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Existence of Ψ-bounded solutions for linear differential systems on time scales

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Abstract

In this paper, we define Ψ-boundedness on time scales and we present necessary and sufficient conditions for the existence of at least one Ψ-bounded solution for the linear non-homogeneous matrix system $x^{\Delta} = A(t)x + f(t)$, where f(t) is a Ψ-bounded matrix valued function on T assuming that f is a Lebesgue Ψ-delta integrable function on time scale T. Finally we give a result in connection with the asymptotic behavior of the Ψ-bounded solutions of this system.

Keywords: Ψ-bounded, Ψ-integrable, Lebesgue Ψ-delta integrable.

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1. Introduction

Matrix differential equations arise in a number of areas of applied mathematics such as control systems, dynamic programming, optimal filters, quantum mechanics and systems engineering. The analytical, numerical solutions and control aspects of matrix differential equations have been studied by many authors. The aim of this paper is to give necessary and sufficient conditions that the non-homogeneous linear matrix system

$$x^{\Delta} = A(t)x + f(t), \tag{1.1}$$

where $x(t)$, $f(t)$ are in T^d and A is continuous $d \times d$ matrix valued function, has at least one Ψ-bounded solution for every Lebesgue Ψ-delta integrable function f on the time scale T. Here Ψ- is a rd-continuous matrix function, instead of a scalar function. The calculus of time scales was initiated by Stefan Hilger [13] in order to create a theory that can unify discrete and continuous analysis. The study of dynamic

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equations on time scales, is an area of mathematics that has recently received a lot of attention and sheds new light on the discrepancies between continuous differential equations and discrete difference equations. It also prevents one from proving a result twice, once for differential equations and once for difference equations.

The general idea, which is the main goal of Bohner and Peterson's excellent introductory text [2], is to prove a result for a dynamic equation where the domain of the unknown function is so called time scale. If T= R, the general result obtained yields the same result concerning an ordinary differential equation. If T = Z, the general result is the same result one would obtain concerning a difference equation. However, since there are infinitely many other time scales that one may work besides the real and the integers, one has a much more general result.

The present work unifies the results of existence of Ψ-bounded solutions of linear differential equations [7] and linear difference equations [12, 16] and also generalizes to matrix differential systems on time scales. In Section 2, we review most of the results and definitions on timescales and Ψ-boundedness. In Section 3, first, we present a necessary and sufficient condition for the existence of at least one Ψ-bounded solution for linear non-homogeneous matrix differential equation on time scales (1.1) for every Lebesgue Ψ-delta integrable function f, on time scale $\mathbb{T}$. Further, we obtain a result relating to the asymptotic behavior of solutions of (1.1).

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, [1, 2, 4, 5, 7–10, 12]. In [7–9], the author proposes the novel concept of Ψ-boundedness of solutions, Ψ being a continuous matrix-valued function, allows a better identification of various types of asymptotic behavior of the solutions on R.

Similarly, we can consider solutions of (1.1), which are Ψ-bounded not only on $\mathbb{T}^+$ but on $\mathbb{T}$. In this case, the condition for the existence of at least one Ψ-bounded solution are rather complicated, as shown in [10] and below. In [10], the authors gave a necessary and sufficient condition so that the system (1.1) has at least one Ψ-bounded solution on $\mathbb{T}$ for every continuous and Ψ-bounded function f on $\mathbb{T}$.

The norms used in this paper are taken from an excellent survey presented in [17].

2. preliminaries

The purpose of this section is to review some useful results, definitions and basic properties on time scales and Ψ boundedness which are needful for later discussion.

Let $\mathbb{T}$ be a time scale, i.e., an arbitrary non-empty closed subset of real numbers. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be unbounded above and below. In this paper we introduce some notations: $\mathbb{T}^+ = (0,\infty)\cap\mathbb{T}$, $\nu = \min\{[0,\infty)\cap\mathbb{T}\}$. For $x = (x_1, x_2, x_3, \dots, x_d)^T \in \mathbb{T}^d$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_d|\}$ be the norm of x. For a $d\times d$ real matrix $A = (a_{ij})$, we define the norm

$$|A| = \sup_{\|x\|\leqslant 1} \|Ax\|.$$

Definition 2.1. A matrix P is said to be a projection if $P^2 = P$. If P is the projection, then I-P is also a Projection. Two such projections, whose sum is I and hence whose product is zero are said to be supplementary.

Result 2.2. If A is differentiable at $t \in \mathbb{T}^k$, then $A(\sigma(t)) = A(t) + \mu(t)A^\Delta(t)$ [3].

For basic calculus on time scales and theorems on time scales mentioned in this section we refer to [2, 3].

Definition 2.3. A mapping $f : \mathbb{T} \longrightarrow X$, where X is a Banach space, is called rd-continuous if

(i) it is continuous at each right-dense $t \in \mathbb{T}$;

(ii) at each left dense point the left side limit f(t) exists.

Remark 2.4. If condition (ii) is replaced by f being continuous at each left-dense point, then f is said to be continuous function on T.

Result 2.5. If f is Δ-differentiable, then f is continuous. Also if t is right scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Definition 2.6. A function $F : T^k \longrightarrow R$ is called an anti-derivative of $f : T^k \longrightarrow R$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T^k$. We then define the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(s).$$

Theorem 2.7. *Assume* $f : T \longrightarrow R$ *is a function and let* $t \in T^k$. *Then we have the following:*

(i) *if f is differentiable at* t, *then is continuous at* t;
(ii) *if f is continuous at* t *and* t *is right-scattered, then f is differentiable at* t *with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

(iii) *if* t *is right-dense, then* f *is differentiable at* t *if the limit*

$$\lim_{t \longrightarrow s} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number, in this case

$$f^{\Delta}(t) = \lim_{t \longrightarrow s} \frac{f(t) - f(s)}{t - s};$$

(iv) *if* f *is differentiable at* t, *then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Definition 2.8. A function $f : T \longrightarrow R$ is called regulated provided its right-sided limits exists (finite) at all right-dense points in T and its left-sided limits exist (finite) at all left dense points in T.

Definition 2.9. A function $f : T \longrightarrow R$ is called rd-continuous provided it is continuous at right-dense points in T and its left-sided limits exist (finite) at left-dense points in T. The set rd-continuous functions $f : T \longrightarrow R$ will be denoted by

$$C_{rd} = C_{rd}(T) = C_{rd}(T, R).$$

The set of functions $f : T \longrightarrow R$ that are differentiable and whose derivative is rd continuous is denoted by

$$C^1_{rd} = C^1_{rd}(T) = C^1_{rd}(T, R).$$

Result 2.10. Assume $f : T \longrightarrow R$.

(i) If f is continuous, then f is rd-continuous.
(ii) If f is rd- continuous, then f is regulated.
(iii) If the jump operator σ is rd- continuous, then so is f^{σ}.
(iv) Assume f is continuous. If $g : T \longrightarrow R$ is regulated or rd-continuous, then $f o g$ has that property too.

Theorem 2.11 (Existence of pre-antiderivatives)**.** *Let* f *be regulated. Then there exists a function* F *which is pre-differentiable with region of differentiation D such that* $F^{\Delta}(t) = f(t)$ *holds for all* $t \in D$.

Definition 2.12. Assume $f : T \longrightarrow R$ is a regulated function. Any function F as in theorem (above) is called a pre-anti derivative of f. We define the indefinite integral of a regulated function f by

$$\int f(\tau)\Delta t = F(t) + C,$$

where C is an arbitrary constant and f is a pre-anti derivative of f. We define the Cauchy integral by

$$\int_r^s \Delta(t) = F(s) - F(r)$$

for all $r, s \in T$.

A function $F : T \longrightarrow R$ is called an anti derivative of $f : T \longrightarrow R$ provided

$$F^{\Delta}(t) = f(t)$$

holds for all $t \in T^k$.

Definition 2.13. If $a \in T$, $\sup T = \infty$ and f is rd-continuous on $[a, \infty)$ then we define the improper integral by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \longrightarrow \infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Theorem 2.14. *Let f be a function which is Δ-integrable from a to b, for $t \in [a, b]$, define $F(t) = \int_a^t f(\tau)\Delta\tau$. Then F is continuous on $[a, b]$. If $t_0 \in [a, b)$ and if f is continuous at t_0 provided t_0 is right-dense, then F is Δ-differentiable at t_0 and $F^{\Delta}(t_0) = f(t_0)$.*

Theorem 2.15. *Any set of d-linearly independent solutions $y_1, y_2, \dots, y_d$ of*

$$y^{\Delta} = A(t)y \tag{2.1}$$

is called a fundamental set of solutions and the matrix with $y_1, y_2, \dots, y_d$ as its columns is called a fundamental matrix for the equation (2.1) and is denoted by Φ. The fundamental matrix Φ is non-singular.

Theorem 2.16. *Let $A \in R$ be an $d \times d$ matrix-valued function on T and suppose that $f : T \longrightarrow R^d$ is rd-continuous. Let $t_0 \in T$ and $y_0 \in R^d$. Then the initial value problem*

$$y^{\Delta}(t) = A(t)y(t) + f(t), y(t_0) = y_0$$

has a unique solution $y : T \longrightarrow R^d$. Moreover, this solution is given by

$$y(t) = \Phi_A(t, t_0)y_0 + \int_{t_0}^t \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau,$$

where $\Phi_A(t, t_0)$ is a fundamental matrix.

Also we introduce some definitions, results, and notations of Ψ boundedness.

Let R^d be the Euclidean d-space. For $x = (x_1, x_2, x_3, \dots, x_d)^T \in R^d$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\| \leqslant 1} \|Ax\|$.

Let $\Psi_i : T \to (0, \infty)$, $i = 1, 2, \dots d$, be continuous functions and

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_d].$$

Definition 2.17. A function $\varphi : T \to R^{d\times d}$ is said to be Ψ-bounded on T if the matrix function $\Psi\varphi$ is bounded on T (i.e., there exists $L_2 > 0$ such that $|\Psi(t)\varphi(t)| \leqslant L_2$ for all $t \in T$).

Definition 2.18. A function $\varphi : T \to R^{d\times d}$ is said to be Lebesgue Ψ-integrable on T if φ is Δ measurable and $\Psi\varphi$ is Lebesgue integrable on T.

By a solution of (1.1), we mean an absolutely continuous function satisfying (1.1) for almost all $t \in T$.

Let the vector space R^d be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(t)$ of (2.1) is Ψ-bounded on T if and only if $y(0) \in X_0$ and Ψ-bounded on $T^+ = [0, \infty)$ if and only if $y(0) \in X_- \oplus X_0$. Also, let P_-, P_0, and P_+ denote the corresponding projections of R^d onto X_-, X_0, and X_+, respectively.

In the next section we establish our main results. It may be noted that, by a fundamental sequence in a Banach space, X, we mean a sequence in X whose span Y is dense in X. In other words, every element in X can be approximated by an element in the span. That is, given $x \in X$, we can find a $y \in Y$ such that $\|x - y\|$ is small. This concept is used in our main results.

3. Main result

Theorem 3.1. *If* A *is a continuous* $d \times d$ *real matrix on* T*, then* (1.1) *has at least one* Ψ*-bounded solution on* T *for every Lebesgue* Ψ*-delta integrable function* $f : T \to T^d$ *on* T *if and only if there exists a positive constant* K *such that*

$$\begin{aligned}
|\Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t > 0,\ \sigma(s) \leqslant 0,\\
|\Psi(t)\Phi(t)(P_0 + P_-)\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t > 0,\ \sigma(s) > 0, \sigma(s) < t,\\
|\Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t > 0, \sigma(s) > 0,\ \sigma(s) \geqslant t,\\
|\Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t \leqslant 0,\ \sigma(s) < t,\\
|\Psi(t)\Phi(t)(P_0 + P_+)\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t \leqslant 0,\ \sigma(s) \geqslant t,\ \sigma(s) < 0,\\
|\Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| &\leqslant K \quad \textit{for } t \leqslant 0,\ \sigma(s) \geqslant t,\ \sigma(s) \geqslant 0.
\end{aligned} \tag{3.1}$$

Proof. For "only if" part, suppose that the system (1.1) has at least one Ψ-bounded solution on T for every Lebesgue Ψ-Δ integrable function f: $T \to T^d$ on T. We shall define by

i) C_Ψ: the Banach space of all Ψ-bounded and continuous functions $x : T \to T^d$ with the norm $\|x\|_{C_\Psi} = \sup_{t\in T} \|\Psi(t)x(t)\|$;
ii) B: the Banach space of all Lebesgue Ψ-Δ integrable functions $x : T \to T^d$ with the norm $\|x\|_B = \int_{-\infty}^{+\infty} \|\Psi(t)x(t)\|\Delta t$;
iii) D: the set of all functions $x : T \to T^d$ which are absolutely continuous on all intervals $J \subset T$, Ψ-bounded on T, $x(0) \in X_- \oplus X_+$ and $x^\Delta - Ax \in B$.

Obviously, D is a vector space and $x \to \|x\|_D = \|x\|_{C_\Psi} + \|x^\Delta - Ax\|_B$ is a norm on D.

Step 1. $(D, \|\cdot\|_D)$ is a Banach space. Let $(x_n)_{n\in N}$ be a fundamental sequence of elements of D. Then, it is a fundamental sequence in C_Ψ. Therefore, there exists a continuous and Ψ-bounded function $x : T \to T^d$ such that $\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t)$, uniformly on T. From the inequality

$$\begin{aligned}
\|x_n(t) - x(t)\| &\leqslant \|\Psi^{-1}(t)\Psi(t)x_n(t) - \Psi^{-1}(t)\Psi(t)x(t)\|, \quad t \in T,\\
&\leqslant |\Psi^{-1}(t)|\|\Psi(t)x_n(t) - \Psi(t)x(t)\|, \quad t \in T,
\end{aligned}$$

it follows that $\lim_{n\to\infty} x_n(t) = x(t)$, uniformly on every compact subset of T. Thus, $x(0) \in X_- \oplus X_+$.

On the other hand, the sequence $(f_n)_{n\in N}$, where $f_n(t) = x_n^\Delta(t) - A(t)x_n(t)$, is a fundamental sequence in the Banach space B. Thus, there exists $f \in B$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \|\Psi(t)(f_n(t) - f(t))\|\Delta t = 0.$$

For a fixed, but arbitrary, $t \in T$, we have

$$\begin{aligned} x(t) - x(0) &= \lim_{n\to\infty} \big(x_n(t) - x_n(0)\big) \\ &= \lim_{n\to\infty} \int_0^t x_n^{\Delta}(s)\Delta s \\ &= \lim_{n\to\infty} \int_0^t [x_n^{\Delta}(s) - A(s)x_n(s) + A(s)x_n(s)]\Delta s \\ &= \lim_{n\to\infty} \int_0^t [\Psi^{-1}(s)\Psi(s)[x_n^{\Delta}(s) - A(s)x_n(s)] + A(s)x_n(s)]\Delta s \\ &= \lim_{n\to\infty} \int_0^t [\Psi^{-1}(s)\Psi(s)f_n(s) + A(s)x_n(s)]\Delta s \\ &= \lim_{n\to\infty} \int_0^t [\Psi^{-1}(s)\Psi(s)(f_n(s) - f(s) + f(s) + A(s)x_n(s))]\Delta s \\ &= \int_0^t \big(f(s) + A(s)x(s)\big)\Delta s. \end{aligned}$$

It follows that $x^{\Delta} - Ax = f \in B$ and x is absolutely continuous on all intervals $J \subset T$. Thus, $x \in D$. Now, from

$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } T$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \|\Psi(t)[(x_n^{\Delta}(t) - A(t)x_n(t)) - (x^{\Delta}(t) - A(t)x(t))]\|\Delta t = 0,$$

it follows that

$$\begin{aligned} \lim_{n\to\infty} \Psi(t)\|x_n - x\|_D &= \sup_{t\in T} \|\Psi(t)(x_n - x)\| + \|(x_n - x)^{\Delta} - A(t)(x_n - x)\|_B \\ &= \sup_{t\in T} \|\Psi(t)(x_n - x)\| + \int_0^{+\infty} \|\Psi(t)[\|(x_n)^{\Delta} - A(t)(x_n) - (x^{\Delta} - A(t)x)]\|\Delta t = 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} x_n(t) = x(t).$$

Thus, $(D, \|\cdot\|_D)$ is a Banach space.

Step 2. There exists a positive constant K such that, for every $f \in B$ and for corresponding solution $x \in D$ of (2.1), we have

$$\sup_{t\in T} \|\Psi(t)x(t)\| \leqslant K \int_{-\infty}^{+\infty} \|\Psi(t)f(t)\|\Delta t. \tag{3.2}$$

For this, define the mapping $T : D \to B$, $Tx = x^{\Delta} - Ax$. This mapping is clearly linear and bounded, with $\|T\| \leqslant 1$.

Let $Tx = 0$. Then, $x^{\Delta} = Ax$, $x \in D$. This shows that x is a Ψ-bounded solution on T of (2.1). Then, $x(0) \in X_0 \cap \big(X_- \oplus X_+\big) = \{0\}$. Thus, $x = 0$, such that the mapping T is "one-to-one".

Let $f \in B$ and let x be the Ψ-bounded solution on T of the system (1.1). Let z be the solution of the Cauchy problem

$$z^{\Delta}(t) = A(t)z + f(t), \quad z(0) = (P_- + P_+)x(0).$$

Then $u(t) = x(t) - z(t)$ is a solution of (2.1) with $u(0) = x(0) - (P_- + P_+)x(0) = P_0x(0)$. From the Definition of X_0, it follows that $z(t)$ is Ψ-bounded on T. Thus, z is Ψ-bounded on T. Therefore, z belongs to D and $\Theta z = f$. Consequently, the mapping Θ is "onto".

From a fundamental result of Banach space: "If Θ is a bounded one-to-one linear operator of one Banach space onto another, then the inverse operator Θ^{-1} is also bounded" (this is a consequence of the open mapping theorem in Banach spaces which can be found in Simmons [18]), we have $\|\Theta^{-1}f\|_D \leqslant \|\Theta^{-1}\|\|f\|_B$, for all $f \in B$.

For a given $f \in B$, let $x = \Theta^{-1}f$ be the corresponding solution $x \in D$ of (1.1). We have

$$\|x\|_D = \|x\|_{C_\Psi} + \|x^\Delta - Ax\|_B = \|x\|_{C_\Psi} + \|f\|_B \leqslant \|\Theta^{-1}\|\|f\|_B$$

or

$$\|x\|_{C_\Psi} \leqslant \left(\|\Theta^{-1}\| - 1\right)\|f\|_B = K\|f\|_B.$$

This inequality is equivalent to (3.1). Thus, the end of the proof.

Step 3. Let $\Theta_1 < 0 < \Theta_2$ be a fixed points but arbitrarily, and let $f : T \to T^d$ a function in B which vanishes on $(-\infty, \Theta_1] \cup [\Theta_2, +\infty)$. It is easy to see that the function $x : T \to T^d$ defined by

$$x(t) = \begin{cases} -\int_{\Theta_1}^{0} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_{\Theta_1}^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s, & t < \Theta_1, \\ \int_{\Theta_1}^{t} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_{0}^{t} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s, & \Theta_1 \leqslant t \leqslant \Theta_2, \\ \int_{\Theta_1}^{\Theta_2} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_{0}^{\Theta_2} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s, & t > \Theta_2, \end{cases}$$

is the solution in D of the system (1.1). Now, we put

$$G(t,s) = \begin{cases} \Phi(t)P_-\Phi^{-1}(\sigma(s)), & \sigma(s) \leqslant 0 < t, \\ \Phi(t)(P_0 + P_-)\Phi^{-1}(\sigma(s)), & 0 < \sigma(s) < t, \\ -\Phi(t)P_+\Phi^{-1}(\sigma(s)), & 0 < t \leqslant \sigma(s), \\ \Phi(t)P_-\Phi^{-1}(\sigma(s)), & \sigma(s) < t \leqslant 0, \\ -\Phi(t)(P_0 + P_+)\Phi^{-1}(\sigma(s)), & t \leqslant \sigma(s) < 0, \\ -\Phi(t)P_+\Phi^{-1}(\sigma(s)), & t \leqslant 0 \leqslant \sigma(s). \end{cases}$$

This function is continuous on T^2 except on the line $t = \sigma(s)$, where it has a jump discontinuity. Then, we have $x(t) = \int_{\Theta_1}^{\Theta_2} G(t, \sigma(s))f(s)\Delta s$, $t \in T$.

- For $t < \Theta_1$, we have

$$\begin{aligned} \int_{\Theta_1}^{\Theta_2} G(t,\sigma(s))f(s)\Delta s &= -\int_{\Theta_1}^{0} \Phi(t)(P_0 + P_+)\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_{0}^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &= -\int_{\Theta_1}^{0} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_{\Theta_1}^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s = x(t). \end{aligned}$$

- For $t \in [\Theta_1, 0)$, we have

$$\begin{aligned} \int_{\Theta_1}^{\Theta_2} G(t,\sigma(s))f(s)\Delta s &= \int_{\Theta_1}^{t} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{0} \Phi(t)(P_0 + P_+)\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &\quad - \int_{0}^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &= \int_{\Theta_1}^{t} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_{0}^{t} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &\quad - \int_{t}^{T_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s = x(t). \end{aligned}$$

- For $t \in (0, \Theta_2]$, we have

$$\begin{aligned}\int_{\Theta_1}^{\Theta_2} G(t,\sigma(s))f(s)\Delta s &= \int_{\Theta_1}^{0} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^t \Phi(t)(P_0+P_-)\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &\quad - \int_t^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &= \int_{\Theta_1}^{t} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^t \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &\quad - \int_t^{\Theta_2} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s = x(t).\end{aligned}$$

- For $t > \Theta_2$, we have

$$\begin{aligned}\int_{\Theta_1}^{\Theta_2} G(t,\sigma(s))f(s)\Delta s &= \int_{\Theta_1}^{0} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^{\Theta_2} \Phi(t)(P_0+P_-)\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &= \int_{\Theta_1}^{\Theta_2} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^{\Theta_2} \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &= x(t).\end{aligned}$$

Now, the inequality (3.1) becomes

$$\sup_{t\in T} \|\Psi(t)\int_{\Theta_1}^{\Theta_2} G(t,\sigma(s))f(s)\Delta s\| \leqslant K\int_{\Theta_1}^{\Theta_2} \|\Psi(t)f(t)\|\Delta t.$$

For a fixed points $\sigma(s) \in T$, $\delta > 0$ and $\xi \in T^d$, but arbitrarily, let f the function defined by

$$f(t) = \begin{cases}\Psi^{-1}(t)\xi, & \text{for } \sigma(s) \leqslant t \leqslant \sigma(s)+\delta,\\ 0, & \text{elsewhere.}\end{cases}$$

Clearly, $f \in B$, $\|f\|_B = \delta\|\xi\|$. The above inequality becomes

$$\|\int_s^{s+\delta} \Psi(t)G(t,\sigma(u))\Psi^{-1}(u)\xi\Delta u\| \leqslant K\delta\|\xi\| \quad \text{for all } t \in T.$$

Dividing by δ and letting $\delta \to 0$, we obtain for any $t \neq s$,

$$\|\Psi(t)G(t,\sigma(s))\Psi^{-1}(s)\xi\| \leqslant K\|\xi\| \quad \text{for all } t \in T,\ \xi \in T^d.$$

Hence, $|\Psi(t)G(t,\sigma(s))\Psi^{-1}(s)| \leqslant K$, which is equivalent to (3.1). By continuity, (3.1) remains valid also in the excepted case $t = \sigma(s)$.

Now, we prove the "if" part. Suppose that the fundamental matrix Y of (2.1) satisfies the condition (3.1) for some $K > 0$. Let $f : T \to T^d$ be a Lebesgue Ψ-delta integrable function on T. We consider the function $u : T \to T^d$ defined by

$$u(t) = \int_{-\infty}^{t} \Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^t \Phi(t)P_0\Phi^{-1}(\sigma(s))f(s)\Delta s - \int_t^{\infty} \Phi(t)P_+\Phi^{-1}(\sigma(s))f(s)\Delta s. \tag{3.3}$$

Step 4. The function u is well-defined on T. Indeed, for $v < t \leqslant 0$, we have

$$\int_v^t \|\Phi(t)P_-\Phi^{-1}(\sigma(s))f(s)\|\Delta s = \int_v^t \|\Psi^{-1}(t)\Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\|\Delta s$$

$$\begin{aligned}
&\leqslant |\Psi^{-1}(t)| \int_{\nu}^{t} |\Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)| \|\Psi(s)f(s)\|\Delta s \\
&\leqslant K|\Psi^{-1}(t)| \int_{\nu}^{t} \|\Psi(s)f(s)\|\Delta s,
\end{aligned}$$

which shows that the integral $\int_{-\infty}^{t} \Phi(t)P_{-}\Phi^{-1}(\sigma(s))f(s)\Delta s$ is absolutely convergent. For $t > 0$, we have the same result.

Similarly, the integral $\int_{t}^{\infty} \Phi(t)P_{+}\Phi^{-1}(\sigma(s))f(s)\Delta s$ is absolutely convergent. Thus, the function u is well-defined and is an absolutely continuous function on all intervals $J \subset T$.

Step 5. The function u is a solution of (1.1). Indeed, for almost all $t \in T$, we have

$$\begin{aligned}
u^{\Delta}(t) = &\int_{-\infty}^{t} \Phi^{\Delta}(t)P_{-}\Phi^{-1}(\sigma(s))f(s)\Delta s + [\Phi(\sigma(t))P_{-}\Phi^{-1}(t)f(t) - 0] \\
&+ \int_{0}^{t} \Phi^{\Delta}(t)P_{0}\Phi^{-1}(\sigma(s))f(s)\Delta s + [0 - \Phi(\sigma(t))(t)P_{0}\Phi^{-1}(t)f(t)] \\
&- \int_{t}^{\infty} \Phi^{\Delta}(t)P_{+}\Phi^{-1}(\sigma(s))f(s)\Delta s - [0 - \Phi(\sigma(t))(t)P_{+}\Phi^{-1}(t)f(t) \\
= &\int_{-\infty}^{t} A(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))f(s)\Delta s + Y(\sigma(t))(t)P_{-}\Phi^{-1}(t)f(t) \\
&+ \int_{0}^{t} A(t)\Phi(\sigma(t))P_{0}\Phi^{-1}(\sigma(s))f(s)\Delta s + \Phi(\sigma(t))P_{0}\Phi^{-1}(t)f(t) \\
&- \int_{t}^{\infty} A(t)\Phi(t)P_{+}\Phi^{-1}(\sigma(s))f(s)\Delta s + \Phi(\sigma(t))P_{+}\Phi^{-1}(t)f(t) \\
= &A(t)u(t) + \Phi(\sigma(t))(P_{-} + P_{0} + P_{+})\Phi^{-1}(t)f(t) = A(t)u(t) + f(t).
\end{aligned}$$

This shows that the function u is a solution of (1.1).

Step 6. The solution u is Ψ-bounded on T. Indeed, for $t < 0$, we have

$$\begin{aligned}
\Psi(t)u(t) = &\int_{-\infty}^{t} \Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s \\
&+ \int_{0}^{t} \Psi(t)\Phi(t)P_{0}\Phi^{-1}(s)\Psi^{-1}(\sigma(s))\Psi(s)f(s)\Delta s \\
&- \int_{t}^{\infty} \Psi(t)\Phi(t)P_{+}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s \\
= &\int_{-\infty}^{t} \Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s \\
&- \int_{t}^{0} \Psi(t)\Phi(t)(P_{0} + P_{+})\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s \\
&- \int_{0}^{\infty} \Psi(t)\Phi(t)P_{+}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\,.
\end{aligned}$$

Then

$$\|\Psi(t)u(t)\| \leqslant K \cdot \int_{-\infty}^{\infty} \|\Psi(s)f(s)\|\Delta s.$$

For $t \geqslant 0$, we have

$$\Psi(t)u(t) = \int_{-\infty}^{t} \Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s$$

$$
\begin{aligned}
&+\int_0^t \Psi(t)\Phi(t)P_0\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s-\int_t^\infty \Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\
=&\int_{-\infty}^0 \Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\
&+\int_0^t \Psi(t)\Phi(t)(P_0+P_-)\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\
&-\int_t^\infty \Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\,.
\end{aligned}
$$

Then

$$\|\Psi(t)u(t)\|\leqslant K\cdot\int_{-\infty}^{\infty}\|\Psi(s)f(s)\|\Delta s.$$

Hence

$$\sup_{t\in T}\|\Psi(t)u(t)\|\leqslant K\cdot\int_{-\infty}^{\infty}\|\Psi(s)f(s)\|\Delta s,$$

which shows that the solution u is Ψ-bounded on T. The proof is now complete. □

In a particular case, we have the following result.

Theorem 3.2. *If the homogeneous equation (2.1) has no nontrivial Ψ-bounded solution on T, then the (1.1) has a unique Ψ-bounded solution on T for every Lebesgue Ψ-delta integrable function $f : T \to T^d$ on T if and only if there exists a positive constant K such that*

$$
\begin{aligned}
&|\Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\leqslant K\sigma(s) \quad \text{for } -\infty<\sigma(s)<t<+\infty,\\
&|\Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\leqslant Kt\leqslant\sigma(s) \quad \text{for } -\infty<t\leqslant\sigma(s)<+\infty.
\end{aligned}
\tag{3.4}
$$

In this case, $P_0 = 0$ and the proof is as above.

Next, we prove a theorem in which we will see that the asymptotic behavior of solutions to (1.1) is determined completely by the asymptotic behavior of the fundamental matrix Y.

Theorem 3.3. *Suppose that:*

(1) *the fundamental matrix $\Phi(t)$ of (3.4) satisfies:*
  - (a) *condition (3.1) is satisfied for some $K > 0$;*
  - (b) *the following conditions are satisfied:*
    - (i) $\lim_{t\to\pm\infty} |\Psi(t)\Phi(t)P_0| = 0$;
    - (ii) $\lim_{t\to-\infty} |\Psi(t)\Phi(t)P_+| = 0$;
    - (iii) $\lim_{t\to+\infty} |\Psi(t)\Phi(t)P_-| = 0$;

(2) *the function $f : T \to T^d$ is Lebesgue Ψ-delta integrable on T.*

Then, every Ψ-bounded solution x of (1.1) is such that

$$\lim_{t\to\pm\infty}\|\Psi(t)x(t)\|=0.$$

Proof. By Theorem 3.1, for every Lebesgue Ψ-integrable function $f : T \to T^d$, the equation (1.1) has at least one Ψ-bounded solution on T.

Let x be a Ψ-bounded solution on T of (1.1). Let u be defined by (3.3). The function u is a Ψ-bounded solution on T of (1.1).

Now, let the function $y(t) = x(t) - u(t) - \Phi(t)P_0(x(0) - u(0))$, $t \in T$. Obviously, y is a solution on T of (2.1). Because $\Psi(t)\Phi(t)P_0$ is bounded on T, y is Ψ-bounded on T. Thus, $y(0) \in X_0$. On the other hand,

$$y(0)=x(0)-u(0)-Y(0)P_0(x(0)-u(0))=(P_-+P_+)(x(0)-u(0))\in X_-\oplus X_+.$$

Therefore, $y(0) \in X_0 \cap (X_- \oplus X_+) = \{0\}$ and then, $y = 0$. It follows that

$$x(t) = \Phi(t)P_0(x(0) - u(0)) + u(t), t \in T.$$

Now, we prove that $\lim_{t\to\pm\infty} \|\Psi(t)u(t)\| = 0$. For $t \geqslant 0$, we write again

$$\begin{aligned}\Psi(t)u(t) = &\int_{-\infty}^{0} \Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\ &+ \int_0^t \Psi(t)\Phi(t)(P_0 + P_-)\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\ &- \int_t^{\infty} \Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s.\end{aligned}$$

Let $\varepsilon > 0$. From the hypotheses: there exists $t_0 < 0$ such that

$$\int_{-\infty}^{t_0} \|\Psi(s)f(s)\|\Delta s < \frac{\varepsilon}{5K};$$

there exists $t_1 > 0$ such that, for all $t \geqslant t_1$,

$$|\Psi(t)\Phi(t)P_-| < \frac{\varepsilon}{5}(1 + \int_{t_0}^{0} \|\Phi^{-1}(s)f(s)\|\Delta s)^{-1};$$

there exists $t_2 > t_1$ such that, for all $t \geqslant t_2$,

$$\int_t^{\infty} \|\Psi(s)f(s)\|\Delta s < \frac{\varepsilon}{5K};$$

there exists $t_3 > t_2$ such that, for all $t \geqslant t_3$,

$$|\Psi(t)\Phi(t)(P_0 + P_-)| < \frac{\varepsilon}{5}(1 + \int_0^{t_2} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)^{-1}.$$

Then, for $t \geqslant t_3$, we have

$$\begin{aligned}\|\Psi(t)u(t)\| \leqslant &\int_{-\infty}^{t_0} |\Psi(t)\Phi(t)P_-\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ &+ \int_{t_0}^{0} |\Psi(t)\Phi(t)P_-|\|\Phi^{-1}(\sigma(s))f(s)\|\Delta s + \int_0^{t_2} |\Psi(t)\Phi(t)(P_0\\ &+ P_-)|\|\Phi^{-1}(\sigma(s))f(s)\|\Delta s + \int_{t_2}^{t} |\Psi(t)\Phi(t)(P_0 + P_-)\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ &+ \int_t^{\infty} |\Psi(t)\Phi(t)P_+\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ < &\, K\int_{-\infty}^{t_0} \|\Psi(s)f(s)\|\Delta s + \frac{\varepsilon}{5(1 + \int_{t_0}^{0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)} \int_{t_0}^{0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s\\ &+ \frac{\varepsilon}{5(1 + \int_0^{t_2} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)} \int_0^{t_2} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s\\ &+ K\int_{t_2}^{t} \|\Psi(s)f(s)\|\Delta s + K\int_t^{\infty} \|\Psi(s)f(s)\|\Delta s\\ < &\, K\frac{\varepsilon}{5K} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + K(\int_{t_2}^{t} \|\Psi(s)f(s)\|\Delta s + \int_t^{\infty} \|\Psi(s)f(s)\|\Delta s) < \frac{3\varepsilon}{5} + K\frac{\varepsilon}{5K} < \varepsilon.\end{aligned}$$

This shows that $\lim_{t\to+\infty} \|\Psi(t)u(t)\| = 0$.

For $t < 0$, we write again

$$\begin{aligned}\Psi(t)u(t) &= \int_{-\infty}^{t} \Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\ &\quad - \int_{t}^{0} \Psi(t)\Phi(t)(P_0+P_{+})\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s\\ &\quad - \int_{0}^{\infty} \Psi(t)\Phi(t)P_{+}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)\Psi(s)f(s)\Delta s.\end{aligned}$$

Let $\varepsilon > 0$. From the hypotheses, we have: there exists $t_0 > 0$ such that

$$\int_{t^0}^{+\infty} \|\Psi(s)f(s)\|\Delta s < \frac{\varepsilon}{5K};$$

there exists $t_4 < 0$ such that, for all $t < t_4$,

$$|\Psi(t)\Phi(t)P_{+}| < \frac{\varepsilon}{5}(1+\int_{0}^{t^0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)^{-1};$$

there exists $t_5 < t_4$ such that, for all $t \leqslant t_5$,

$$\int_{-\infty}^{t} \|\Psi(s)f(s)\|\Delta s < \frac{\varepsilon}{5K};$$

there exists $t_6 < t_5$ such that, for all $t \leqslant t_6$,

$$|\Psi(t)\Phi(t)(P_0+P_{+})| < \frac{\varepsilon}{5}(1+\int_{t_5}^{0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)^{-1}.$$

Then, for $t \leqslant t_6$, we have

$$\begin{aligned}\|\Psi(t)u(t)\| &\leqslant \int_{-\infty}^{t} |\Psi(t)\Phi(t)P_{-}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ &\quad + \int_{t}^{t_5} |\Psi(t)\Phi(t)(P_0+P_{+})\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ &\quad + \int_{t_5}^{0} |\Psi(t)\Phi(t)(P_0+P_{+})|\|\Phi^{-1}(\sigma(s))f(s)\|\Delta s + \int_{0}^{t^0} |\Psi(t)\Phi(t)P_{+}|\|\Phi^{-1}(\sigma(s))f(s)\|\Delta s\\ &\quad + \int_{t^0}^{+\infty} |\Psi(t)\Phi(t)P_{+}\Phi^{-1}(\sigma(s))\Psi^{-1}(s)|\|\Psi(s)f(s)\|\Delta s\\ &< K\int_{-\infty}^{t} \|\Psi(s)f(s)\|\Delta s + K\int_{t}^{t_5} \|\Psi(s)f(s)\|\Delta s\\ &\quad + \frac{\varepsilon}{5(1+\int_{t_5}^{0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)} \int_{t_5}^{0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s\\ &\quad + \frac{\varepsilon}{5(1+\int_{0}^{t^0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s)} \int_{0}^{t^0} \|\Phi^{-1}(\sigma(s))f(s)\|\Delta s + K\int_{t^0}^{+\infty} \|\Psi(s)f(s)\|\Delta s\\ &< K(\int_{-\infty}^{t} \|\Psi(s)f(s)\|\Delta s + \int_{t}^{t_5} \|\Psi(s)f(s)\|\Delta s) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + K\frac{\varepsilon}{5K} < K\frac{\varepsilon}{5K} + \frac{3\varepsilon}{5} < \varepsilon.\end{aligned}$$

This shows that $\lim_{t\to-\infty} \|\Psi(t)u(t)\| = 0$.

Now, it is easy to see that $\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 0$. The proof is now complete. □

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