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# Some Results on the Generalized Rough Lie Subalgebras

S. B. Hosseini[1,a]; A. Kazemi [b]

Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran.

E-mail address: [a] sbhosseini@iausari.ac.ir ; [b] Arefe.kazemi65@gmail.com

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***Abstract***

The main purpose of this paper is to introduce and discuss the concept of *T*-roughness in Lie subalgebra and generalized *T*-rough Lie subalgebras. We define a set-valued homomorphism on a Lie algebra and study some of their properties and useful applications.

**Keywords:** Lower approximation; Upper approximation; *T*-rough set; Set-valued homomorphism; Lie algebras.

## 1. Introduction

The notion of rough sets has been introduced by Z. Pawlak [11, 12], Z. Pawlak and A. Skowron [13] and T. Iwinski [8]. It soon invoked a natural question concerning a possible connection between rough sets and algebraic systems. The algebraic approach to rough sets have been studied by Z. Bonikowaski [2]. R. Biswas and S. Nanda [1] introduced the notion of rough subgroups. N. Kuroki [10] introduced the notion of rough ideals in a semigroup. B. Davvaz [4] introduced the notion of rough subring with respect to an ideal of a ring. O. Kazanci, B. Davvaz [9] discussed the structure on the rough prime (primary) ideals. In [15], W. Zhang, W. Wu considered some other results. B. Davvaz [3] introduced *T*- rough set and *T*- rough homomorphism in a group. In [14], S. Yamak, O. Kazanci, B. Davvaz introduced the generalized lower and upper approximation in a ring. S. B. Hosseini et al. [6, 7] introduced *T*-rough ideal in a semigroup and in a commutative ring. The rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and the upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty

[1]Corresponding author.

intersection with the set. The rough sets are a suitable mathematical model of vague concepts, i.e., concepts without sharp boundaries. In this paper, a set-valued homomorphism on a Lie algebra and the concept $T$-rough Lie subalgebra are introduced and some interesting properties are proved. Suppose that $U$ is a non-empty set. A partition or classification of $U$ is a family $\Theta$ of non-empty subsets of $U$ such that each element of $U$ is contained in exactly one element of $\Theta$. It is vitally important to recall that an equivalence relation $\theta$ on a set $U$ is a reflexive, symmetric and transitive binary relation on $U$. Each partition induces an equivalence relation on $U$. If $\theta$ is an equivalence relation on $U$, then for every $x \in U, [x]_\theta$ denotes the equivalence class of $\theta$ determined by $x$. For any $X \subseteq U$, we write $X^c$ to denote the complement of $X$ in $U$ that is the set $U \setminus X$. A pair $(U,\theta)$ where $U \neq \varnothing$ and $\theta$ is an equivalence relation on $U$ is called an approximation space. Let $P(U)$ be the set of all subsets of $U$ and for an approximation space $(U,\theta)$ by a rough approximation in $(U,\theta)$ we mean a mapping

$$Apr : P(U) \to P(U) \times P(U) \text{ defined by for every } X \in P(U),\ Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$$

where

$$\underline{Apr}(X) = \{x \in U \mid [x]_\theta \subseteq X\} \ ; \ \overline{Apr}(X) = \{x \in U \mid [x]_\theta \cap X \neq \varnothing\}.$$

$\underline{Apr}(X)$ is called the lower rough approximation of $X$ in $(U,\theta)$ whereas $\overline{Apr}(X)$ is called the upper rough approximation of $X$ in $(U,\theta)$.

Given an approximation space $(U,\theta)$ a pair $(A,B)$ in $P(U) \times P(U)$ is called a rough set in $(U,\theta)$ if $(A,B) = (\underline{Apr}(X), \overline{Apr}(X))$ for some $X \in P(U)$.

# 2 Set-valued Lie homomorphism and $T$-rough Lie subalgebra

In this section, we define the concept of a set-valued Lie homomorphism and give some important examples of a set-valued mapping. We also investigate some basic properties of the generalized lower and upper approximation operators in a Lie algebra. We generalize the rough Lie subalgebra called $T$-rough Lie subalgebra. We apply the notion of $T$-rough sets in a Lie algebra and extend some theorems which have been proved in [3, 4, 6, 7].

Throughout in this section and the next, the set of all non-empty subsets of $Y$ is denoted by $P^*(Y)$.

**Definition 2.1** [3] *Let $X$ and $Y$ be two non-empty sets and $\varnothing \neq B \subseteq Y$. Let $T : X \to P^*(Y)$ be a set-valued mapping. The lower inverse and upper inverse of $B$ under $T$ are defined by*

$$L_T(B) = \{x \in X \mid T(x) \subseteq B\} \ ; \ U_T(B) = \{x \in X \mid T(x) \cap B \neq \varnothing\},$$

respectively.

**Definition 2.2** [3] *Let $X$ and $Y$ be two non-empty sets and $B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping. $(L_T(B), U_T(B))$ is called a $T$ - rough set with respect to $B$.*

**Proposition 2.3** [3,6,7] *Let $X$ and $Y$ be two non-empty sets and $A, B \subseteq Y$. Let $T : X \to P^*(Y)$ be a set-valued mapping, then the following holds:*

(*i*) $U_T(A \bigcup B) = U_T(A) \bigcup U_T(B)$;

(*ii*) $L_T(A \bigcap B) = L_T(A) \bigcap L_T(B)$;

(*iii*) $A \subseteq B$ *implies* $L_T(A) \subseteq L_T(B)$ *and* $U_T(A) \subseteq U_T(B)$;

(*iv*) $L_T(A) \bigcup L_T(B) \subseteq L_T(A \bigcup B)$ *and* $U_T(A \bigcap B) \subseteq U_T(A) \bigcap U_T(B)$.

**Example 2.4** (*i*) *Let $(U, \theta)$ be an approximation space and $T : U \to P^*(U)$ be a set-valued mapping where $T(x) = [x]_\theta$, then for any $B \subseteq U$, $L_T(B) = \underline{Apr}(B)$ and $U_T(B) = \overline{Apr}(B)$. So, rough sets are $T$-rough sets. In fact, $T$-rough sets are a generalization of rough sets.*

(*ii*) Let $\mathbb{Z}$ be integer numbers set and $T : \mathbb{Z} \to P^*(\mathbb{Z})$ be a set-valued mapping where $T(n) = n\mathbb{Z}$ for all $n \in \mathbb{Z}$. If $A = 2\mathbb{Z}$, then $L_T(A) = 2\mathbb{Z}$ and $U_T(A) = \mathbb{Z}$.

**Definition 2.5** *Let $F$ be a field. A Lie algebra over $F$ is an $F$-vector space $L$, together with a bilinear map, the Lie bracket*

$$L \times L \to L, \quad (x, y) \to [x, y],$$

satisfying the following properties:

- $[x, x] = 0$ *for all* $x \in L$; $\qquad (L1)$
- $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ *for all* $x, y, z \in L$. $\qquad (L2)$

The Lie bracket $[x, y]$ is often referred to as the *commutator* of $x$ and $y$.

Condition $(L2)$ is known as the *Jacobi identity*.

**Definition 2.6** (*i*) *If $L$ is a Lie algebra. We defined a Lie subalgebra of $L$ to be a vector subspace $\emptyset \neq K \subseteq L$ such that*

$$[x, y] \in K \quad \text{for all } x, y \in K.$$

(*ii*) If $A$ and $B$ be two Lie subalgebras of $L$ then we define $[A, B]$ as follows:

$$[A,B] = span\{[a,b] \mid a \in A, b \in B\} = \{\sum_{i=1}^{n} \lambda_i [a_i, b_i] \mid \lambda_i \in F, a_i \in A, b_i \in B, n \in N\}.$$

(*iii*) A subspace $I$ of a Lie algebra $L$ is called an ideal if

$$[x, y] \in I \quad \text{for all } x \in L, y \in I.$$

($iv$) A Lie algebra $L$ is called commutative when $[x,y]=0$ for all $x,y\in L$.

($v$) If $L_1$ and $L_2$ are Lie algebras over $F$, then we say that a map $\varphi: L_1 \to L_2$ is a Lie homomorphism if $\varphi$ is a linear map and

$$\varphi([x,y])=[\varphi(x),\varphi(y)] \quad \text{forall}\, x,y\in L_1.$$

**Definition 2.7** *Let $L$ and $L^{'}$ be two Lie algebras over field $F$ and $T: L\to P^*(L^{'})$ be a set-valued mapping. $T$ is called a set-valued Lie homomorphism if*

- $T(x+y)=T(x)+T(y)$;
- $T(\lambda x)=\lambda T(x)$;
- $\{[a,b] \mid a\in T(x), b\in T(y)\}=T[x,y]$.

for all $\lambda\in F$ and $x,y\in L$.

It is clear that $T(0)=\{0\}$ and $T(-x)=-T(x)$ for all $x\in L$.

**Example 2.8** ($i$) *Let $L$ is a Lie algebra and $T: L\to P^*(\frac{L}{I})$ be a set-valued mapping and $T(x)=\{x+I\}$ for all $x\in L$ which $I$ is an ideal of $L$. Then $T$ is a set-valued Lie homomorphism. Here, $\frac{L}{I}$ is a Lie algebra over $F$ and its the Lie bracket defined by*

$$[x+I, y+I]=[x,y]+I \ \text{ for all}\, x,y\in L.$$

$\frac{L}{I}$ is called the quotient algebra of $L$ by $I$.

($ii$) Let $f: L\to L^{'}$ be a Lie algebra homomorphism and $T: L\to P^*(L^{'})$ defined by $T(x)=\{f(x)\}$, then $T$ is a set-valued Lie homomorphism.

($iii$) Let $T: L\to P^*(L^{'})$ be a set-valued mapping such that $T(x)=\{0\}$, then $T$ is a set-valued Lie homomorphism.

The following corollaries are clear.

**Corollary 2.9** *Let $T: L\to P^*(L^{'})$ be a set-valued Lie homomorphism and $f: L^{''}\to L$ be a Lie homomorphism, then $Tof$ is a set-valued Lie homomorphism from $L^{''}$ to $P^*(L^{'})$ such that $U_{Tof}(B)=f^{-1}(U_T(B))$ and $L_{Tof}(B)=f^{-1}(L_T(B))$ for all $B\in P^*(L^{'})$.*

**Corollary 2.10** *Let $T: L\to P^*(L^{'})$ be a set-valued Lie homomorphism and $f: L^{'}\to L^{''}$ be a Lie homomorphism, then $T_f$ is a set-valued Lie homomorphism from $L$ to $P^*(L^{''})$ defined by $T_f(m)=f(T(m))$ such that $L_{T_f}(A)=L_T(f^{-1}(A))$ and $U_{T_f}(A)=U_T(f^{-1}(A))$ for all $A\in P^*(L^{''})$ and $m\in L$.*

**Lemma 2.11** *Let $W$ be an ideal $L^{'}$ and $T: L \to P^{*}(L^{'})$ be a set-valued Lie homomorphism, then $L_T(W)$ and $U_T(W)$ are ideals of $L$.*

**Proof.** It is clear that $0 \in L_T(W)$, therefore $L_T(W) \neq \varnothing$. Now we show that $L_T(W)$ is a subspace. Let $x, y \in L_T(W)$ and $\lambda \in F$, so $T(x) \subseteq W$ and $T(y) \subseteq W$. Therefore $\lambda T(x) \subseteq W$, hence $\lambda T(x) + T(y) \subseteq W$. Since $T$ is a Lie homomorphism, so $T(\lambda x) + T(y) \subseteq W$. It yields that $T(\lambda x + y) \subseteq W$. It shows that $\lambda x + y \in L_T(W)$. Now to show that $L_T(W)$ is an ideal, we need to check that $[x, y] \in L_T(W)$ for all $x \in L_T(W)$ and $y \in L$, or $T([x, y]) \subseteq W$. In fact we show that $\{[a,b] \mid a \in T(x), b \in T(y)\} \subseteq W$. Let $[u,v] \in \{[a,b] \mid a \in T(x), b \in T(y)\}$ since $u \in T(x) \subseteq W$, and $W$ is an ideal, then $[u,v] \in W$.

A little change to above proving, we can obtain the next result.

**Lemma 2.12** *Let $T: L \to P^{*}(L^{'})$ be a set-valued Lie homomorphism, then $kerT = \{x \in L \mid T(x) = \{0\}\}$ is an ideal of $L$.*

**Corollary 2.13** *Let $T: L \to P^{*}(L^{'})$ be a set-valued Lie homomorphism. If $A$ and $B$ be two non-empty sets of $L^{'}$, then $[U_T(A), U_T(B)] \subseteq U_T([A,B])$ and $[L_T(A), L_T(B)] \subseteq L_T([A,B])$.*

The following example shows that in general, the converse of above relations do not hold:

**Example 2.14** *Let $L$ be a commutative Lie algebra and $A = L = B$ and $T: L \to P^{*}(L)$ defined by $T(x) = \{0\}$, then we have*

$(i)$ $L_T[A,B] = L_T[L,L] = L_T\{0\} = \{x \in L \mid T(x) \subseteq \{0\}\} = L$. On the other hand, $L_T(L) = \{x \in L \mid T(x) \subseteq L\} = L$, hence $[L_T(L), L_T(L)] = [L,L] = 0$.

$(ii)$ $U_T(L) = \{x \in L \mid T(x) \cap L \neq \varnothing\} = L$, hence $[U_T(L), U_T(L)] = [L,L] = 0$. On the other hand, $U_T[L,L] = U_T\{0\} = \{x \in L \mid T(x) \cap \{0\} \neq \varnothing\} = L.$

**Corollary 2.15** *Let $T: L \to P^{*}(L^{'})$ be a set-valued Lie homomorphism. If $A$ is a Lie subalgebra of $L^{'}$, then $U_T(A)$ and $L_T(A)$ are Lie subalgebras of $L$.*

**Proof.** The proof is straightforward.

**Theorem 2.16** *Let $f: L \to L^{'}$ be an isomorphism and $T_2: L^{'} \to P^{*}(L^{'})$ be a set-valued Lie homomorphism. If $T_1(x) = \{u \in L \mid f(u) \in T_2(f(x))\}$ for all $x \in L$, then $T_1$ is a set-valued Lie homomorphism from $L$ to $P^{*}(L)$.*

**Proof.** First, we show that $T_1$ is a well-defined mapping. Suppose $x_1 = x_2$, we have

$$y_1 \in T_1(x_1) \Leftrightarrow f(y_1) \in T_2(f(x_1)) = T_2(f(x_2)) \Leftrightarrow y_1 \in T_1(x_2).$$

Then $T_1(x_1)=T_1(x_2)$. Now we show that $T_1(x_1+x_2)=T_1(x_1)+T_1(x_2)$.
Suppose $y\in T_1(x_1+x_2)$, then

$$f(y)\in T_2(f(x_1+x_2))=T_2(f(x_1)+f(x_2))=T_2(f(x_1))+T_2(f(x_2)).$$

Hence there exist $a\in T_2(f((x_1))$ and $b\in T_2(f((x_2))$ such that $f(y)=a+b$. Since $f$ is onto, then there exist $d,c\in L$ such that $f(c)=a,\ f(d)=b$. On the other hand, we have $f(c)\in T_2(f(x_1))$, then $c\in T_1(x_1)$ and also $f(d)\in T_2(f(x_2))$. Therefore $d\in T_1(x_2)$ and $f(y)=a+b=f(c)+f(d)=f(c+d)$. Since $f$ is one to one, it implies $y=c+d$. So $y\in T_1(x_1)+T_1(x_2)$. It follows $T_1(x_1+x_2)\subseteq T_1(x_1)+T_1(x_2)$.

Conversely, assume that $y\in T_1(x_1)+T_1(x_2)$, then there are $a\in T_1(x_1), b\in T_1(x_2)$ such that $y=a+b$. Hence

$$f(y)=f(a)+f(b)=f(a+b)\in T_2(f(x_1))+T_2(f(x_2))=T_2(f(x_1+x_2)).$$
$$\Rightarrow y\in T_1(x_1+x_2).$$

So $T_1(x_1)+T_1(x_2)\subseteq T_1(x_1+x_2)$. Also we show that $T_1(\lambda x)=\lambda T_1(x)$. Suppose $u\in T_1(\lambda x)$. So $f(u)\in T_2(f(\lambda x))=\lambda T_2(f(x))$. Then there exists $z\in T_2(f(x))$ such that $f(u)=\lambda z$. Since $f$ is onto, then there is $m\in L_1$ such that $z=f(m)$. Therefore we have $f(u)=\lambda f(m)=f(\lambda m)$ and since $f$ is one to one, it implies that $u=\lambda m$. So $z=f(m)\in T_2(f(x))$. It shows that $m\in T_1(x)$. Therefore $\lambda m\in\lambda T_1(x)$. Then $u\in\lambda T_1(x)$. Now for proving $\lambda T_1(x)\subseteq T_1(\lambda x)$, let $u\in\lambda T_1(x)$. By definition, there exists $z\in T_1(x)$ such that $u=\lambda z$. So $f(u)=\lambda f(z)$. Since $f(z)\in T_2(f(x))$ and $f(u)\in\lambda T_2(f(x))=T_2(f(\lambda x))$, hence $f(u)\in T_2(f(\lambda x))\Rightarrow u\in T_1(\lambda x)$.

Now we show that $T_1$ preserves the Lie bracket, that means

$$T_1[x,y]=\{[a,b]\mid a\in T_1(x), b\in T_1(y)\}.$$

First, we show that $f[a,b]\in T_2(f[x,y])$ for all $a\in T_1(x)$ and $b\in T_1(y)$. Let $f(a)\in T_2(f(x))$ and $f(b)\in T_2(f(y))$. Since $f$ is a Lie homomorphism, therefore

$$f[a,b]=[f(a),f(b)]\in\{[u,v]\mid u\in T_2(f(x)), v\in T_2(f(y))\}$$
$$=T_2[f(x),f(y)].$$

It implies $\{[a,b]\mid a\in T_1(x), b\in T_1(y)\}\subseteq T_1[x,y]$. Now we show

$$T_1[x,y]\subseteq\{[a,b]\mid a\in T_1(x), b\in T_1(y)\}.$$

We have $T_1[x,y]=\{u\in L\mid f(u)\in T_2(f[x,y])\}$

$$=\{u\in L\mid f(u)\in T_2([f(x),f(y)])\}$$
$$=\{u\in L\mid f(u)\mid\{[a,b]\mid a\in T_2(f(x)), b\in T_2(f(y))\}.$$

Now let $f(u)=[a,b]$ then there exist $a\in T_2(f(x))$ and $b\in T_2(f(y))$. Since $f$ is onto, then there exist $d,c\in L$ such that $f(c)=a$ and $f(d)=b$. It is clear that $c\in T_1(x)$ and $d\in T_1(y)$ and $f(u)=[f(c),f(d)]=f[c,d]$. Since $f$ is one to one, then

$$u=[c,d]\in\{[t_1,t_2]\mid t_1\in T_1(x), t_2\in T_1(y)\}\subseteq\{[a,b]\mid a\in T_1(x), b\in T_1(y)\}.$$

**Theorem 2.17** *Let* $f : L \to L'$ *be a Lie algebra isomorphism and let* $T_2 : L' \to P^*(L')$ *be a set-valued Lie homomorphism. If for any* $x \in L$, $T_1(x) = \{u \in L \mid f(u) \in T_2(f(x))\}$ *and* $A$ *is a subalgebra of* $L'$, *then*

$(i)$ $f(L_{T_1}(A)) = L_{T_2}(f(A))$;

$(ii)$ $f(U_{T_1}(A)) = U_{T_2}(f(A))$.

**Proof.** $(i)$ If $y \in f(L_{T_1}(A))$, then there exists $x \in L_{T_1}(A)$ such that $y = f(x)$. But if $x \in L_{T_1}(A)$, then we have $T_1(x) \subseteq A$. Now if $w \in T_2(f(x))$, since $f$ is onto, then there exists $z \in L$ such that $w = f(z)$. So,

$$\begin{aligned} & w = f(z) \in T_2(f(x)) \Rightarrow z \in T_1(x) \subseteq A \\ & \Rightarrow w = f(z) \in f(A) \\ & \Rightarrow T_2(f(x)) \subseteq f(A) \\ & \Rightarrow y \in L_{T_2}(f(A)). \end{aligned}$$

Therefore $f(L_{T_1}(A)) \subseteq L_{T_2}(f(A))$.

Conversely, if $y \in L_{T_2}(f(A))$, then $T_2(y) \subseteq f(A)$. On the other hand, $f$ is onto, then there is $x \in L$ such that $y = f(x)$. Hence, we have $T_2(f(x)) \subseteq f(A)$.

Let $u \in T_1(x)$, then $f(u) \in f(A)$, therefore there exists $a \in A$ such that $f(u) = f(a)$. But $f$ is one to one, so $u = a$. Hence we have

$$u \in A \Rightarrow T_1(x) \subseteq A \Rightarrow x \in L_{T_1}(A) \Rightarrow y \in f(L_{T_1}(A)).$$

So, $L_{T_2}(f(A)) \subseteq f(L_{T_1}(A))$.

$(ii)$ If $y \in f(U_{T_1}(A))$, then there exists $x \in U_{T_1}(A)$ such that $y = f(x)$. But if $x \in U_{T_1}(A)$, then $T_1(x) \cap A \neq \varnothing$. Let $a \in T_1(x) \cap A$. Therefore

$$\begin{aligned} & f(a) \in T_2(f(x)) \cap f(A) \Rightarrow T_2(f(x)) \cap f(A) \neq \varnothing \\ & \Rightarrow f(x) \in U_{T_2}(f(A)) \\ & \Rightarrow y \in U_{T_2}(f(A)). \end{aligned}$$

It means that $f(U_{T_1}(A)) \subseteq U_{T_2}(f(A))$. Conversely, if $y \in U_{T_2}(f(A))$, since $f$ is onto, then there exist $x \in L$ such that $y = f(x)$, and $T_2(y) \cap f(A) \neq \varnothing$. So, we have $T_2(f(x)) \cap f(A) \neq \varnothing$. Hence there is $z \in T_2(f(x)) \cap f(A)$. It means that there exists $a \in A$ such that $z = f(a) \in T_2(f(x))$. Then $a \in T_1(x) \cap A \neq \varnothing$. It obtains that $x \in U_{T_1}(A)$. Then $y = f(x) \in f(U_{T_1}(A))$. It follows that $U_{T_2}(f(A)) \subseteq f(U(_{T_1}(A))$.

**Definition 2.18** *A congruence* $\theta$ *on* $L$ *is called complete if for any* $x, y \in L$ *and* $r \in F$

(*i*) $[x]_\theta + [y]_\theta = [x+y]_\theta$;

(*ii*) $r[x]_\theta = [rx]_\theta$;

(iii) $[[x, y]]_\theta = \{[a,b] \mid a \in [x]_\theta, b \in [y]_\theta\}$.

By using the above theorems and definition, we obtain the following:

**Corollary 2.19** *Let $\theta_2$ be a complete congruence relation on Lie algebra $L_2$ and $f: L_1 \to L_2$ be a Lie algebra isomorphism and $\theta_1 = \{(x,y) \in L_1 \times L_1 \mid f(x), f(y) \in \theta_2\}$, then $\theta_1$ is a complete congruence relation on $L_1$ that $\emptyset \neq A \subseteq L_1$*

$(i)$ $f(\underline{Apr}_{\theta_1}(A)) = \underline{Apr}_{\theta_2}(f(A))$;

$(ii)$ $f(\overline{Apr}_{\theta_1}(A)) = \overline{Apr}_{\theta_2}(f(A))$.

# 3 Generalized $T$-rough Lie algebras

In this section, we define a $T$-rough Lie algebra with respect to a Lie subalgebra of a Lie algebra, is called the generalized $T$-rough Lie algebra and study some of their appealing properties.

**Definition 3.1** *Let $A$ and $B$ be two Lie subalgebras of $L'$ and $T: L \to P^*(L')$ be a set-valued Lie homomorphism, then*

$$L_T^A(B) = \{x \in L \mid (T(x) + A) \subseteq B\} \ ; \ U_T^A(B) = \{x \in L \mid (T(x) + A) \cap B \neq \emptyset\}$$

are called the generalized lower and upper approximations of $B$, respectively, with respect to the Lie subalgebra $A$.

In the special case, if $A = 0$, then $L_T^A(B) = L_T(B)$ and $U_T^A(B) = U_T(B)$. Furthermore, if $0 \in A \subseteq B$, then $L_T^A(B) \subseteq L_T(B)$ and $U_T^A(B) \subseteq U_T(B)$.

**Definition 3.2** *Let $L$ be a Lie algebra. If $A$ be a Lie subalgebra of $L$ and $\emptyset \neq S \subseteq F$, then $SA$ denotes the set of $\{\sum_{i=1}^n s_i a_i \mid s_i \in S, a_i \in A, n \in \mathbf{N}\}$.*

**Theorem 3.3** *Let $T: L \to P^*(L')$ be a set-valued Lie homomorphism, the following holds:*

$(i)$ If $A$ and $B$ are two Lie subalgebras of $L'$ such that $A \subseteq B$, then $L_T^A(B)$ is Lie subalgebra of $L$;

$(ii)$ If $A$ is an ideal of $L'$ and $B$ is a Lie subalgebra of $L'$, then $U_T^A(B)$ is Lie subalgebra of $L$.

**Proof.** $(i)$ First, we show that $L_T^A(B)$ is a subspace. It is clear that $0 \in L_T^A(B)$. Now suppose that $x, y$ be two elements of $L_T^A(B)$ and $\lambda \in F$. If $x \in L_T^A(B)$, since $A$ is a Lie subalgebra, hence $0 \in A$ then $T(x) \subseteq B$, so $\lambda T(x) \subseteq B$. On the other hand, $A \subseteq B$. Therefore $\lambda T(x) + A \subseteq B$, and so $T(\lambda x) + A \subseteq B$. Also $T(y) + A \subseteq B$. Hence $T(\lambda x) + T(y) + A \subseteq B$. It deduces that $T(\lambda x + y) + A \subseteq B$. Therefore $\lambda x + y \in L_T^A(B)$. Now we show that if $x, y \in L_T^A(B)$, then $[x, y] \in L_T^A(B)$. Since $T[x,y] = \{[a,b] \mid a \in T(x), b \in T(y)\}$, from $a \in T(x)$ and $b \in T(y)$, we have

$a \in B$ and $b \in B$, therefore $[a,b] \in B$. It means that $\{[a,b] \mid a \in T(x), b \in T(y)\} \subset B$, so $T[x,y] \subset B$. On the other hand, $A \subseteq B$ which implies $T[x,y] + A \subseteq B$. It follows $[x,y] \in L_T^A(B)$.

$(ii)$ First, we show that $U_T^A(B)$ is a subspace. It is clear that $0 \in U_T^A(B)$. Now suppose that $x, y$ be two elements of $U_T^A(B)$ and $\lambda \in F$. If $x \in U_T^A(B) \Rightarrow (T(x)+A) \cap B \neq \varnothing$ and since $y \in U_T^A(B)$, then $(T(y)+A) \cap B \neq \varnothing$, so there exist $a, b \in L'$ such that $a \in (T(x)+A) \cap B$ and $b \in (T(y)+A) \cap B$. It implies that $\lambda a \in T(\lambda x) + \lambda A \subseteq T(\lambda x) + A$. On the other hand, $\lambda a \in \lambda B \subset B$, so $\lambda a \in B$. It deduces that $\lambda a \in (T(\lambda x)+A) \cap B$, since $b \in (T(y)+A) \cap B$, therefore $\lambda a + b \in (T(\lambda x + y)+A) \cap B$. It shows that $(T(\lambda x + y)+A) \cap B \neq \varnothing$. It follows $\lambda x + y \in U_T^A(B)$.

Now we show that if $x, y \in U_T^A(B)$, then $[x,y] \in U_T^A(B)$. Since $x \in U_T^A(B)$, then $(T(x)+A) \cap B \neq \varnothing$, so there exists $u \in (T(x)+A) \cap B$, and since $y \in U_T^A(B)$, then $(T(y)+A) \cap B \neq \varnothing$ so there exists $v \in (T(x)+A) \cap B$. It implies that $[u,v] \in B$. Let $u = x_1 + a_1$ such that $x_1 \in T(x), a_1 \in A$ and $v = x_2 + a_2$ such that $x_2 \in T(y), a_2 \in A$, we have

$$\begin{aligned}
[u,v] &= [x_1 + a_1, x_2 + a_2] \\
&= [x_1, x_2] + [x_1, a_2] + [a_1, x_2] + [a_1, a_2] \in B.
\end{aligned}$$

Since $A$ is ideal of Lie algebra $L'$, so $[x_1, a_2], [a_1, x_2], [a_1, a_2] \in A$. Hence we have

$$\begin{aligned}
\{[x_1, x_2] + A \mid x_1 \in T(x), x_2 \in T(y)\} \subset B &\Rightarrow T[x,y] + A \subset B \\
&\Rightarrow T[x,y] + A \cap B \neq \varnothing \\
&\Rightarrow [x,y] \in U_T^A(B).
\end{aligned}$$

**Notice**: In the above theorem, the condition $A \subseteq B$ is necessary, because $0 \notin L_T^A(B)$.

The following example shows that in condition $(ii)$ to $A$'s being ideal is a necessity.

**Example 3.4** *Let $L_1 = L_2 = gl(n,F)$ be the set of all $n \times n$ matrices over $F$ and $T : gl(n,F) \to P^*(gl(n,F))$ and for any $x \in gl(n,F)$ we define $T(x) = \{x\}$. Now if $A = b(n,F)$ is the upper triangular matrices and $B = \{0\}$, then $A$ is not an ideal of $gl(n,F)$ and $U_T^A(B) = A$. for*

$$\begin{aligned}
U_T^A(B) &= \{x \in L \mid (T(x)+A) \cap B \neq \varnothing\} \\
&= \{x \in L \mid \{x\} + b(n,F) \cap \{0\} \neq \varnothing\} = b(n,F) = A.
\end{aligned}$$

Now we have $T$ is a Lie algebra homomorphism

$(i)$ $T(A+B) = \{A+B\} = \{A\} + \{B\} = T(A) + T(B)$;
$(ii)$ $T(\lambda A) = \{\lambda A\} = \lambda\{A\} = \lambda T(A)$;
$(iii)$ $T[A,B] = \{[A,B]\} = \{[a,b] \mid a \in T(A), b \in T(B)\}$.

**Lemma 3.5** *Let $L$ be a Lie algebra and $A, B$ be non-empty Lie subalgebras of $L'$ such that $A \subseteq B$ and let $S$ be a subspace of $L'$. If $T : L \to P^*(L')$ be a set-valued Lie homomorphism, then*
$(i)$ $L_T^B(S) \subseteq L_T^A(S)$;

$(ii)$ $U_T^A(S) \subseteq U_T^B(S)$.

**Proof.** The proof is straightforward.

The following corollary follows by Lemma 3.5.

**Corollary 3.6** *Let* $A, B$ *be Lie subalgebras of* $L'$ *and* $S$ *be a non-empty subset of* $L'$. *If* $T : L \to P^*(L')$ *be a set-valued Lie homomorphism, then*

$(i)$ $L_T^A(S) \cap L_T^B(S) \subseteq L_T^{A\cap B}(S)$;

$(ii)$ $U_T^{A\cap B}(S) \subseteq U_T^A(S) \cap U_T^B(S)$.

**Theorem 3.7** *Suppose* $S$ *be a non-empty subset of* $F$ *and* $B$ *be Lie subalgebra of* $L'$ *and* $A$ *be a subspace of* $L'$. *If* $T : L \to P^*(L')$ *be a set-valued Lie homomorphism, then*

$(i)$ If $A \subseteq B$, then $SL_T^A(B) \subseteq L_T^A(SB)$;

$(ii)$ $SU_T^A(B) \subseteq U_T^A(SB)$.

**Proof.** $(i)$ Let $x$ be any element of $SL_T^A(B)$, then $x = \Sigma_{i=1}^n s_i b_i$ for some $s_i \in S, b_i \in L_T^A(B)$ and $n \in \mathbf{N}$. Now, we have $T(b_i) + A \subseteq B$, and so $s_i T(b_i) + A \subseteq s_i B \subseteq SB$, for all $i = 1,2,...,n$. Then we have $T(s_i b_i) + A \subseteq SB$ which implies $s_i b_i \in L_T^A(SB)$. Therefore $x = \Sigma_{i=1}^n s_i b_i \in L_T^A(SB)$ and so $SL_T^A(B) \subseteq L_T^A(SB)$.

$(ii)$ Let $x$ be any element of $SU_T^A(B)$, then $x = \Sigma_{i=1}^n s_i b_i$ for some $s_i \in S, b_i \in U_T^A(B)$ and $n \in \mathbf{N}$. Now, we have $(T(b_i) + A) \cap B \neq \varnothing$ for all $i = 1,2,...,n$. So there exists $a_i \in (T(b_i) + A) \cap B$. Hence $s_i a_i \in B$ and $s_i a_i \in s_i T(b_i) + A = T(s_i b_i) + A$. So $\Sigma_{i=1}^n s_i a_i \in SB$ and $\Sigma_{i=1}^n s_i a_i \in T(\Sigma_{i=1}^n s_i b_i) + A$. Therefore $\Sigma_{i=1}^n s_i a_i \in (T(x) + A) \cap SB$. Thus $(T(x) + A) \cap SB \neq \varnothing$ which implies $x \in U_T^A(SB)$, and so $SU_T^A(B) \subseteq U_T^A(SB)$.

**Theorem 3.8** *Suppose that* $A, B$ *and* $C$ *be Lie subalgebras of* $L'$. *If* $T : L \to P^*(L')$ *be a set-valued Lie homomorphism, then*

$(i)$ $L_T^A(C) + L_T^B(C) = L_T^{A+B}(C)$;

$(ii)$ $U_T^A(C) + U_T^B(C) = U_T^{A+B}(C)$.

**Proof.** $(i)$ Since $A \subseteq A + B$ and $B \subseteq A + B$, then by Lemma 3.5,

$$L_T^{A+B}(C) \subseteq L_T^A(C) \subseteq L_T^A(C) + L_T^B(C).$$

Now, let $x \in L_T^A(C) + L_T^B(C)$, then $x = y + z$ for some $y \in L_T^A(C)$ and $z \in L_T^B(C)$. Hence $T(y) + A \subseteq C$ and $T(z) + B \subseteq C$, then $T(y+z) + A + B \subseteq C$ which implies $x \in L_T^{A+B}(C)$. Therefore we obtain $L_T^A(C) + L_T^B(C) = L_T^{A+B}(C)$.

$(ii)$ Since $A \subseteq A + B$ and $B \subseteq A + B$, by Lemma 3.5, $U_T^A(C) \subseteq U_T^{A+B}(C)$ and $U_T^B(C) \subseteq U_T^{A+B}(C)$ and so $U_T^A(C) + U_T^B(C) \subseteq U_T^{A+B}(C)$. Also $U_T^{A+B}(C) \subseteq U_T^A(C) \subseteq U_T^A(C) + U_T^B(C)$.
And the equality in relation $(ii)$ is true when $A + B$ is an ideal. Now we have $U_T^{A+B}(C) \subseteq U_T^A(C) + U_T^B(C)$. Let $x \in U_T^{A+B}(C)$, then $(T(x) + A + B) \cap C \neq \emptyset$, so there exists $u \in T(x) + A + B \cap C$, therefore we have $u = x_1 + a + b$ such that $x_1 \in T(x), a \in A, b \in B$, so $u - b = x_1 + a \in T(x) + A$. Since $A + B$ is an ideal, then $U_T^{A+B}(C)$ is a Lie subalgebra, so $A + B \subset C$. On the other hand, $A \subset A + B \subset C$ and $B \subset A + B \subset C$. Hence $A \subset C$ and $B \subset C$. So

$$(T(x) + A) \cap C \neq \emptyset \Rightarrow x \in U_T^A(C) \subset U_T^A(C) + U_T^B(C).$$

**Proposition 3.9** *Let $L$ be a Lie algebra and $A$ is a Lie subalgebra of $L'$ and $B$ is non-empty subset of $L'$, then*

$(i)$ $L_T^A(B^c) = (U_T^A(B))^c$;

$(ii)$ $U_T^A(B^c) = (L_T^A(B))^c$.

**Proof.** $(i)$ We have

$$x \in L_T^A(B^c) \Leftrightarrow T(x) + A \subseteq B^c \Leftrightarrow (T(x) + A) \cap B = \emptyset$$
$$\Leftrightarrow x \notin U_T^A(B) \Leftrightarrow x \in (U_T^A(B))^c.$$

$(ii)$ By substitution $B^c$ for $B$ in $(i)$ we get $U_T^A(B^c) = (L_T^A(B))^c$.

# 4 Conclusion

In this work, the lower $T$-rough and upper $T$-rough Lie subalgebras are formulated in the context of Lie algebra theory. We introduced the notion of the set-valued Lie homomorphism and generalized $T$-rough Lie subalgebra in a Lie algebra which is an extended notion of Lie homomorphism and Lie subalgebra of a Lie algebra. We hope that this extended research may provide a powerful tool in approximate reasoning. We strongly believe that $T$-rough Lie algebra offered here will turn out to be more useful in the theory and applications of the rough sets.

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