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Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for p-convex functions via new fractional conformable integral operators

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Abstract

In this paper, we obtained the Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for p-convex functions via new fractional conformable integral operators. We also gave some new Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for convex functions and harmonically convex functions via new fractional conformable integral operators.

Keywords: Hermite-Hadamard inequalities, Hermite-Hadamard-Fejer inequalities, Riemann-Liouville fractional integral, fractional conformable integral operators, convex functions, p-convex functions, harmonically convex functions.

2010 MSC: 26A51, 26A33, 26D10, 26D07, 26D15.

1. Introduction

A function $\varphi : \mathbb{K} \to \mathbb{R}$ on a real interval, for all $k_1, k_2 \in \mathbb{K}$ and $\tau \in [0,1]$, is called convex if

$$\varphi(\tau k_1 + (1-\tau)k_2) \leqslant \tau\varphi(k_1) + (1-\tau)\varphi(k_2)$$

holds. Many authors gave results for convex functions due to its importance. The most well known inequality for convex functions is called The Hermite-Hadamard inequality [5] given as

$$\varphi\left(\frac{k_1+k_2}{2}\right) \leqslant \frac{1}{k_2-k_1}\int_{k_1}^{k_2}\varphi(s)ds \leqslant \frac{\varphi(k_1)+\varphi(k_2)}{2}, \tag{1.1}$$

where $k_1, k_2 \in \mathbb{K}$, $k_1 < k_2$. Then Fejer [4] introduced the weighted generalization of (1.1) as follows

$$\varphi\left(\frac{k_1+k_2}{2}\right)\int_{k_1}^{k_2} g(s)ds \leqslant \frac{1}{k_2-k_1}\int_{k_1}^{k_2}\varphi(s)g(s)ds \leqslant \frac{\varphi(k_1)+\varphi(k_2)}{2}\int_{k_1}^{k_2} g(s)ds,$$

where $g : [k_1, k_2] \to \mathbb{R}$ is nonnegative, integrable, and symmetric to $(k_1+k_2)/2$. These two inequalities are then generalized in different ways. There are many generalization of convex functions.

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Definition 1.1 ([7]). Consider an interval $\mathbb{K} \subset (0,\infty) = \mathbb{R}_+$, and $p \in \mathbb{R}\backslash\{0\}$. A function $\varphi : \mathbb{K} \to \mathbb{R}$ is called p-convex if

$$\varphi\left([\tau k_1^p + (1-\tau)k_2^p]^{\frac{1}{p}}\right) \leqslant \tau\varphi(k_1) + (1-\tau)\varphi(k_2) \tag{1.2}$$

for all $k_1, k_2 \in \mathbb{K}$ and $\tau \in [0,1]$. If (1.2) is reversed then φ is called p-concave.

Authors (see [1–3, 16, 18–21]) gave Hermite-Hadamard and Hermite-Hadamard-Fejer inequalities in other fractional integrals including Riemann-Liouville, Hadamard, Katugampola, etc.. These integrals are defined as follows.

Definition 1.2 ([11]). Let $\varphi \in L[k_1, k_2]$. The right- and left-hand side Riemann- Liouville fractional integrals $J^{\alpha}_{k_1+}\varphi$ and $J^{\alpha}_{k_2-}\varphi$ of order $\alpha > 0$, $k_2 > k_1 \geqslant 0$, are expressed as:

$$J^{\alpha}_{k_1+}\varphi(s) = \frac{1}{\Gamma(\alpha)}\int_{k_1}^{s}(s-\tau)^{\alpha-1}\varphi(\tau)d\tau,\ s > k_1, \tag{1.3}$$

and

$$J^{\alpha}_{k_2-}\varphi(s) = \frac{1}{\Gamma(\alpha)}\int_{s}^{k_2}(\tau-s)^{\alpha-1}\varphi(\tau)d\tau,\ s < k_2, \tag{1.4}$$

respectively, where $\Gamma(\cdot)$ is the Gamma function expressed as $\Gamma(\alpha) = \int_0^{\infty} e^{-\tau}\tau^{\alpha-1}d\tau$.

Definition 1.3 ([14]). Let $\alpha > 0$ with $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, and $1 < s < k_2$. The left- and right-side Hadamard fractional integrals of order α of function φ are respectively detailed as:

$$H^{\alpha}_{k_1+}\varphi(s) = \frac{1}{\Gamma(\alpha)}\int_{k_1}^{s}\left(\ln\frac{s}{\tau}\right)^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau, \tag{1.5}$$

and

$$H^{\alpha}_{k_2-}\varphi(s) = \frac{1}{\Gamma(\alpha)}\int_{s}^{k_2}\left(\ln\frac{\tau}{s}\right)^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau. \tag{1.6}$$

Definition 1.4 ([10]). Let $[k_1, k_2] \subset \mathbb{R}$ is an interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha(>0)$ of $\varphi \in X^p_c(k_1, k_2)$ are described as:

$${}^{\rho}I^{\alpha}_{k_1+}\varphi(s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{k_1}^{s}(s^{\rho}-\tau^{\rho})^{\alpha-1}\tau^{\rho-1}\varphi(\tau)d\tau, \tag{1.7}$$

and

$${}^{\rho}I^{\alpha}_{k_2-}\varphi(s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{s}^{k_2}(\tau^{\rho}-k^{\rho})^{\alpha-1}\tau^{\rho-1}\varphi(\tau)d\tau, \tag{1.8}$$

with $k_1 < s < k_2$ and $\rho > 0$.

Jarad et al. [9] has defined the following new fractional integral operator.

Definition 1.5 ([9]). Let $\beta \in \mathbb{C}$, then the left and right sided fractional conformable integral operators of order $\alpha > 0$ are respectively characterized as:

$${}^{\beta}_{k_1}\mathfrak{J}^{\alpha}\varphi(s) = \frac{1}{\Gamma(\beta)}\int_{k_1}^{s}\left(\frac{(s-k_1)^{\alpha}-(\tau-k_1)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\varphi(\tau)}{(\tau-k_1)^{1-\alpha}}d\tau, \tag{1.9}$$

$${}^{\beta}\mathfrak{J}^{\alpha}_{k_2}\varphi(s) = \frac{1}{\Gamma(\beta)}\int_{s}^{k_2}\left(\frac{(k_2-s)^{\alpha}-(k_2-\tau)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\varphi(\tau)}{(k_2-\tau)^{1-\alpha}}d\tau. \tag{1.10}$$

Note that the fractional conformable integral operator (1.9) gives (1.3), (1.5), and (1.7) by taking $\alpha = 1$, $k_1 = 0$ and $\alpha \to 0$, and $k_1 = 0$, respectively. Similarly, the fractional conformable integral operator (1.10) gives (1.4), (1.6) and (1.8) by taking $\alpha = 1$, $k_2 = 0$ and $\alpha \to 0$, and $k_2 = 0$, respectively.

In this paper, we established new Hermite-Hadamard and Hermite-Hadamard-Fejer inequalities for p-convex functions via new fractional conformable integrals.

2. Hermite-Hadamard type inequalities

The intention of this section is to prove new inequalities for p-convex functions via new fractional conformable integrals.

Theorem 2.1. *Let* $\varphi : [k_1, k_2] \subset (0, \infty) \to \mathbb{R}$ *be a* p*-convex function such that* $\varphi \in L[k_1, k_2]$*. Then*

(i) *for* $p > 0$ *we have*

$$\varphi\left(\left[\frac{k_1^p + k_2^p}{2}\right]^{1/p}\right) \leqslant \frac{\alpha^\beta \Gamma(\beta+1)}{2(k_2^p - k_1^p)^{\alpha\beta}} \left[{}^{\beta}_{k_1^p}\mathcal{J}^{\alpha} \varphi \circ h(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p} \varphi \circ h(k_1^p)\right] \leqslant \frac{\varphi(k_1) + \varphi(k_2)}{2}, \tag{2.1}$$

where $h(s) = s^{\frac{1}{p}}$ *for all* $s \in [k_1^p, k_2^p]$.

(ii) *for* $p < 0$ *we have*

$$\varphi\left(\left[\frac{k_1^p + k_2^p}{2}\right]^{1/p}\right) \leqslant \frac{\alpha^\beta \Gamma(\beta+1)}{2(k_1^p - k_2^p)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{k_1^p} \varphi \circ h(k_2^p) + {}^{\beta}_{k_2^p}\mathcal{J}^{\alpha} \varphi \circ h(k_1^p)\right] \leqslant \frac{\varphi(k_1) + \varphi(k_2)}{2},$$

where $h(s) = s^{\frac{1}{p}}$, $s \in [k_2^p, k_1^p]$.

Proof. Since φ is p-convex on $[k_1, k_2]$, we can have

$$\varphi\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \frac{\varphi(x) + \varphi(y)}{2}.$$

Taking $x^p = \tau k_1^p + (1-\tau)k_2^p$ and $y^p = (1-\tau)k_1^p + \tau k_2^p$ with $\tau \in [0, 1]$, we get

$$\varphi\left(\left[\frac{k_1^p + k_2^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \frac{\varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) + \varphi\left(\left[(1-\tau)k_1^p + \tau k_2^p\right]^{\frac{1}{p}}\right)}{2}. \tag{2.2}$$

Multiplying (2.2) by $\left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1}$ on both sides with $\tau \in (0, 1)$, $\alpha > 0$ and then integrating along τ over $\in [0, 1]$, we obtain

$$\begin{aligned} \varphi\left(\left[\frac{k_1^p + k_2^p}{2}\right]^{\frac{1}{p}}\right) \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} d\tau &\leqslant \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &\quad + \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[(1-\tau)k_1^p + \tau k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= I_1 + I_2. \end{aligned} \tag{2.3}$$

By setting $u = \tau k_1^p + (1-\tau)k_2^p$, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \int_{k_2^p}^{k_1^p} \left(\frac{1 - \left(\frac{u-k_2^p}{k_1^p-k_2^p}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{u-k_2^p}{k_1^p-k_2^p}\right)^{\alpha-1} \varphi \circ h(u) \frac{du}{k_1^p - k_2^p} \\ &= \frac{1}{(k_2^p - k_1^p)^{\alpha\beta}} \int_{k_1^p}^{k_2^p} \left(\frac{(k_2^p - k_1^p)^\alpha - (k_2^p - u)^\alpha}{\alpha}\right)^{\beta-1} (k_2^p - u)^{\alpha-1} \varphi \circ h(u) du \\ &= \frac{\Gamma(\beta)}{(k_2^p - k_1^p)^{\alpha\beta}} \, {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p} \varphi \circ h(k_1^p). \end{aligned}$$

Similarly, by setting $u = \tau k_2^p + (1-\tau)k_1^p$, we have

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[(1-\tau)k_1^p + \tau k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \int_{k_1^p}^{k_2^p} \left(\frac{1-\left(\frac{u-k_1^p}{k_2^p-k_1^p}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{u-k_1^p}{k_2^p-k_1^p}\right)^{\alpha-1} \varphi\circ h(u)\frac{du}{k_2^p-k_1^p} \\ &= \frac{1}{(k_2^p-k_1^p)^{\alpha\beta}} \int_{k_1^p}^{k_2^p} \left(\frac{(k_2^p-k_1^p)^\alpha-(u-k_1^p)^\alpha}{\alpha}\right)^{\beta-1} (u-k_1^p)^{\alpha-1}\varphi\circ h(u)du \\ &= \frac{\Gamma(\beta)}{(k_2^p-k_1^p)^{\alpha\beta}}\, {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}\varphi\circ h(k_2^p). \end{aligned}$$

Thus by putting values of I_1 and I_2 in (2.3), we get

$$\frac{1}{\alpha^\beta\beta}\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \frac{\Gamma(\beta)}{(k_2^p-k_1^p)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}\varphi\circ h(k_1^p) + {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}\varphi\circ h(k_2^p)\right].$$

This completes the first inequality of (2.1). For second inequality, we know that

$$\varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) + \varphi\left(\left[\tau k_2^p + (1-\tau)k_1^p\right]^{\frac{1}{p}}\right) \leqslant \left[\varphi(k_1) + \varphi(k_2)\right]. \tag{2.4}$$

Multiplying (2.4) by $\left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1}\tau^{\alpha-1}$ on both sides with $\tau \in (0,1)$, $\alpha > 0$ and then integrating along τ over $\in [0,1]$, we obtain

$$\frac{\Gamma(\beta)}{(k_2^p-k_1^p)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}\varphi\circ h(k_1^p) + {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}\varphi\circ h(k_2^p)\right] \leqslant \frac{1}{\alpha^\beta\beta}(\varphi(k_1)+\varphi(k_2)).$$

This completes the second inequality of (2.1). Hence we have the proof.

The proof of (ii) is parallel to (i). □

Remark 2.2. In Theorem 2.1:

1. by allowing $p = 1$, we achieve Theorem 2.1 in [17];
2. by allowing $p = 1$ and $\alpha = 1$, we achieve Theorem 2 in [15];
3. by allowing $p = -1$ and $\alpha = 1$, we achieve Theorem 4 in [8].

Corollary 2.3. *With the parallel assumption of Theorem 2.1, if we take $p = -1$, then we get*

$$\varphi\left(\frac{2k_1k_2}{k_1+k_2}\right) \leqslant \frac{(k_1k_2)^{\alpha\beta}\alpha^\beta\Gamma(\beta+1)}{2(k_2-k_1)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{1/k_1}\varphi\circ h\left(\frac{1}{k_2}\right) + {}^{\beta}_{1/k_2}\mathcal{J}^{\alpha}\varphi\circ h\left(\frac{1}{k_1}\right)\right] \leqslant \frac{\varphi(k_1)+\varphi(k_2)}{2},$$

where $h(s) = \frac{1}{s}$, $s \in \left[\frac{1}{k_2}, \frac{1}{k_1}\right]$.

Lemma 2.4. *Let $\varphi : [k_1, k_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (k_1, k_2) with $k_1 < k_2$ such that $\varphi' \in L[k_1, k_2]$, then*

(i) *for* $p > 0$

$$\begin{aligned} &{}_1\Delta_\varphi(k_1, k_2; \alpha; \beta; \mathcal{J}) \\ &= \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\int_0^1\left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right] A_\tau^{\frac{1}{p}-1}\varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau, \end{aligned} \tag{2.5}$$

where $A_\tau = \left[\tau k_1^p + (1-\tau)k_2^p\right]$ *and*

$$ {}_1\Delta_\varphi(k_1, k_2; \alpha; \beta; \mathcal{J}) = \left(\frac{\varphi(k_1^p) + \varphi(k_2^p)}{2}\right) - \frac{\Gamma(\beta+1)\alpha^\beta}{2(k_2^p - k_1^p)^{\alpha\beta}} \left[{}^{\beta}_{k_1^p}\mathcal{J}^\alpha \varphi \circ h(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p} \varphi \circ h(k_1^p)\right]; $$

(ii) *for* $p < 0$

$$ {}_2\Delta_\varphi(k_1, k_2; \alpha; \beta; \mathcal{J}) = \frac{(k_1^p - k_2^p)\alpha^\beta}{2p} \int_0^1 \left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right] B_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_2^p + (1-\tau)k_1^p\right]^{\frac{1}{p}}\right) d\tau, $$

where $B_\tau = \left[\tau k_2^p + (1-\tau)k_1^p\right]$ *and*

$$ {}_2\Delta_\varphi(k_1, k_2; \alpha; \beta; \mathcal{J}) = \left(\frac{\varphi(k_1^p) + \varphi(k_2^p)}{2}\right) - \frac{\Gamma(\beta+1)\alpha^\beta}{2(k_1^p - k_2^p)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^\alpha_{k_1^p} \varphi \circ h(k_2^p) + {}^{\beta}_{k_2^p}\mathcal{J}^{\alpha} \varphi \circ h(k_1^p)\right]. $$

Proof.

(i) Consider,

$$ \begin{aligned} &\int_0^1 \left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right] A_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta A_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &\quad - \int_0^1 \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta A_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau = I_1 - I_2. \end{aligned} $$

Then applying by parts integration, we achieve

$$ \begin{aligned} I_1 &= \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta A_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta \frac{p}{k_1^p - k_2^p} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right)\Big|_0^1 \\ &\quad - \frac{p}{k_2^p - k_1^p} \int_0^1 \beta \left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \frac{p}{\alpha^\beta (k_2^p - k_1^p)} \varphi(k_2^p) - \frac{p\beta}{k_2^p - k_1^p} \frac{\Gamma(\beta)}{(k_2^p - k_1^p)^{\alpha\beta}} {}^{\beta}\mathcal{J}^\alpha_{k_2^p} \varphi \circ h(k_1^p) \\ &= \frac{p}{k_2^p - k_1^p} \left[\frac{\varphi(k_2^p)}{\alpha^\beta} - \frac{\Gamma(\beta+1)}{(k_2^p - k_1^p)^{\alpha\beta}} {}^{\beta}\mathcal{J}^\alpha_{k_2^p} \varphi \circ h(k_1^p)\right], \end{aligned} $$

and similarly,

$$ \begin{aligned} I_2 &= \int_0^1 \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta A_\tau^{\frac{1}{p}-1} \varphi'\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \\ &= \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta \frac{p}{k_1^p - k_2^p} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right)\Big|_0^1 \\ &\quad - \frac{p}{k_1^p - k_2^p} \int_0^1 \beta \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^{\beta-1} (1-\tau)^{\alpha-1} \varphi\left(\left[\tau k_1^p + (1-\tau)k_2^p\right]^{\frac{1}{p}}\right) d\tau \end{aligned} $$

$$= -\frac{p}{\alpha^\beta(k_2^p-k_1^p)}\varphi(k_1^p) + \frac{p\beta}{k_2^p-k_1^p}\frac{\Gamma(\beta)}{(k_2^p-k_1^p)^{\alpha\beta}}\, {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}\varphi\circ h(k_2^p)$$
$$= -\frac{p}{k_2^p-k_1^p}\left[\frac{\varphi(k_2^p)}{\alpha^\beta} - \frac{\Gamma(\beta+1)}{(k_2^p-k_1^p)^{\alpha\beta}}\, {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}\varphi\circ h(k_2^p)\right],$$

where we used the changes of variable with $x = 1-\tau$. Thus by adding I_1, $-I_2$ and then by multiplying both sides by $\frac{\alpha^\beta(k_2^p-k_1^p)}{2p}$, we get the required result (2.5).

(ii) The proof is similar to (i). □

Remark 2.5. In Lemma 2.4,

1. if we take $p=1$ we get Lemma 3.1 in [17];
2. if we take $p=1$ and $\alpha=1$ we get Lemma 2 in [15].

Theorem 2.6. *Let $\varphi : [k_1,k_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (k_1,k_2), $k_1 < k_2$, such that $\varphi' \in L[k_1,k_2]$. If $|\varphi'|^q$, where $q \geqslant 1$, is p-convex, then*

(i) *for $p>0$*

$$|\,{}_1\Delta_\varphi(k_1,k_2;\alpha;\beta;\mathcal{J})|$$
$$\leqslant \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\left(\frac{k_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{k_1^p}{k_2^p}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha^{\beta+1}}B\left(\frac{2}{\alpha},\beta+1\right)[|\varphi'(k_1)|^q+|\varphi'(k_2)|^q]\right)^q; \tag{2.6}$$

(ii) *for $p<0$*

$$|\,{}_2\Delta_\varphi(k_1,k_2;\alpha;\beta;\mathcal{J})|$$
$$\leqslant \frac{(k_1^p-k_2^p)\alpha^\beta}{2p}\left(\frac{k_1^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{k_2^p}{k_1^p}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha^{\beta+1}}B\left(\frac{2}{\alpha},\beta+1\right)[|\varphi'(k_1)|^q+|\varphi'(k_2)|^q]\right)^q.$$

Where B and ${}_2F_1$ are classical Beta function and Hypergeometric function, respectively.

Proof. Applying Lemma 2.4, modulus property, Holder's inequality, and p-convexity of $|\varphi'|^q$, we achieve

$$|\,{}_1\Delta_\varphi(k_1,k_2;\alpha;\beta;\mathcal{J})|$$
$$= \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\left|\int_0^1\left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right]A_\tau^{\frac{1}{p}-1}\varphi'\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)d\tau\right|$$
$$\leqslant \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\left|\int_0^1\left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta + \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right]A_\tau^{\frac{1}{p}-1}\varphi'\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)\right|d\tau$$
$$\leqslant \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\left(\int_0^1 A_\tau^{\frac{1}{p}-1}d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta + \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right]|\varphi'\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)|^q d\tau\right)^q \tag{2.7}$$
$$\leqslant \frac{(k_2^p-k_1^p)\alpha^\beta}{2p}\left(\int_0^1 A_\tau^{\frac{1}{p}-1}d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left[\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta + \left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right]\left(\tau|\varphi'(k_1)|^q+(1-\tau)|\varphi'(k_2)|^q\right)d\tau\right)^q$$

$$= \frac{(k_2^p - k_1^p)\alpha^\beta}{2p}\mu^{1-\frac{1}{q}}\left(|\varphi'(k_1)|^q \int_0^1 \left[\tau\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta + \tau\left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right] d\tau\right.$$
$$\left. + |\varphi'(k_2)|^q \int_0^1 \left[(1-\tau)\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta + (1-\tau)\left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta\right] d\tau\right)^q,$$

where

$$\mu = \int_0^1 A_\tau^{\frac{1}{p}-1} d\tau = \frac{k_2^{1-p}}{2}\, {}_2F_1\left(1-\frac{1}{p}, 1; 2; 1-\frac{k_1^p}{k_2^p}\right),$$

and by using changes of variables as $x = \tau^\alpha$ and $y = (1-\tau)^\alpha$,

$$\int_0^1 \tau\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta d\tau = \frac{1}{\alpha^{\beta+1}} B\left(\frac{2}{\alpha}, \beta+1\right),$$
$$\int_0^1 \tau\left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta d\tau = \frac{1}{\alpha^{\beta+1}}\left[B\left(\frac{1}{\alpha}, \beta+1\right) - B\left(\frac{2}{\alpha}, \beta+1\right)\right],$$
$$\int_0^1 (1-\tau)\left(\frac{1-\tau^\alpha}{\alpha}\right)^\beta d\tau = \frac{1}{\alpha^{\beta+1}}\frac{1}{\alpha^{\beta+1}}\left[B\left(\frac{1}{\alpha}, \beta+1\right) - B\left(\frac{2}{\alpha}, \beta+1\right)\right],$$
$$\int_0^1 (1-\tau)\left(\frac{1-(1-\tau)^\alpha}{\alpha}\right)^\beta d\tau = \frac{1}{\alpha^{\beta+1}} B\left(\frac{2}{\alpha}, \beta+1\right).$$

Thus by substituting all above equalities in (2.7), we get the inequality (2.6).

Proof of (ii) is similar to (i). □

Corollary 2.7. *Under identical consideration of Theorem 2.6, if we take* $p = -1$*, then we get*

$$\left|\left(\frac{\varphi\left(\frac{1}{k_1}\right) + \varphi\left(\frac{1}{k_2}\right)}{2}\right) - \frac{(k_1k_2)^{\alpha\beta}\Gamma(\beta+1)\alpha^\beta}{2(k_2-k_1)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{1/k_1}\varphi\circ h\left(\frac{1}{k_2}\right) + {}^\beta_{1/k_2}\mathcal{J}^\alpha\varphi\circ h\left(\frac{1}{k_1}\right)\right]\right|$$
$$\leqslant \frac{(k_2-k_1)\alpha^\beta}{-2k_1k_2}\left(\frac{k_1^2}{2}\, {}_2F_1\left(2, 1; 2; 1-\frac{k_1}{k_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha^{\beta+1}} B\left(\frac{2}{\alpha}, \beta+1\right)\left[|\varphi'(k_1)|^q + |\varphi'(k_2)|^q\right]\right)^q,$$

where $h(s) = 1/s$, $s \in \left[\frac{1}{k_2}, \frac{1}{k_1}\right]$.

3. Hermite-Hadamard-Fejer type inequalities

In this section our intention is to prove some Hermite-Hadamard-Fejer type inequalities via new fractional conformable integral operators. Kunt and Iscan [12] defined following useful definition.

Definition 3.1 ([12]). Let $p \in \mathbb{R}\backslash\{0\}$. A function $g : [k_1, k_2] \subseteq (0, \infty) \to \mathbb{R}$ is called p-symmetric along $\left[\frac{k_1^p + k_2^p}{2}\right]^{1/p}$, if

$$g(s) = g\left(\left[k_1^p + k_2^p - s^p\right]^{\frac{1}{p}}\right)$$

holds for all $s \in [k_1, k_2]$.

In order to give result involving Hermite-Hadamard-Fejer type inequality we need following lemma.

Lemma 3.2. *Let* $p \in \mathbb{R}\backslash\{0\}$*. If* $g : [k_1, k_2] \subseteq (0, \infty) \to \mathbb{R}$ *is integrable and* p*-symmetric along* $\left[\frac{k_1^p + k_2^p}{2}\right]^{1/p}$*, then*

(i) *for* $p > 0$

$$ {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}(g\circ h)(k_2^p) = {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(g\circ h)(k_1^p) = \frac{1}{2}\left[{}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}(g\circ h)(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(g\circ h)(k_1^p)\right], $$

with $\alpha > 0$ *and where* $h(s) = s^{\frac{1}{p}}$ *for all* $s \in [k_1^p, k_2^p]$;

(ii) *for* $p < 0$

$$ {}^{\beta}_{k_2^p}\mathcal{J}^{\alpha}(g\circ h)(k_1^p) = {}^{\beta}\mathcal{J}^{\alpha}_{k_1^p}(g\circ h)(k_2^p) = \frac{1}{2}\left[{}^{\beta}_{k_2^p}\mathcal{J}^{\alpha}(g\circ h)(k_1^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_1^p}(g\circ h)(k_2^p)\right], $$

with $\alpha > 0$ *and where* $h(s) = s^{\frac{1}{p}}$ *for all* $s \in [k_2^p, k_1^p]$.

Proof. Since g is p-symmetric along $\left[\frac{k_1^p+k_2^p}{2}\right]^{1/p}$, then by definition we have $g(s^{\frac{1}{p}}) = g\left(\left[k_1^p + k_2^p - s\right]^{\frac{1}{p}}\right)$ for all $s \in [k_1^p, k_2^p]$. In the following integral, setting $u = k_1^p + k_2^p - s$ gives

$$ \begin{aligned} {}^{\beta}_{k_1^p}\mathcal{J}^{\alpha} g\circ h(k_2^p) &= \frac{1}{\Gamma(\beta)}\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(u-k_1^p)^{\alpha}}{\alpha}\right)^{\beta-1}(u-k_1^p)^{\alpha-1}g(u^{\frac{1}{p}})du \\ &= \frac{1}{\Gamma(\beta)}\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-s)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-s)^{\alpha-1}g\left(\left[k_1^p+k_2^p-s\right]^{\frac{1}{p}}\right)ds \\ &= \frac{1}{\Gamma(\beta)}\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-s)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-s)^{\alpha-1}g\left(s^{\frac{1}{p}}\right)ds \\ &= {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(g\circ h)(k_1^p). \end{aligned} $$

The proof of (ii) is parallel to (i). □

Remark 3.3. In Lemma 3.2:

1. by allowing $\alpha = 1$, we get Lemma 1 in [13];
2. by allowing $\alpha = 1$ and $p = 1$, we get Lemma 3 in [6].

Corollary 3.4. *Under the assumption of Lemma 3.2:*

1. *if we take* $p = 1$, *we get parallel result for convex function;*
2. *if we take* $p = -1$, *we get parallel result for harmonically convex function.*

Theorem 3.5. *Let* $p \in \mathbb{R}\backslash\{0\}$. *Consider a function* $\varphi : [k_1, k_2] \subset (0,\infty) \to \mathbb{R}$ *is* p*-convex, with* $k_1 < k_2$, *and* $\varphi \in L[k_1, k_2]$. *If* $g : [k_1, k_2] \subseteq (0,\infty) \to \mathbb{R}$ *is nonnegative, integrable, and* p*-symmetric along* $\left[\frac{k_1^p+k_2^p}{2}\right]^{1/p}$, *then*

(i) *for* $p > 0$

$$ \begin{aligned} &\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{1/p}\right)\left[{}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}(g\circ h)(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(g\circ h)(k_1^p)\right] \\ &\leqslant \left[{}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}(\varphi g\circ h)(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(\varphi g\circ h)(k_1^p)\right] \\ &\leqslant \frac{\varphi(k_1)+\varphi(k_2)}{2}\left[{}^{\beta}_{k_1^p}\mathcal{J}^{\alpha}(g\circ h)(k_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k_2^p}(g\circ h)(k_1^p)\right], \end{aligned} \tag{3.1} $$

with $\alpha > 0$ *and* $h(s) = s^{\frac{1}{p}}$ *for all* $s \in [k_1^p, k_2^p]$;

(ii) *for* $p<0$

$$\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{1/p}\right)\left[{}^{\beta}_{k_2^p}\mathcal{J}^{\alpha}(g\circ h)(k_1^p)+{}^{\beta}\mathcal{J}^{\alpha}_{k_1^p}(g\circ h)(k_2^p)\right]$$
$$\leqslant\left[{}^{\beta}_{k_2^p}\mathcal{J}^{\alpha}(\varphi g\circ h)(k_1^p)+{}^{\beta}\mathcal{J}^{\alpha}_{k_1^p}(\varphi g\circ h)(k_2^p)\right]\leqslant\frac{\varphi(k_1)+\varphi(k_2)}{2}\left[{}^{\beta}_{k_2^p}\mathcal{J}^{\alpha}(g\circ h)(k_1^p)+{}^{\beta}\mathcal{J}^{\alpha}_{k_1^p}(g\circ h)(k_2^p)\right],$$

with $\alpha>0$ *and* $h(s)=s^{\frac{1}{p}}$ *for all* $s\in[k_2^p,k_1^p]$.

Proof. Since φ is p-convex on $[k_1,k_2]$, we have

$$\varphi\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right)\leqslant\frac{\varphi(x)+\varphi(y)}{2}.$$

Taking $x^p=\tau k_1^p+(1-\tau)k_2^p$ and $y^p=(1-\tau)k_1^p+\tau k_2^p$ with $\tau\in[0,1]$, we get

$$\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{\frac{1}{p}}\right)\leqslant\frac{\varphi\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)+\varphi\left(\left[(1-\tau)k_1^p+\tau k_2^p\right]^{\frac{1}{p}}\right)}{2}. \tag{3.2}$$

Multiplying (3.2) by $\left(\frac{1-\tau^{\alpha}}{\alpha}\right)^{\beta-1}\tau^{\alpha-1}g\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)$ on both sides with $\tau\in(0,1)$, $\alpha>0$, and then integrating along τ over $\in[0,1]$, we obtain

$$\begin{aligned}
&2\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1\left(\frac{1-\tau^{\alpha}}{\alpha}\right)^{\beta-1}\tau^{\alpha-1}g\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)d\tau\\
&\leqslant\int_0^1\left(\frac{1-\tau^{\alpha}}{\alpha}\right)^{\beta-1}\tau^{\alpha-1}\varphi\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)g\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)d\tau\\
&\quad+\int_0^1\left(\frac{1-\tau^{\alpha}}{\alpha}\right)^{\beta-1}\tau^{\alpha-1}\varphi\left(\left[(1-\tau)k_1^p+\tau k_2^p\right]^{\frac{1}{p}}\right)g\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)d\tau.
\end{aligned}$$

Since g is nonnegative, integrable, and p-symmetric about $\left[\frac{k_1^p+k_2^p}{2}\right]^{1/p}$, then

$$g\left(\left[\tau k_1^p+(1-\tau)k_2^p\right]^{\frac{1}{p}}\right)=g\left(\left[\tau k_2^p+(1-\tau)k_1^p\right]^{\frac{1}{p}}\right).$$

Also, by choosing $u=\tau k_1^p+(1-\tau)k_2^p$

$$\begin{aligned}
&\frac{2}{(k_2^p-k_1^p)^{\alpha\beta}}\varphi\left(\left[\frac{k_1^p+k_2^p}{2}\right]^{\frac{1}{p}}\right)\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-u)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-u)^{\alpha-1}g\left(u^{\frac{1}{p}}\right)du\\
&\leqslant\frac{1}{(k_2^p-k_1^p)^{\alpha\beta}}\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-u)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-u)^{\alpha-1}\varphi\left(u^{\frac{1}{p}}\right)g\left(u^{\frac{1}{p}}\right)du\\
&\quad+\frac{1}{(k_2^p-k_1^p)^{\alpha\beta}}\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-u)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-u)^{\alpha-1}\varphi\left(\left[k_1^p+k_2^p-u\right]^{\frac{1}{p}}\right)g\left(u^{\frac{1}{p}}\right)du\\
&=\frac{1}{(k_2^p-k_1^p)^{\alpha\beta}}\left[\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(k_2^p-u)^{\alpha}}{\alpha}\right)^{\beta-1}(k_2^p-u)^{\alpha-1}\varphi\left(u^{\frac{1}{p}}\right)g\left(u^{\frac{1}{p}}\right)du\right.\\
&\quad\left.+\int_{k_1^p}^{k_2^p}\left(\frac{(k_2^p-k_1^p)^{\alpha}-(u-k_1^p)^{\alpha}}{\alpha}\right)^{\beta-1}(u-k_1^p)^{\alpha-1}\varphi\left(u^{\frac{1}{p}}\right)g\left(\left[k_1^p+k_2^p-u\right]^{\frac{1}{p}}\right)\right]du
\end{aligned}$$

$$= \frac{1}{(k_2^p - k_1^p)^{\alpha\beta}} \bigg[\int_{k_1^p}^{k_2^p} \left(\frac{(k_2^p - k_1^p)^\alpha - (k_2^p - u)^\alpha}{\alpha} \right)^{\beta-1} (k_2^p - u)^{\alpha-1} \varphi\left(u^{\frac{1}{p}}\right) g\left(u^{\frac{1}{p}}\right) du$$
$$+ \int_{k_1^p}^{k_2^p} \left(\frac{(k_2^p - k_1^p)^\alpha - (u - k_1^p)^\alpha}{\alpha} \right)^{\beta-1} (u - k_1^p)^{\alpha-1} \varphi\left(u^{\frac{1}{p}}\right) g\left(u^{\frac{1}{p}}\right) \bigg] du.$$

Thus by Lemma 3.2, we have

$$\varphi\left(\left[\frac{k_1^p + k_2^p}{2} \right]^{\frac{1}{p}} \right) \left[{}^{\beta}_{k_1^p}\mathcal{J}^\alpha (g \circ h)(k_2^p) + {}^{\beta}\mathcal{J}^\alpha_{k_2^p} (g \circ h)(k_1^p) \right] \leqslant \left[{}^{\beta}_{k_1^p}\mathcal{J}^\alpha (\varphi g \circ h)(k_2^p) + {}^{\beta}\mathcal{J}^\alpha_{k_2^p} (\varphi g \circ h)(k_1^p) \right].$$

This completes the first inequality of (3.1). For the second inequality, as φ is p-convex, then we have

$$\varphi\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right) + \varphi\left(\left[\tau k_2^p + (1-\tau) k_1^p \right]^{\frac{1}{p}} \right) \leqslant [\varphi(k_1) + \varphi(k_2)]. \tag{3.3}$$

Multiplying (3.3) by $\left(\frac{1-\tau^\alpha}{\alpha}\right)^{\beta-1} \tau^{\alpha-1} g\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right)$ on both sides with $\tau \in (0,1)$, $\alpha > 0$, and then integrating along τ over $\in [0,1]$, we obtain

$$\int_0^1 \left(\frac{1-\tau^\alpha}{\alpha} \right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right) g\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right) d\tau$$
$$+ \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha} \right)^{\beta-1} \tau^{\alpha-1} \varphi\left(\left[(1-\tau) k_1^p + \tau k_2^p \right]^{\frac{1}{p}} \right) g\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right) d\tau$$
$$\leqslant [\varphi(k_1) + \varphi(k_2)] \int_0^1 \left(\frac{1-\tau^\alpha}{\alpha} \right)^{\beta-1} \tau^{\alpha-1} g\left(\left[\tau k_1^p + (1-\tau) k_2^p \right]^{\frac{1}{p}} \right) d\tau.$$

That is

$$\frac{1}{(k_2^p - k_1^p)^{\alpha\beta}} \left[{}^{\beta}_{k_1^p}\mathcal{J}^\alpha (\varphi g \circ h)(k_2^p) + {}^{\beta}\mathcal{J}^\alpha_{k_2^p} (\varphi g \circ h)(k_1^p) \right]$$
$$\leqslant \frac{1}{(k_2^p - k_1^p)^{\alpha\beta}} \left[{}^{\beta}_{a^p}\mathcal{J}^\alpha (g \circ h)(k_2^p) + {}^{\beta}\mathcal{J}^\alpha_{k_2^p} (g \circ h)(k_1^p) \right] \left[\frac{\varphi(k_1) + \varphi(k_2)}{2} \right].$$

Hence we have the proof. □

Remark 3.6. In Theorem 3.5:

1. by allowing $\alpha = 1$, we get Theorem 9 in [13];
2. by allowing $\alpha = 1$ and $p = 1$, we get Theorem 4 in [6].

Corollary 3.7. *Under parallel conditions of Theorem 3.5*

1. *if we take* $p = 1$, *we get*

$$\varphi\left(\frac{k_1 + k_2}{2} \right) \left[{}^{\beta}_{k_1}\mathcal{J}^\alpha g(k_2) + {}^{\beta}\mathcal{J}^\alpha_{k_2} g(k_1) \right] \leqslant \left[{}^{\beta}_{k_1}\mathcal{J}^\alpha \varphi g(k_2) + {}^{\beta}\mathcal{J}^\alpha_{k_2} \varphi g(k_1) \right]$$
$$\leqslant \frac{\varphi(k_1) + \varphi(k_2)}{2} \left[{}^{\beta}_{k_1}\mathcal{J}^\alpha g(k_2) + {}^{\beta}\mathcal{J}^\alpha_{k_2} g(k_1) \right];$$

2. *if we take* $p = -1$, *we get*

$$\varphi\left(\frac{k_1 + k_2}{2k_1k_2} \right) \left[{}^{\beta}_{1/k_2}\mathcal{J}^\alpha (g \circ h)\left(\frac{1}{k_1}\right) + {}^{\beta}\mathcal{J}^\alpha_{1/k_1} (g \circ h)\left(\frac{1}{k_2}\right) \right]$$
$$\leqslant \left[{}^{\beta}_{1/k_2}\mathcal{J}^\alpha (\varphi g \circ h)\left(\frac{1}{k_1}\right) + {}^{\beta}\mathcal{J}^\alpha_{1/k_1} (\varphi g \circ h)\left(\frac{1}{k_2}\right) \right]$$
$$\leqslant \frac{\varphi(k_1) + \varphi(k_2)}{2} \left[{}^{\beta}_{1/k_2}\mathcal{J}^\alpha (g \circ h)\left(\frac{1}{k_1}\right) + {}^{\beta}\mathcal{J}^\alpha_{1/k_1} (g \circ h)\left(\frac{1}{k_2}\right) \right],$$

where $h(s) = 1/s$ *for all* $s \in \left[\frac{1}{k_2}, \frac{1}{k_1}\right]$.

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