t_m^-)\big)\right) \\ \qquad + \displaystyle\int_{t_m}^t T_\alpha(t-s)[f\big(s,u(s)\big) + \int_0^s q(s-\tau)g(\tau,u(\tau))d\tau]ds, & t \in (t_m, T]. \end{cases}$$

Note that Γ is well defined on PC(I,X). Our proof will be divided up into five steps.

Step 1. Γ is continuous.

Let (x_n) be a sequence in PC(I,X) such that $x_n \to x$ in PC(I,X). Noting that the function f, g are continuous on I× X, we have

$$f\big(s,x_n(s)\big) \to f\big(s,x(s)\big), \qquad g\big(s,x_n(s)\big) \to g\big(s,x(s)\big) \quad \text{as} \quad n \to \infty,$$

that is for all $\epsilon > 0$, there exists N, when n>N, we have

$$\left\|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\right\|<\epsilon, \qquad \left\|g\big(s,x_n(s)\big)-g\big(s,x(s)\big)\right\|<\epsilon.$$

Now, for every $t \in [0,t_1]$, when n>N, we have

$$\begin{aligned}\|(\Gamma x_n)(t)-(\Gamma x)(t)\| &\le \int_0^{t_1}\|T_\alpha(t-s)\|\ \Big[\left\|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\right\| \\ &\qquad +q\Big\|\int_0^s\left\|g\big(\tau,x_n(\tau)\big)-g\big(\tau,x(\tau)\big)\right\|d\tau\Big\|ds\Big] \\ &\le \frac{1}{\alpha}T^\alpha\widetilde{M}_T\Big[\left\|f\big(s,x_n(s)\big)-f\big(s,x(s)\big)\right\| \\ &\qquad +qT\left\|g\big(\tau,x_n(\tau)\big)-g\big(\tau,x(\tau)\big)\right\|\Big] \\ &\le \frac{1}{\alpha}T^\alpha\widetilde{M}_T(1+qT)\epsilon. \end{aligned} \tag{4.9}$$

Moreover, for any $t \in (t_i,t_{i+1}]$, $i=2,\dots,m-1$, we have

$$\begin{aligned}\|(\Gamma x_n)(t)-(\Gamma x)(t)\| \le \widetilde{M}_s[&\|x_n(t_i^-)-x(t_i^-)\|+\|I_i(x_n(t_i^-))-I_i(x(t_i^-))\|] \\ &+T^{\alpha-2}\widetilde{M}_K\big[J_i\big(x_n(t_i^-)\big)-J_i\big(x(t_i^-)\big)\big] \\ &+\frac{1}{\alpha}T^\alpha\widetilde{M}_T(1+qT)\epsilon, \end{aligned} \tag{4-10}$$

and

$$\begin{aligned}\|(\Gamma x_n)(t)-(\Gamma x)(t)\| \le \widetilde{M}_s[&\|x_n(t_m^-)-x(t_m^-)\|+\|I_m(x_n(t_m^-))-I_m(x(t_m^-))\|] \\ &+T^{\alpha-2}\widetilde{M}_K\big[J_m\big(x_n(t_m^-)\big)-J_m\big(x(t_m^-)\big)\big] \\ &+\frac{1}{\alpha}T^\alpha\widetilde{M}_T(1+qT)\epsilon, \qquad t\in(t_m,T]. \end{aligned} \tag{4-11}$$

By the continuity of $I_k, J_k (k=1,\dots,m)$, as well (4.9) – (4.11), we have

$$\lim_{n\to\infty}\|\Gamma x_n-\Gamma x\|_{PC}=0.$$

Step 2. Γ maps bounded sets into bounded sets in PC(I,X).

It suffices to prove that for any r>0 there exists $\delta > 0$ such that $\|\Gamma x\|_{PC} \le \delta$ for each

$B_r=\{x\in PC(I,X) : \|x\|_{PC}\le r\}$. Define on B_r the operator Γ_1 and Γ_2 by:

$$\Gamma_1 x(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1, & t \in [0, t_1], \\ S_\alpha(t-t_1)\left(u(t_1^-) + I_1\left(u(t_1^-)\right)\right) + K_\alpha(t-t_1)\left(u'(t_1^-) + J_1\left(u(t_1^-)\right)\right), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t-t_m)\left(u(t_m^-) + I_m\left(u(t_m^-)\right)\right) + K_\alpha(t-t_m)\left(u'(t_m^-) + J_m\left(u(t_m^-)\right)\right), & t \in (t_m, T] \end{cases}$$

and

$$\Gamma_2 x(t) = \begin{cases} \int_0^t T_\alpha(t-s)[f\left(s, u(s)\right) + \int_0^s q(s-\tau)g\left(\tau, u(\tau)\right)d\tau]ds, & t \in [0, t_1], \\ \int_{t_1}^t T_\alpha(t-s)[f\left(s, u(s)\right) + \int_0^s q(s-\tau)g\left(\tau, u(\tau)\right)d\tau]ds, & t \in (t_1, t_2], \\ \vdots \\ \int_{t_m}^t T_\alpha(t-s)[f\left(s, u(s)\right) + \int_0^s q(s-\tau)g\left(\tau, u(\tau)\right)d\tau]ds, & t \in (t_m, T], \end{cases}$$

and for any k>0 positive functions $\mu_k, \nu_k \in L^\infty([0, T], R^+)$ such that

$$\sup_{\|u\| \le k}\|f(t, u)\| \le \mu_k(t), \quad \sup_{\|u\| \le k}\|f(t, u)\| \le \nu_k(t),$$

$$\sup_{1 \le k \le m, x \in B_r}\|I_k(x)\| \le \rho, \quad \sup_{1 \le k \le m, x \in B_r}\|J_k(x)\| \le \lambda.$$

Then, for any $x \in B_r, t \in [0, t_1]$

$$\begin{aligned} \|(\Gamma_2 x)(t)\| &\le \int_0^{t_1}\|T_\alpha(t-s)\|\left[\|f(s, x(s))\| + q\int_0^s\|q(\tau, x(\tau))\|d\tau\right]ds \\ &\le \frac{1}{\alpha}T^\alpha \widetilde{M}_T\left[\|\mu_r\|_{L^\infty(I, R^+)} + qT\|\nu_r\|_{L^\infty(I, R^+)}\right] < \infty \end{aligned}$$

and

$$\begin{aligned} \|(\Gamma_1 x)(t)\| &\le \|S_\alpha(t)\|\|u_0\| + \|K_\alpha(t)\|\|u_1\| \\ &\le \widetilde{M}_s r + T^{\alpha-2}\widetilde{M}_k r < \infty. \end{aligned}$$

Then

$$\|(\Gamma x)(t)\| \le \widetilde{M}_s r + T^{\alpha-2}\widetilde{M}_k r + \frac{1}{\alpha}T^\alpha \widetilde{M}_T\left[\|\mu_r\|_{L^\infty(I, R^+)} + qT\|\nu_r\|_{L^\infty(I, R^+)}\right]. \tag{4.12}$$

Similary, for any $t \in (t_i, t_{i+1}], i = 1, \dots, m$

$$\|(\Gamma x)(t)\| \le \widetilde{M}_s(r+\rho) + T^{\alpha-2}\widetilde{M}_K(r+\lambda)$$

$$(4-13) \qquad +\frac{1}{\alpha}T^{\alpha}\widetilde{M}_T\left[\|\mu_r\|_{L^{\infty}(I,R^+)}+qT\|\nu_r\|_{L^{\infty}(I,R^+)}\right].$$

From (4.12) and (4.13), we obtain

$$\|(\Gamma x)(t)\| \le \widetilde{M}_s(r+\rho)+T^{\alpha-2}\widetilde{M}_K(r+\lambda)$$

$$+\frac{1}{\alpha}T^{\alpha}\widetilde{M}_T\left[\|\mu_r\|_{L^{\infty}(I,R^+)}+qT\|\nu_r\|_{L^{\infty}(I,R^+)}\right] := \delta, \qquad t\in[0,T].$$

Step 3. $\Gamma(B_r)$ is equicontinuous, where (B_r), Γ_1 and Γ_2 are defined in step 2. Define

$$\mu=\max\|f(t,x)\|, \qquad \nu=\max\|g(t,x)\|.$$

For this purpose, for all $s_1,s_2\in[0,t_1]$, with $s_1<s_2$ by $\|S_\alpha(t)\|\le M_1e^{wt}$ and $\|K_\alpha(t)\|\le M_2e^{wt}(1+t^{\alpha-2})$ and Theorem 3.4, we have

$$\begin{aligned}
\|(\Gamma x)(s_2)-(\Gamma x)(s_1)\| &\le \|(\Gamma_1 x)(s_2)-(\Gamma_1 x)(s_1)\|+\|(\Gamma_2 x)(s_2)-(\Gamma_2 x)(s_1)\|\\
&\le \|S_\alpha(s_2)-S_\alpha(s_1)\|\|u_0\|+\|K_\alpha(s_2)-K_\alpha(s_1)\|\|u_1\|\\
&\quad+\int_0^{s_2}\|T_\alpha(s_1-s)-T_\alpha(s_2-s)\|\Big[\big\|f\big(s,u(s)\big)\big\|\\
&\quad+q\int_0^{s}\big\|g\big(\tau,u(\tau)\big)d\tau\big\|\Big]ds\\
&\quad+\int_{s_2}^{s_1}\|T_\alpha(s_1-s)\|\Big[\big\|f\big(s,u(s)\big)\big\|\\
&\quad+q\int_0^{s}\big\|g\big(\tau,u(\tau)\big)d\tau\big\|\Big]ds\\
&\le M_1\|u_0\||e^{ws_2}-e^{ws_1}|+M_2\|u_1\|\left|e^{ws_2}s_2^{\alpha-2}-e^{ws_1}s_1^{\alpha-2}\right|\\
&\quad+\left(\|\mu\|_{L^{\infty}(I,R^+)}+qT\|\nu\|_{L^{\infty}(I,R^+)}\right)\\
&\quad\times\left[\frac{\widetilde{M}_T(s_1^{\alpha}+s_2^{\alpha}-(s_1-s_2)^{\alpha})}{\alpha}\right]\\
&\quad+\left(\|\mu\|_{L^{\infty}(I,R^+)}+qT\|\nu\|_{L^{\infty}(I,R^+)}\right)\left[\frac{\widetilde{M}_T(s_2^{\alpha}-s_1^{\alpha})}{\alpha}\right].
\end{aligned}$$

Similary $\forall s_1,s_2\in(t_i,t_{i+1}]$, with $s_1<s_2$, $i=1,\dots,m$, we have

$$\|(\Gamma x)(s_2)-(\Gamma x)(s_1)\|\le M_1(r+\rho)e^{-wt_1}|e^{ws_2}-e^{ws_1}|$$

$$+M_2(r+\lambda)e^{-wt_1}\left|e^{ws_2}s_2^{\alpha-2}-e^{ws_1}s_1^{\alpha-2}\right|$$

$$+\left(\|\mu\|_{L^{\infty}(I,R^+)}+qT\|\nu\|_{L^{\infty}(I,R^+)}\right)$$

$$\times\left[\frac{\widetilde{M}_T(s_1^\alpha + s_2^\alpha - (s_1 - s_2)^\alpha)}{\alpha}\right]$$
$$+\left(\|\mu\|_{L^\infty(I,R^+)} + qT\|\nu\|_{L^\infty(I,R^+)}\right)\left[\frac{\widetilde{M}_T(s_2^\alpha - s_1^\alpha)}{\alpha}\right].$$

Thus, from the above inequalities, we have $\lim_{s_1\to s_2}\|(\Gamma x)(s_2) - (\Gamma x)(s_1)\| = 0$. So $\Gamma(B_r)$ is equicontinuous.

Step 4. Γ maps B_r into a compact set in X.

To this end, we decompose $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_1 are defined in step 2. We now prove that $\{(\Gamma_2 x)(t) : x \in B_r\}$ is compact in X.

In step 2, we have proved that $\{(\Gamma_2 x)(t) : x \in B_r\}$ is bounded and in step 3, we have proved that $\{(\Gamma_2 x)(t) : x \in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, Γ_2 is compact.

Next, we show that $\{(\Gamma_1 x)(t) : x \in B_r\}$ is compact in X. For all $t \in [0, t_1]$, since $(\Gamma_1 x)(t) = S_\alpha(t)u_0 + K_\alpha(t)u_1$, by (A_6), it follows that $\{(\Gamma_1 x)(t) : t \in [0, t_1],\ x \in B_r\}$ is a compact subset of X. On the other hand, for $t \in (t_i, t_{i+1}], i \geq 1$ and $x \in B_r$, there exists $\hat{r} > 0$ such that

$$\{\Gamma'_1 x\}_i(t) \in \begin{cases} S_\alpha(t - t_i)\left(u(t_i^-) + I_i\left(u(t_i^-)\right)\right) \\ \qquad + K_\alpha(t - t_i)\left(u'(t_i^-) + J_i\left(u(t_i^-)\right)\right), & t \in (t_i, t_{i+1}), u, u' \in B_{\hat{r}} \\ S_\alpha(t_{i+1} - t_i)\left(u(t_i^-) + I_i\left(u(t_i^-)\right)\right) \\ \qquad + K_\alpha(t_{i+1} - t_i)\left(u'(t_i^-) + J_i\left(u(t_i^-)\right)\right), & t = t_{i+1}, u, u' \in B_{\hat{r}} \\ u(t_i^-) + I_i\left(u(t_i^-)\right) + u'(t_i^-) + J_i\left(u(t_i^-)\right)), & t = t_i, u, u' \in B_{\hat{r}} \end{cases}$$

where $B_{\hat{r}}$ is an open ball of radius $\hat{r}$. From (A_5) and (A_6), it follows that $\{\Gamma'_1 x\}_i(t)$ is relatively compact in X, for all $t \in [t_i, t_{i+1}]$, $i \geq 1$. Moreover, by the compactness of I_i and $J_i, (i = 1, \dots, m)$ and the continuity of the evolution operators $S_\alpha(t)$ and $K_\alpha(t)$, for all $t \in [0, T]$, we conclude that operator Γ_1 is also compact. Therefore, so is $\Gamma = \Gamma_1 + \Gamma_2$.

Step 5. We show that the set

$$E = \{x \in PC(I, X) :\ x = \lambda\, \Gamma(x)\ \text{ for some } 0 < \lambda < 1\}$$

is bounded in PC(I,X).

Let $x_\lambda \in E$. Then $x_\lambda(t) = (\Gamma x_\lambda)(t)$ for some $0 < \lambda < 1$. Thus

$$\|x_\lambda(t)\| \le \begin{cases} \lambda\Big[\widetilde{M}_s\|u_0\| + T^{\alpha-2}\widetilde{M}_K\|u_1\| \\ \qquad + \widetilde{M}_T\displaystyle\int_0^t (t-s)^{\alpha-1}[m(s)\,W(x_\lambda(s)) \\ \qquad + q\displaystyle\int_0^s m'(\tau)\,W\big(x_\lambda(\tau)\big)d\tau]ds\Big], & t \in [0,t_1], \\ \lambda\big[\widetilde{M}_s(\|x_\lambda\| + N_1) + T^{\alpha-2}\widetilde{M}_K(\|x_\lambda\| + N'_1) \\ \qquad + \widetilde{M}_T\displaystyle\int_{t_1}^t (t-s)^{\alpha-1}[m(s)\,W\big(x_\lambda(s)\big) \\ \qquad + q\displaystyle\int_0^s m'(\tau)\,W\big(x_\lambda(\tau)\big)d\tau]ds\Big], & t \in [t_1,t_2], \\ \vdots \\ \lambda\big[\widetilde{M}_s(\|x_\lambda\| + N_m) + T^{\alpha-2}\widetilde{M}_K(\|x_\lambda\| + N'_m) \\ \qquad + \widetilde{M}_T\displaystyle\int_{t_m}^t (t-s)^{\alpha-1}[m(s)\,W\big(x_\lambda(s)\big) \\ \qquad + q\displaystyle\int_0^s m'(\tau)\,W\big(x_\lambda(\tau)\big)d\tau]ds\Big], & t \in [t_m,T]. \end{cases}$$

By the Young inequality [[3], page 6], for $t \in (t_i, t_{i+1})$, $i = 1, \dots, m$, we get that

$$\|x_\lambda(t)\| \le \widetilde{M}_s(\|x_\lambda\| + N_i) + T^{\alpha-2}\widetilde{M}_K(\|x_\lambda\| + N'_i) \\ + \frac{T^\alpha}{\alpha}\widetilde{M}_T\int_{t_i}^t \big(m(s)\,W(\|x_\lambda(s)\|) + qTm'(\tau)\,W(\|x_\lambda(\tau)\|)\big)\,ds,$$

and for $t \in [0,t_1]$, we have

$$\|x_\lambda(t)\| \le \widetilde{M}_s\|x_\lambda\| + T^{\alpha-2}\widetilde{M}_K\|x_\lambda\| \\ + \frac{T^\alpha}{\alpha}\widetilde{M}_T\int_0^t \big(m(s)\,W(\|x_\lambda(s)\|) + qTm'(\tau)\,W(\|x_\lambda(\tau)\|)\big)\,ds.$$

Then, for all $t \in [0,T]$, we have

$$\|x_\lambda(t)\| \le \beta_\lambda(t) \cong c + \frac{T^\alpha}{(1-\widetilde{M}_s)\alpha}\widetilde{M}_T\int_0^t \big(m(s)\,W(\|x_\lambda(s)\|) + qTm'(\tau)\,W(\|x_\lambda(\tau)\|)\big)\,ds,$$

where

$$c = \max_{1\le i\le m}\left\{\frac{\widetilde{M}_s\|u_0\| + \widetilde{M}_s N_i + T^{\alpha-2}\widetilde{M}_K\|u_0\| + T^{\alpha-2}\widetilde{M}_K N'_i}{1-\widetilde{M}_s}\right\}.$$

Computing $\beta'_\lambda(t)$ for $t \in [0,T]$, we arrive at

$$\beta'_\lambda(t) \leq \frac{T^\alpha}{(1-\widetilde{M}_s)\alpha}\widetilde{M}_T\big(m(t)\,W(\|x_\lambda(t)\|) + qTm'(t)\,W(\|x_\lambda(t)\|)\big).$$

Thus,

$$\frac{d\beta_\lambda(t)}{W(\|\beta_\lambda(t)\|)} \leq \frac{d\beta_\lambda(t)}{W(\|x_\lambda(t)\|)} \leq \frac{T^\alpha}{(1-\widetilde{M}_s)\alpha}\big(m(t) + qTm'(t)\big)dt.$$

Since W(s) is positive and non-decreasing. Integrating both sides of the above inequality over [0, T], we have

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W(s)} \leq \frac{T^\alpha}{(1-\widetilde{M}_s)\alpha}\int_0^T \big(m(s) + qTm'(s)\big)ds < \int_c^\infty \frac{ds}{W(s)},$$

where we have used the facts $\beta_\lambda(0) = c$, $\beta_\lambda(t)$ is positive and non-decreasing. Hence, by the above inequality, we conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded. This implies that $E = \{x \in PC(I,X) : x = \lambda\Gamma x,\ 0 < \lambda < 1\}$ is bounded in X. Since we have already proven that Γ is continuous and compact, by Lemma 4.4, Γ has a fixed point which is a mild solution of (1.1). ■

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