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Geometric meaning of conformable derivative via fractional cords

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Abstract

In this paper, we answer the question that many researchers did ask us about: "what is the geometrical meaning of the conformable derivative?". We answer the question using the concept of fractional cords. Fractional orthogonal trajectories are also introduced. Some examples illustrating the concepts of fractional cords and fractional orthogonal trajectories are given.

Keywords: Fractional derivatives, fractional cords, orthogonal fractional trajectory.

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1. Introduction

There are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [6].

Definition 1.1 (Riemann-Liouville definition)**.** For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Definition 1.2 (Caputo definition)**.** For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Such definitions have many setbacks such as the following.

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(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0$ ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:
$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:
$$D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:
$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}(g(t))g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assume that the function f is differentiable.

We refer the reader to [6] for more results on Caputo and Riemann-Liouville Definitions.

Recently, the authors in [5], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using such definition. We refer to [1–5] and references there in for recent results on conformable fractional derivative. The definition goes as follows.

Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$, then for all $t > 0$, $\alpha \in (0, 1)$, let
$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

T_α is called the conformable fractional derivative of f of order α.

Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$. If f is α-differentiable in some $(0, b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define
$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, see [5].

1. $T_\alpha(1) = 0$;
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$;
3. $T_\alpha(\sin at) = at^{1-\alpha} \cos at, \quad a \in \mathbb{R}$;
4. $T_\alpha(\cos at) = -at^{1-\alpha} \sin at, \quad a \in \mathbb{R}$;
5. $T_\alpha(e^{at}) = at^{1-\alpha} e^{at}, \quad a \in \mathbb{R}$.

Further, many functions behave as in the usual derivative. Here are some formulas:

1. $T_\alpha(\frac{1}{\alpha} t^\alpha) = 1$;
2. $T_\alpha(e^{\frac{1}{\alpha} t^\alpha}) = e^{\frac{1}{\alpha} t^\alpha}$;
3. $T_\alpha(\sin \frac{1}{\alpha} t^\alpha) = \cos(\frac{1}{\alpha} t^\alpha)$;
4. $T_\alpha(\cos \frac{1}{\alpha} t^\alpha) = -\sin(\frac{1}{\alpha} t^\alpha)$.

2. Fractional cords

Let $f(x, y) = c$ be an equation that represents some curve in the xy-plane with $x > 0$. For a point (x_0, y_0) on the curve, the equation $\frac{y-y_0}{x-x_0} = y'(x_0)$ represents the equation of the tangent line to the curve $f(x, y) = c$ at the point (x_0, y_0).

Definition 2.1. Let $f(x, y) = c$ be an equation that represents some curve in the xy-plane with $x > 0$. The equation

$$\frac{y^\alpha - y_0^\alpha}{x^\alpha - x_0^\alpha} = \frac{y_0^{\alpha-1}}{x_0^{\alpha-1}} \, y^{(\alpha)}(x_0)$$

represents a curve passing through the point (x_0, y_0). Such a curve will be called fractional cord of the curve $f(x, y) = c$ at the point (x_0, y_0).

Remark 2.2. If $\alpha = 1$, then the fractional cord equation is $\frac{y-y_0}{x-x_0} = y'(x_0)$ which is exactly the tangent line to the curve at (x_0, y_0).

Thus fractional cords represent deviation curves from the tangent line, in the sense

$$\lim_{\alpha \to 1} \frac{y^\alpha - y_0^\alpha}{x^\alpha - x_0^\alpha} = \lim_{\alpha \to 1} \frac{y_0^{\alpha-1}}{x_0^{\alpha-1}} \, y^{(\alpha)}(x_0),$$

which means

$$\frac{y - y_0}{x - x_0} = y'(x_0).$$

What is more interesting is, geometrical meaning of the conformable fractional derivative.

Theorem 2.3. *The conformable fractional derivative $y^{(\alpha)}(x_0)$ of the function $y(x)$ in the equation $f(x, y) = c$, is the slope of the tangent line to the fractional cords associated with the curve $f(x, y) = c$ at (x_0, y_0).*

Indeed, if we differentiate the equation of the fractional cord

$$\frac{y^\alpha - y_0^\alpha}{x^\alpha - x_0^\alpha} = \frac{y_0^{\alpha-1}}{x_0^{\alpha-1}} y^{(\alpha)}(x_0)$$

with respect to x at (x_0, y_0), we get $y'(x_0) = y^{(\alpha)}(x_0)$.

Example 2.4. Consider $f(x, y) = y - x^2 = 0$.

(i) The fractional cord for $\alpha = 1/2$ at $(2, 4)$ is:

$$\frac{\sqrt{y} - 2}{\sqrt{x} - \sqrt{2}} = \frac{\sqrt{2}}{2} \, y^{(\frac{1}{2})}(2) = \frac{\sqrt{2}}{2} (4\sqrt{2}) = 4,$$

i.e.,

$$\sqrt{y} - 2 = 4(\sqrt{x} - \sqrt{2}). \tag{2.1}$$

The slope of the tangent to the fractional cord (2.1) is $y'(2) = 4\sqrt{2}$. This is $y^{(\frac{1}{2})}(2)$ for the original curve $y = x^2$, see Figure 1, where the fractional cord is the blue one.

(ii) The fractional cord for $\alpha = 1/3$ at $(2, 4)$ is

$$\frac{\sqrt[3]{y} - \sqrt[3]{4}}{\sqrt[3]{x} - \sqrt[3]{2}} = \frac{\sqrt[3]{2}}{\sqrt[3]{4}} \left(2y^{(\frac{1}{3})}(2)\right) = 4.$$

The slope of the tangent line to the fractional cord is $y'(2) = 4\sqrt[3]{4}$ which is exactly $y^{(\frac{1}{3})}(2)$ for the curve

$y = x^2$, see Figure 2, where the fractional cord is the blue one.

3. Fractional orthogonal curves

Let $f(x, y) = \lambda$, $g(x, y) = \eta$ be two families of curves. We say $f(x, y) = \lambda$ is α-fractionally orthogonal to $g(x, y) = \eta$ at (x, y) if $y^{(\alpha)}(x)$ at (x, y) for $f(x, y) = \lambda$ is equal to $-\dfrac{1}{y^{(\alpha)}(x)}$ for $g(x, y) = \eta$.

Let

$$\frac{y^\alpha - y_0^\alpha}{x^\alpha - x_0^\alpha} = \frac{y_0^{\alpha-1}}{x_0^{\alpha-1}} y^{(\alpha)}(x_0) \tag{3.1}$$

be the fractional cord at (x_0, y_0) for the curve $f(x, y) = 0$. Then

$$\frac{y^\alpha - y_0^\alpha}{x^\alpha - x_0^\alpha} = \frac{y_0^{\alpha-1}}{x_0^{\alpha-1}} \left(\frac{-1}{y^{(\alpha)}(x_0)}\right) \tag{3.2}$$

is the orthogonal α-fractional cord at (x_0, y_0) for the curve $f(x, y) = 0$. Indeed, the slope of the tangent to (3.1) at (x_0, y_0) is $y^{(\alpha)}(x_0)$ while the slope of the tangent to (3.2) is $\dfrac{-1}{y^{(\alpha)}(x_0)}$.

Example 3.1. Consider the following family of curves $y(x) = c\, e^{-2\sqrt{x}}$. Let us find the $\frac{1}{2}$-orthogonal trajectories at (x, y). Observe that

$$y^{(1/2)} = -c\, e^{-2\sqrt{x}} \Longrightarrow y^{(1/2)} = -y.$$

To find the $\frac{1}{2}$-orthogonal trajectories, we set

$$\frac{-1}{y^{(1/2)}} = -y, \quad \text{or} \quad y^{(1/2)} = \frac{1}{y}.$$

Then $\sqrt{x}\frac{dy}{dx} = \frac{1}{y}$, i.e., $\frac{1}{2}y^2 = 2\sqrt{x} + c$.

Example 3.2. Consider the family of curves $y = x + c$. Then $y^{(1/2)} = \sqrt{x}$. Put $\dfrac{-1}{y^{(1/2)}} = \sqrt{x}$. Then $xy' = -1$. Hence $y = -\ln x + C$, $x > 0$. This is the $\frac{1}{2}$-orthogonal trajectories to the family $y = x + c$ at (x, y).

Example 3.3. Consider the curve Γ given by $y = \sin x$. Now $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \in \Gamma$. Take $0 < \alpha \leqslant 1$. Then

$$y^{(\alpha)}\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^{1-\alpha} \frac{1}{\sqrt{2}}.$$

Since the fractional cord of Γ at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is given by

$$\frac{y^\alpha - (\frac{1}{\sqrt{2}})^\alpha}{x^\alpha - (\frac{\pi}{4})^\alpha} = \frac{1}{\sqrt{2}} \left(\frac{\pi}{4}\right)^{1-\alpha} \left(\frac{4}{\sqrt{2}\,\pi}\right)^{1-\alpha},$$

the orthogonal fractional cord at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is:

$$\frac{y^\alpha - (\frac{1}{\sqrt{2}})^\alpha}{x^\alpha - (\frac{\pi}{4})^\alpha} = \sqrt{2} \left(\frac{\pi}{4}\right)^{1-\alpha} \left(\frac{4}{\sqrt{2}\,\pi}\right)^{1-\alpha}.$$

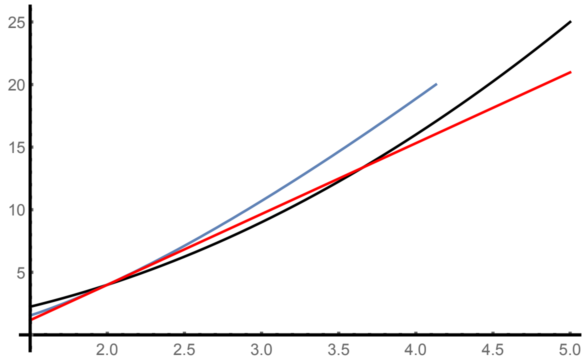

Figure 1: $(x_0, y_0) = (2, 4)$, $\alpha = 1/2$.

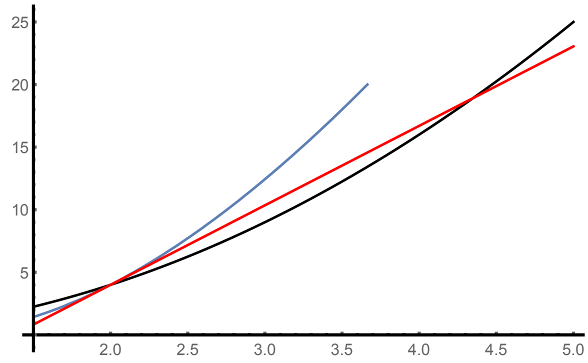

Figure 2: $(x_0, y_0) = (2, 4)$, $\alpha = 1/3$.

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