Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 20 (2020), 30–42
Research Article

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A new quintic B-spline approximation for numerical treatment of Boussinesq equation

Tahir Nazir[a], Muhammad Abbas[a,*], Muhammad Kashif Iqbal[b]

[a]*Department of Mathematics, University of Sargodha, 40100 Sargodha, Pakistan.*

[b]*Department of Mathematics, Government College University, 38000 Faisalabad, Pakistan.*

Abstract

In this work, we have presented a new quintic B-spline approximation technique for numerical solution of Boussinesq equation. Usual finite difference formulation has been applied to discretize the problem in temporal domain, whereas, the typical fifth degree B-spline functions, equipped with a new approximation for fourth order derivative, have been utilized to interpolate the unknown function in spatial direction. The stability and error analysis of the proposed numerical algorithm have been studied rigorously. Two test examples are considered to affirm the performance and accuracy of the new scheme. The computational outcomes are found to be better than the existing numerical techniques on the topic.

Keywords: Quintic B-spline functions, theta weighted scheme, quintic B-spline collocation method, Boussinesq equation, Von-Neumann stability.

2010 MSC: 65M70, 65Z05, 65D05, 65D07, 65N12.

1. Introduction

The Boussinesq equation (BE) belongs to the family of Korteweg-de Vries equations. It describes several physical phenomena such as propagation of waves, non-linear plasma waves exhibiting some dissipative effects, propagation of bores and wave diffraction in shallow waters [9, 14, 15]. The shallow water wave equations have several applications including storm and Tsunami propagation, river flooding and dam breaks [25]. The generalized BE is given by

$$u_{tt} + \alpha u_{zz} + \beta u_{zzzz} + \gamma (u^2)_{zz} = 0, \quad z \in [a, b], \quad t \in [0, T], \tag{1.1}$$

with the initial conditions

$$u(z, 0) = g_1(z), \quad u_t(z, 0) = g_2(z) \tag{1.2}$$

and boundary conditions

$$\begin{cases} u(a, t) = \psi_1(t), \quad u(b, t) = \psi_2(t), \\ u_z(a, t) = \psi_3(t), \quad u_z(b, t) = \psi_4(t), \quad \text{or} \quad u_{zz}(a, t) = \psi_3(t), \quad u_{zz}(b, t) = \psi_4(t), \end{cases} \tag{1.3}$$

where $u = u(z, t)$ denotes the wave displacement at point z and time t, α, β, γ are real constants and

*Corresponding author
Email address: muhammad.abbas@uos.edu.pk (Muhammad Abbas)

doi: 10.22436/jmcs.020.01.04

Received: 2019-04-19 Revised: 2019-05-30 Accepted: 2019-07-01

g_i's, ψ_i's are known to be continuous functions. In the last couple of decades, several numerical and analytical methods have been proposed for solving BE. Ismail and Bratsos [12] proposed a pair of predictor correctors for approximate solution of BE. The authors in [4] presented a linearized numerical algorithm for solving BE. Bratsos et al. [3] proposed a parametric finite difference method (FDM) for approximate solution of one dimensional shallow sea waves. Wazwaz [23] employed tanh-coth method to explore multi soliton solutions for BE. Chavez et al. [5] studied the analytical solution of Boussinesq type equations with their applications to the agricultural drainage. Uddin et al. [21] proposed radial based functions based pseudo-spectral method for solving good BE. Akinlar and Secer [2] proposed Petrov Galerkin wavelet method for approximate solution of BE. A fourth order FDM was developed by Ismail and Mosally [13] to investigate the numerical solution of good BE. Siddiqi and Arshad [20] proposed quintic B-spline (QnBS) collocation approach for approximate solution of classical non-linear BE. The authors in [18] employed homotopy analysis method (HAM) to explore the analytical solution of BE with variable coefficients. Zakaria et al. [25] used quintic trigonometric B-spline collocation method (QnTBSM) to study the numerical solution of BE. Jang [16] introduced a new semi analytic dispersion-relation preserving method to integrate the Boussinesq like equations. Darvishi et al. [6] derived the traveling wave solutions for four different Boussinesq type equations.

In this work, the approximate solution of non-linear BE has been explored by means of QnBS collocation method. The usual finite difference scheme and QnBS functions are employed for temporal and spatial discretization respectively. The accuracy of QnBS collocation method has been enhanced by introducing of a new approximation for fourth order derivative. The stability of the presented numerical scheme has been studied subject to Von-Neumann stability analysis.

This work has been assembled as follows. In Section 2, we shall review some preliminaries of QnBS interpolation. The formulation of new QnBS approximation for $u^{(4)}(x)$ is demonstrated in Section 3. The numerical scheme is presented in Section 4. The stability and error analysis have been given in Sections 5 and 6. Section 7 covers the numerical results and discussion.

2. Quintic B-spline functions

We uniformly partition the interval $[a, b]$ by $n+1$ equidistant knots $z_i = z_0 + ih$, $i = 0(1)n$, where $n \in \mathbb{Z}^+$, $a = z_0$, $b = z_n$ and $h = \frac{1}{n}(b-a)$. The j^{th} spline basis of degree p and order $p+1$, is defined as [7]

- for $p = 0$

$$B_{0,j}(z) = \begin{cases} 1, & \text{if } z \in [z_j, z_{j+1}], \\ 0, & \text{otherwise;} \end{cases} \tag{2.1}$$

- for $p > 0$

$$B_{p,j}(z) = \frac{(z - z_j)}{(z_{j+p} - z_j)} B_{p-1,j}(z) + \Big(1 - \frac{(z - z_{j+1})}{(z_{j+1+p} - z_{j+1})}\Big) B_{p-1,j+1}(z), \quad z \in [z_j, z_{j+1+p}]. \tag{2.2}$$

Using (2.1)-(2.2), the fifth degree B-spline functions are defined as [17, 24]

$$B_j(z) = \frac{1}{120h^5}\begin{cases} (z - z_{j-3})^5, & z \in [z_{j-3}, z_{j-2}], \\ h^5 + 5h^4(z - z_{j-2}) + 10h^3(z - z_{j-2})^2 + 10h^2(z - z_{j-2})^3 + 5h(z - z_{j-2})^4 - 5(z - z_{j-2})^5, & z \in [z_{j-2}, z_{j-1}], \\ 26h^5 + 50h^4(z - z_{j-1}) + 20h^3(z - z_{j-1})^2 - 20h^2(z - z_{j-1})^3 - 20h(z - z_{j-1})^4 + 10(z - z_{j-1})^5, & z \in [z_{j-1}, z_j], \\ 26h^5 + 50h^4(z_{j+1} - z) + 20h^3(z_{j+1} - z)^2 - 20h^2(z_{j+1} - z)^3 - 20h(z_{j+1} - z)^4 + 10(z_{j+1} - z)^5, & z \in [z_j, z_{j+1}], \\ h^5 + 5h^4(z_{j+2} - z) + 10h^3(z_{j+2} - z)^2 + 10h^2(z_{j+2} - z)^3 + 5h(z_{j+2} - z)^4 - 5(z_{j+2} - z)^5, & z \in [z_{j+1}, z_{j+2}], \\ (z_{j+3} - z)^5, & z \in [z_{j+2}, z_{j+3}], \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

where $j = -2, -1, 0, \ldots, n+2$. For a sufficiently differentiable function $u(z)$, there corresponds a unique QnBS $U(z)$, s.t.

$$U(z) = \sum_{j=-2}^{n+2} e_j B_j(z), \tag{2.4}$$

where, e_j's are, constants, yet to be calculated. Let U_i, m_i, M_i, T_i and F_i denote the QnBS approximations for $u(z)$ and its first four derivatives at the i^{th} knot, respectively. Using (2.3) and (2.4), we have

$$U_i = U(z_i) = \sum_{j=i-2}^{i+2} e_j B_j(z_i) = \frac{1}{120}\left(e_{i-2} + 26e_{i-1} + 66e_i + 26e_{i+1} + e_{i+2}\right), \tag{2.5}$$

$$m_i = U'(z_i) = \sum_{j=i-2}^{i+2} e_j B_j'(z_i) = \frac{1}{24h}\left(-e_{i-2} - 10e_{i-1} + 10e_{i+1} + e_{i+2}\right), \tag{2.6}$$

$$M_i = U''(z_i) = \sum_{j=i-2}^{i+2} e_j B_j''(z_i) = \frac{1}{6h^2}\left(e_{i-2} + 2e_{i-1} - 6e_i + 2e_{i+1} + e_{i+2}\right), \tag{2.7}$$

$$T_i = U'''(z_i) = \sum_{j=i-2}^{i+2} e_j B_j'''(z_i) = \frac{1}{2h^3}\left(-e_{i-2} + 2e_{i-1} - 2e_{i+1} + e_{i+2}\right), \tag{2.8}$$

$$F_i = U^{(4)}(z_i) = \sum_{j=i-2}^{i+2} e_j B_j^{(4)}(z_i) = \frac{1}{h^4}\left(e_{i-2} - 4e_{i-1} + 6e_i - 4e_{i+1} + e_{i+2}\right). \tag{2.9}$$

Moreover, from (2.5)-(2.9), we can established the following relations [17, 24]

$$m_i = u'(z_i) + \frac{h^6}{5040}u^{(7)}(z_i) - \frac{h^8}{21600}u^{(9)}(z_i) + \cdots, \tag{2.10}$$

$$M_i = u''(z_i) + \frac{h^4}{720}u^{(6)}(z_i) - \frac{h^6}{3360}u^{(8)}(z_i) + \cdots, \tag{2.11}$$

$$T_i = u^{(3)}(z_i) - \frac{h^4}{240}u^{(7)}(z_i) + \frac{11h^6}{30240}u^{(9)}(z_i) + \cdots, \tag{2.12}$$

$$F_i = u^{(4)}(z_i) - \frac{h^2}{12}u^{(6)}(z_i) + \frac{h^4}{240}u^{(8)}(z_i) + \cdots. \tag{2.13}$$

From (2.10)-(2.13), we can write

$$\|m_i - u'(z_i)\|_\infty = \max_{0\leqslant j\leqslant n} \|m_i - u'(z_i)\| = O(h^6), \tag{2.14}$$

$$\|M_i - u''(z_i)\|_\infty = O(h^4), \tag{2.15}$$

$$\|T_i - u^{(3)}(z_i)\|_\infty = O(h^4), \tag{2.16}$$

$$\|F_i - u^{(4)}(z_i)\|_\infty = O(h^2). \tag{2.17}$$

The truncation error in F_i is $O(h^2)$. This provides a solid reason to construct a new approximation for fourth order derivative.

3. The new approximation for $u^{(4)}(z)$

Using (2.13), following expression can be established for F_{i-2} at the knot z_i, $(i = 2, 3, 4, \ldots, n-2)$ [10, 11]

$$F_{i-2} = u^{(4)}(z_{i-2}) - \frac{h^2}{12}u^{(6)}(z_{i-2}) + \frac{h^4}{240}u^{(8)}(z_{i-2}) + \cdots$$

$$= u^{(4)}(z_i) - 2hu^{(5)}(z_i) + \frac{23h^2}{12}u^{(6)}(z_i) - \frac{7h^3}{6}u^{(7)}(z_i) + \cdots.$$

We can derive similar relations for F_{i-1}, F_{i+1} and F_{i+2} at the i^{th} knot as

$$F_{i-1} = u^{(4)}(z_i) - hu^{(5)}(z_i) + \frac{5h^2}{12}u^{(6)}(z_i) - \frac{h^3}{12}u^{(7)}(z_i) + \cdots,$$
$$F_{i+1} = u^{(4)}(z_i) + hu^{(5)}(z_i) + \frac{5h^2}{12}u^{(6)}(z_i) + \frac{h^3}{12}u^{(7)}(z_i) + \cdots,$$
$$F_{i+2} = u^{(4)}(z_i) + 2hu^{(5)}(z_i) + \frac{23h^2}{12}u^{(6)}(z_i) + \frac{7h^3}{6}u^{(7)}(z_i) + \cdots.$$

Let $\widetilde{F}_i$ be the new approximation to $u^{(4)}(z_i)$, s.t.,

$$\widetilde{F}_i = \lambda_1 F_{i-2} + \lambda_2 F_{i-1} + \lambda_3 F_i + \lambda_4 F_{i+1} + \lambda_5 F_{i+2}. \tag{3.1}$$

The above expression returns five equations involving λ_i's as

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \qquad -2\lambda_1 - \lambda_2 + \lambda_4 + 2\lambda_5 = 0,$$
$$23\lambda_1 + 5\lambda_2 - \lambda_3 + 5\lambda_4 + 23\lambda_5 = 0, \qquad -14\lambda_1 - \lambda_2 + \lambda_4 + 14\lambda_5 = 0,$$
$$121\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 121\lambda_5 = 0.$$

Hence, $\lambda_1 = -\frac{1}{240}$, $\lambda_2 = \frac{1}{10}$, $\lambda_3 = \frac{97}{120}$, $\lambda_4 = \frac{1}{10}$, and $\lambda_5 = -\frac{1}{240}$. Substituting λ_i's back into (3.1), we obtain

$$\widetilde{F}_i = \frac{1}{240h^4}\left(-e_{i-4} + 28e_{i-3} + 92e_{i-2} - 604e_{i-1} + 970e_i - 604e_{i+1} + 92e_{i+2} + 28e_{i+3} - e_{i+4}\right). \tag{3.2}$$

Now we approximate $u^{(4)}(z)$ at z_0 using four neighboring values, as

$$\widetilde{F}_0 = \lambda_1 F_0 + \lambda_2 F_1 + \lambda_3 F_2 + \lambda_4 F_3, \tag{3.3}$$

where

$$F_0 = u^{(4)}(z_0) - \frac{h^2}{12}u^{(6)}(z_0) + \frac{h^4}{240}u^{(8)}(z_0) + \cdots,$$
$$F_1 = u^{(4)}(z_0) + hu^{(5)} + \frac{5h^2}{12}u^{(6)}(z_0) + \frac{h^3}{12}u^{(7)}(z_0) + \cdots,$$
$$F_2 = u^{(4)}(z_0) + 2hu^{(5)} + \frac{23h^2}{12}u^{(6)}(z_0) + \frac{7h^3}{6}u^{(7)}(z_0) + \cdots,$$
$$F_3 = u^{(4)}(z_0) + 3hu^{(5)} + \frac{53h^2}{12}u^{(6)}(z_0) + \frac{17h^3}{4}u^{(7)}(z_0) + \cdots.$$

The relation (3.3) returns the following four equations

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \qquad \lambda_2 + 2\lambda_3 + 3\lambda_4 = 0,$$
$$-\lambda_1 + 5\lambda_2 + 23\lambda_3 + 53\lambda_4 = 0, \qquad \lambda_2 + 14\lambda_3 + 51\lambda_4 = 0.$$

Hence, $\lambda_1 = \frac{7}{6}$, $\lambda_2 = -\frac{5}{12}$, $\lambda_3 = \frac{1}{3}$, and $\lambda_4 = -\frac{1}{12}$. Using these values in (3.3), we get

$$\widetilde{F}_0 = \frac{1}{12h^4}\left(14e_{-2} - 61e_{-1} + 108e_0 - 103e_1 + 62e_2 - 27e_3 + 8e_4 - e_5\right). \tag{3.4}$$

Similarly, involving four neighboring knots at z_1, we suppose

$$\widetilde{F}_1 = \lambda_1 F_0 + \lambda_2 F_1 + \lambda_3 F_2 + \lambda_4 F_3, \tag{3.5}$$

where

$$F_0 = u^{(4)}(z_1) - hu^{(5)}(z_1) + \frac{5h^2}{12}u^{(6)}(z_1) - \frac{h^3}{12}u^{(7)}(z_1) + \cdots,$$
$$F_1 = u^{(4)}(z_1) - \frac{h^2}{12}u^{(6)}(z_1) + \frac{h^4}{240}u^{(8)}(z_1) + \cdots,$$
$$F_2 = u^{(4)}(z_1) + hu^{(5)}(z_1) + \frac{5h^2}{12}u^{(6)}(z_1) + \frac{h^3}{12}u^{(7)}(z_1) + \cdots,$$
$$F_3 = u^{(4)}(z_1) + 2hu^{(5)}(z_1) + \frac{23h^2}{12}u^{(6)}(z_1) + \frac{7h^3}{6}u^{(7)}(z_1) + \cdots.$$

The expression (3.5), gives the following equations involving λ_i's

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \qquad -\lambda_1 + \lambda_3 + 2\lambda_4 = 0,$$
$$5\lambda_1 - \lambda_2 + 5\lambda_3 + 23\lambda_4 = 0, \qquad -\lambda_1 + \lambda_3 + 14\lambda_4 = 0.$$

Solving the above system, we get $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{6}$, $\lambda_3 = \frac{1}{12}$, and $\lambda_4 = 0$. Substituting λ_i's back into (3.5), we have

$$\tilde{F}_1 = \frac{1}{12h^4}\big(e_{-2} + 6e_{-1} - 33e_0 + 52e_1 - 33e_2 + 6e_3 + e_4\big). \tag{3.6}$$

Working on the same lines, following approximations at the knots z_{n-1} and z_n are obtained

$$\tilde{F}_{n-1} = \frac{1}{12h^4}\big(e_{n-4} + 6e_{n-3} - 33e_{n-2} + 52e_{n-1} - 33e_n + 6e_{n+1} + e_{n+2}\big), \tag{3.7}$$
$$\tilde{F}_n = \frac{1}{12h^4}\big(-e_{n-5} + 8e_{n-4} - 27e_{n-3} + 62e_{n-2} - 103e_{n-1} + 108e_n - 61e_{n+1} - 14e_{n+2}\big). \tag{3.8}$$

4. Description of the numerical method

In this section, we describe the proposed numerical method for solving BE equation. Applying usual finite difference formula and θ weighted scheme [1], the problem (1.1) is discretized in time direction as

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2} + \theta\left[\alpha u_{zz}^{j+1} + \beta u_{zzzz}^{j+1} + \gamma(u_{zz}^2)^{j+1}\right] + (1-\theta)\left[\alpha u_{zz}^j + \beta u_{zzzz}^j + \gamma(u_{zz}^2)^j\right] = 0, \tag{4.1}$$

where Δt is the temporal inclement, $0 \leqslant \theta \leqslant 1$ and u_i^{j+1} symbolizes $u(z_i, t_j + \Delta t)$. The non-linear term $(u_{zz})^{j+1}$ is linearized as [20]

$$(u_{zz})^{j+1} = 2(u_{zz})^j(u_{zz})^{j+1} - (u_{zz}^2)^j. \tag{4.2}$$

Substituting (4.2) into (4.1), we get

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2} + \theta\left[\alpha u_{zz}^{j+1} + \beta u_{zzzz}^{j+1} + \gamma\left\{2(u_{zz})^j(u_{zz})^{j+1} - (u_{zz}^2)^j\right\}\right]$$
$$+ (1-\theta)\left[\alpha u_{zz}^j + \beta u_{zzzz}^j + \gamma(u_{zz}^2)^j\right] = 0.$$

Rearranging the above relation, we get

$$\begin{aligned} u^{j+1} &+ \left[\alpha\theta\Delta t^2 + 2\gamma\theta\Delta t^2(u_{zz})^j\right](u_{zz})^{j+1} + \beta\theta\Delta t^2 u_{zzzz}^{j+1} \\ &= 2u^j - \Delta t^2(1-\theta)\left[\alpha u_{zz}^j + \beta u_{zzzz}^j + \gamma(u_{zz}^2)^j\right] + \theta\gamma\Delta t^2(u_{zz}^2)^j - u^{j-1}. \end{aligned} \tag{4.3}$$

Now, we uniformly partition the interval $[a, b]$ by $n+1$ equidistant knots $z_i = z_0 + ih$, $i = 0(1)n$, where $n \in \mathbb{Z}^+$, $a = z_0$, $b = z_n$ and $h = \frac{1}{n}(b-a)$. Let us consider that the QnBS solution to (1.1) at j^{th} time level is given by

$$U(z, t_j) = \sum_{m=-2}^{n+2} e_m^j B_m(z),$$

where e_m^j's, the time dependent constants, are to be calculated. Substituting the approximations for u and its derivatives at the knot z_i, equation (4.3) takes the following form

$$U_i^{j+1} + \nu_i^j M_i^{j+1} + \mu F_i^{j+1} = w_i^j, \tag{4.4}$$

where

$$\nu_i^j = \alpha\theta\Delta t^2 + 2\gamma\theta\Delta t^2 (u_{zz})_i^j, \quad \mu = \beta\theta\Delta t^2,$$
$$w_i^j = 2u_i^j - \Delta t^2(1-\theta)\Big[\alpha(u_{zz})_i^j + \beta(u_{zzzz})_i^j + \gamma(u_{zz}^2)_i^j\Big] + \theta\gamma\Delta t^2 (u_{zz}^2)_i^j - u - i^{j-1}.$$

Using (2.5)-(2.8), (3.2), (3.4), and (3.6)-(3.8) in (4.4), for $i = 0, 1, 2, 3, \ldots, n$, we obtain the following linear equations involving $n+5$ unknowns.

$$\begin{aligned}
&\frac{1}{120}(e_{-2}^{j+1} + 26e_{-1}^{j+1} + 66e_0^{j+1} + 26e_1^{j+1} + e_2^{j+1}) + \frac{\nu_0^j}{6h^2}(e_{-2}^{j+1} + 2e_{-1}^{j+1} - 6e_0^{j+1} + 2e_1^{j+1} + e_2^{j+1})\\
&\quad + \frac{\mu}{12h^4}(14e_{-2}^{j+1} - 61e_{-1}^{j+1} + 108e_0^{j+1} - 103e_1^{j+1} + 62e_2^{j+1} - 27e_3^{j+1} + 8e_4^{j+1} - e_5^{j+1}) = w_0^j,\\
&\frac{1}{120}(e_{-1}^{j+1} + 26e_0^{j+1} + 66e_1^{j+1} + 26e_2^{j+1} + e_3^{j+1}) + \frac{\nu_1^j}{6h^2}(e_{-1}^{j+1} + 2e_0^{j+1} - 6e_1^{j+1} + 2e_2^{j+1} + e_3^{j+1})\\
&\quad + \frac{\mu}{12h^4}(e_{-2}^{j+1} + 6e_{-1}^{j+1} - 33e_0^{j+1} + 52e_1^{j+1} - 33e_2^{j+1} + 6e_3^{j+1} + e_4^{j+1}) = w_1^j,\\
&\frac{1}{120}(e_{i-2}^{j+1} + 26e_{i-1}^{j+1} + 66e_i^{j+1} + 26e_{i+1}^{j+1} + e_{i+2}^{j+1}) + \frac{\nu_i^j}{6h^2}(e_{i-2}^{j+1} + 2e_{i-1}^{j+1} - 6e_i^{j+1} + 2e_{i+1}^{j+1} + e_{i+2}^{j+1})\\
&\quad + \frac{\mu}{240h^4}(-e_{i-4}^{j+1} + 28e_{i-3}^{j+1} + 92e_{i-2}^{j+1} - 604e_{i-1}^{j+1} + 970e_i^{j+1} - 604e_{i+1}^{j+1} + 92e_{i+2}^{j+1} + 28e_{i+3}^{j+1} - e_{i+4}^{j+1})\\
&= w_i^j, \; i = 2, 3, 4, \ldots, n-2,\\
&\frac{1}{120}(e_{n-3}^{j+1} + 26e_{n-2}^{j+1} + 66e_{n-1}^{j+1} + 26e_n^{j+1} + e_{n+1}^{j+1}) + \frac{\nu_{n-1}^j}{6h^2}(e_{n-3}^{j+1} + 2e_{n-2}^{j+1} - 6e_{n-1}^{j+1} + 2e_n^{j+1} + e_{n+1}^{j+1})\\
&\quad + \frac{\mu}{12h^4}(e_{n-4}^{j+1} + 6e_{n-3}^{j+1} - 33e_{n-2}^{j+1} + 52e_{n-1}^{j+1} - 33e_n^{j+1} + 6e_{n+1}^{j+1} + e_{n+2}^{j+1}) = w_{n-1}^j,\\
&\frac{1}{120}(e_{n-2}^{j+1} + 26e_{n-1}^{j+1} + 66e_n^{j+1} + 26e_{n+1}^{j+1} + e_{n+2}^{j+1}) + \frac{\nu_n^j}{6h^2}(e_{n-2}^{j+1} + 2e_{n-1}^{j+1} - 6e_n^{j+1} + 2e_{n+1}^{j+1} + e_{n+2}^{j+1})\\
&\quad + \frac{\mu}{12h^4}(-e_{n-5} + 8e_{n-4} - 27e_{n-3} + 62e_{n-2} - 103e_{n-1} + 108e_n - 61e_{n+1} - 14e_{n+2}) = w_n^j.
\end{aligned} \tag{4.5}$$

Four more equations are obtained from the boundary conditions (1.3), as

$$(e_{-2}^{j+1} + 26e_{-1}^{j+1} + 66e_0^{j+1} + 26e_1^{j+1} + e_2^{j+1})/120 = \psi_1(t_{j+1}), \tag{4.6}$$
$$(-e_{-2}^{j+1} - 10e_{-1}^{j+1} + 10e_1^{j+1} + e_2^{j+1})/24h = \psi_3(t_{j+1}), \tag{4.7}$$
$$(-e_{n-2}^{j+1} - 10e_{n-1}^{j+1} + 10e_{n+1}^{j+1} + e_{n+2}^{j+1})/24h = \psi_4(t_{j+1}), \tag{4.8}$$
$$(e_{n-2}^{j+1} + 26e_{n-1}^{j+1} + 66e_n^{j+1} + 26e_{n+1}^{j+1} + e_{n+2}^{j+1})/120 = \psi_2(t_{j+1}). \tag{4.9}$$

The set of equations (4.5)-(4.9) are a system of linear equations which can be expressed in matrix from as

$$AC^{j+1} = B, \tag{4.10}$$

where A denotes the matrix of coefficients with order $n+5$, B is column vector with order $n+5$ and the column vector $C^{j+1} = \left[e_{-2}^{j+1}\ e_{-1}^{j+1}\ e_0^{j+1}\ \cdots\ e_{n+2}^{j+1}\right]^T$ contains the control points at the $(j+1)^{th}$ time level.

Before starting any computation using (4.10), we obtain the following set of equations from initial condition (1.2)

$$\begin{aligned}(-e_{-2}^0 - 10e_{-1}^0 + 10e_1^0 + e_2^0)/24h &= g_1'(z_0),\\ (e_{-2}^0 + 2e_{-1}^0 - 6e_0^0 + 2e_1^0 + e_2^0)/6h^2 &= g_1''(z_0),\\ (e_{i-2}^0 + 26e_{i-1}^0 + 66e_i^0 + 26e_{i+1}^0 + e_{i+2}^0)/120 &= g_1(z_i),\ i = 0:1:n,\\ (e_{n-2}^0 + 2e_{n-1}^0 - 6e_n^0 + 2e_{n+1}^0 + e_{n+2}^0)/6h^2 &= g_1''(z_n),\\ (-e_{n-2}^0 - 10e_{n-1}^0 + 10e_{n+1}^0 + e_{n+2}^0)/24h &= g_1'(z_n).\end{aligned}$$

The above system can be expressed in matrix form as

$$AC^0 = B.$$

The unknown column vector C^0 is determined by a modified form of well known Thomas algorithm. The numerical computations are executed in Mathematica 9.

5. Stability Analysis

For $\theta = 1/2$, the linearized form of (4.1) is given by

$$2u^{j+1} + \alpha\Delta t^2(u_{zz})^{j+1} + \beta\Delta t^2(u_{zzzz})^{j+1} = 4u^j - \alpha\Delta t^2(u_{zz})^j + \beta\Delta t^2(u_{zzzz})^j - 2u^{j-1}. \tag{5.1}$$

By substituting the approximations for u and its derivatives at i^{th} knot in (5.1), we get

$$2U_i^{j+1} + \alpha\Delta t^2 M_i^{j+1} + \beta\Delta t^2 F_i^{j+1} = 4U_i^j - \alpha\Delta t^2 M_i^j - \beta\Delta t^2 F_i^j - 2U_i^{j-1}. \tag{5.2}$$

Using (2.5)-(2.8) and (3.2) in (5.2), we have

$$\begin{aligned}&-d_0e_{i-4}^{j+1} + 28d_0e_{i-3}^{j+1} + (4h^4 + 92d_0 + 40h^2d_1)e_{i-2}^{j+1} + (104h^4 - 604d_0 + 80h^2d_1)e_{i-1}^{j+1}\\ &+ (264h^4 + 970d_0 - 240h^2d_1)e_i^{j+1} + (104h^4 - 604d_0 + 80h^2d_1)e_{i+1}^{j+1} + (4h^4 + 92d_0 + 40h^2d_1)e_{i+2}^{j+1}\\ &+ 28d_0e_{i+3}^{j+1} - d_0e_{i+4}^{j+1}\\ &= d_0e_{i-4}^j - 28d_0e_{i-3}^j + (8h^4 - 92d_0 - 40h^2d_1)e_{i-2}^j\\ &+ (208h^4 + 604d_0 - 80h^2d_1)e_{i-1}^j + (528h^4 - 970d_0 + 240h^2d_1)e_i^j + (208h^4 + 604d_0 - 80h^2d_1)e_{i+1}^j\\ &+ (8h^4 - 92d_0 - 40h^2d_1)e_{i+2}^j - 28d_0e_{i+3}^j + d_0e_{i+4}^j - \frac{1}{120}(e_{i-2}^{j-1} + 26e_{i-1}^{j-1} + 66e_i^{j-1} + 26e_{i+1}^{j-1} + e_{i+2}^{j-1}),\end{aligned} \tag{5.3}$$

where $d_0 = \beta\Delta t^2$ and $d_1 = \alpha\Delta t^2$. Now, following [26], we put $e_i^j = \xi^j e^{\iota m\varphi}$ into (5.3)

$$\begin{aligned}&\xi^{j+1}\Big[-d_0e^{\iota(m-4)\varphi} + 28d_0e^{\iota(m-3)\varphi} + (4h^4 + 92d_0 + 40h^2d_1)e^{\iota(m-2)\varphi} + (104h^4 - 604d_0\\ &+ 80h^2d_1)e^{\iota(m-1)\varphi} + (264h^4 + 970d_0 - 240h^2d_1)e^{\iota m\varphi} + (104h^4 - 604d_0 + 80h^2d_1)e^{\iota(m+1)\varphi} + (4h^4 + 92d_0\\ &+ 40h^2d_1)e^{\iota(m+2)\varphi} + 28d_0e^{\iota(m+3)\varphi} - d_0e^{\iota(m+4)\varphi}\Big]\\ &= \xi^j\Big[d_0e^{\iota(m-4)\varphi} - 28d_0e^{\iota(m-3)\varphi} + (8h^4 - 92d_0 - 40h^2d_1)e^{\iota(m-2)\varphi}\\ &+ (208h^4 + 604d_0 - 80h^2d_1)e^{\iota(m-1)\varphi} + (528h^4 - 970d_0 + 240h^2d_1)e^{\iota m\varphi} + (208h^4\\ &+ 604d_0 - 80h^2d_1)e^{\iota(m+1)\varphi} + (8h^4 - 92d_0 - 40h^2d_1)e^{\iota(m+2)\varphi} - 28d_0e^{\iota(m+3)\varphi} + d_0e^{\iota(m+4)\varphi}\Big]\\ &- \frac{\xi^{j-1}}{120}\Big[e^{\iota(m-2)\varphi} + 26e^{\iota(m-1)\varphi} + 66e^{\iota m\varphi} + 26e^{\iota(m+1)\varphi} + e^{\iota(m+2)\varphi}\Big],\end{aligned} \tag{5.4}$$

where $\iota = \sqrt{-1}$, $\varphi = \zeta h$, and ζ denotes the mode number. Simplifying (5.4), we obtain

$$\begin{aligned}
&\xi^{j+1}\Big[-d_0e^{-4\iota\varphi}+28d_0e^{-3\iota\varphi}+(4h^4+92d_0+40h^2d_1)e^{-2\iota\varphi}+(104h^4-604d_0+80h^2d_1)e^{-\iota\varphi}\\
&+(264h^4+970d_0-240h^2d_1)+(104h^4-604d_0+80h^2d_1)e^{\iota\varphi}+(4h^4+92d_0+40h^2d_1)e^{2\iota\varphi}\\
&+28d_0e^{3\iota\varphi}-d_0e^{4\iota\varphi}\Big]\\
&=\xi^{j}\Big[d_0e^{-4\iota\varphi}-28d_0e^{-3\iota\varphi}+(8h^4-92d_0-40h^2d_1)e^{-2\iota\varphi}\\
&+(208h^4+604d_0-80h^2d_1)e^{-\iota\varphi}+(528h^4-970d_0+240h^2d_1)+(208h^4+604d_0-80h^2d_1)e^{\iota\varphi}\\
&+(8h^4-92d_0-40h^2d_1)e^{2\iota\varphi}-28d_0e^{3\iota\varphi}+d_0e^{4\iota\varphi}\Big]-\frac{\xi^{j-1}}{120}\Big[e^{-2\iota\varphi}+26e^{-\iota\varphi}+66+26e^{\iota\varphi}+e^{2\iota\varphi}\Big].
\end{aligned} \tag{5.5}$$

Simplifying (5.5), we get

$$\begin{aligned}
&\xi^{2}\Big[-d_0\cos(4\varphi)+28d_0\cos(3\varphi)+(4h^4+92d_0+40h^2d_1)\cos(2\varphi)+(104h^4-604d_0+80h^2d_1)\cos\varphi\\
&+(264h^4+485d_0-120h^2d_1)\Big]\\
&=\xi\Big[d_0\cos(4\varphi)-28d_0\cos(3\varphi)+(8h^4-92d_0-40h^2d_1)\cos(2\varphi)\\
&+(208h^4+604d_0-80h^2d_1)\cos\varphi+(264h^4-485d_0+120h^2d_1)\Big]-\frac{1}{60}\Big[\cos(2\varphi)+26\cos\varphi+33\Big].
\end{aligned} \tag{5.6}$$

The expression (5.6) leads to the following characteristic equation

$$p(\xi) = A_1\xi^2 + A_2\xi + A_3,$$

where

$$\begin{aligned}
A_1 &= -d_0\cos(4\varphi)+28d_0\cos(3\varphi)+(4h^4+92d_0+40h^2d_1)\cos(2\varphi)+(104h^4-604d_0+80h^2d_1)\cos\varphi\\
&= +(264h^4+485d_0-120h^2d_1),\\
A_2 &= -d_0\cos(4\varphi)-28d_0\cos(3\varphi)+(8h^4-92d_0-40h^2d_1)\cos(2\varphi)-(208h^4+604d_0-80h^2d_1)\cos\varphi\\
&= -(264h^4-485d_0+120h^2d_1),\\
A_3 &= \frac{1}{60}\Big[\cos(2\varphi)+26\cos\varphi+33\Big].
\end{aligned}$$

Applying the transformation $\xi = \frac{1+\omega}{1-\omega}$ and using Routh-Hurwitz criterion [19], we have

$$(1-\omega^2)p\left(\frac{1+\omega}{1-\omega}\right) = (A_1 - A_2 + A_3)\omega^2 + 2(A_1 - A_3)\omega + (A_1 + A_2 + A_3).$$

It can be easily verified that $(A_1 - A_2 + A_3) > 0$, $(A_1 - A_3) > 0$, $(A_1 + A_2 + A_3) > 0$, therefore, $|\xi| \leqslant 1$, the numerical scheme is unconditionally stable.

6. Error analysis

Using the QnBS approximations (2.5)-(2.8), we can establish the following relations [17, 24]

$$U'(z_i) = u'(z_i) + \frac{h^6}{5040}u^{(7)}(z_i) - \frac{h^8}{21600}u^{(9)}(z_i) + \frac{h^{10}}{1036800}u^{(11)}(z_i) + \cdots,$$

$$U''(z_i) = u''(z_i) + \frac{h^4}{720}u^{(6)}(z_i) - \frac{h^6}{3360}u^{(8)}(z_i) + \frac{h^8}{86400}u^{(10)}(z_i) + \cdots,$$
$$U'''(z_i) = u^{(3)}(z_i) - \frac{h^4}{240}u^{(7)}(z_i) + \frac{11h^6}{30240}u^{(9)}(z_i) - \frac{h^8}{28800}u^{(11)}(z_i) + \cdots.$$

Now, using (2.7), (2.8), and (3.2), we can write

$$\begin{aligned} h^4U^{(4)}(z_i) &= \frac{h^2}{40}[-U''(x_{i-2}) + 114U''(x_{i-1}) - 142U''(z_i) + 30U''(x_{i+1}) - U''(x_{i+2})] \\ &+ \frac{7h^3}{10}[U'''(x_{i-1}) + 2U'''(z_i)]. \end{aligned} \tag{6.1}$$

Employing the operator notation, $E^\lambda(U'(z_i)) = U'(x_{i+\lambda})$, $\lambda \in \mathbb{Z}$, Eq. (6.1) is written as [8]

$$h^4U^{(4)}(z_i) = \frac{h^2}{40}[-E^{-2} + 114E^{-1} - 142 + 30E^1 - E^2]u''(z_i) + \frac{7h^3}{10}[E^{-1} + 2]u^{(3)}(z_i). \tag{6.2}$$

Again, using $E = e^{hD}$ in (6.2), we get

$$h^4U^{(4)}(z_i) = \frac{h^2}{40}\left[-e^{-2hD} + 114e^{-hD} - 142 + 30e^{hD} - e^{2hD}\right]u''(z_i) + \frac{7h^3}{10}\left[e^{-hD} + 2\right]u^{(3)}(z_i).$$

Expanding in powers of hD, we obtain

$$\begin{aligned} h^4U^{(4)}(z_i) &= \frac{h^2}{40}\left[-84hD + 68h^2D^2 - 14h^3D^3 + \frac{14}{3}h^4D^4 + \cdots\right]u''(z_i) \\ &+ \frac{7h^3}{10}\left[3 - hD + \frac{1}{2}h^2D^2 - \frac{1}{6}h^3D^3 + \frac{1}{24}h^4D^4\right]u^{(3)}(z_i). \end{aligned}$$

Simplifying the above relation, we have

$$U^{(4)}(z_i) = u^{(4)}(z_i) + \frac{7h^3}{600}u^{(7)}(z_i) - \frac{19h^4}{3600}u^{(8)}(z_i) + \cdots.$$

Now, the linearized form of BE (1.1) can be written as

$$u_{tt} = G(z, t, u), \tag{6.3}$$

where $G = -\alpha u_{zz} - \beta u_{zzzz}$, with $u_i^{j+1} - 2u_i^j + u_i^{j-1} = \Delta t^2\left[\theta G_i^{j+1} + (1-\theta)G_i^j\right]$. Using Taylor series about $(j+\theta)\Delta t$, we obtain

$$\begin{aligned} &(u_{tt})_i + \theta\Delta t(u_{ttt})_i + \frac{6\theta^2+1}{12}\Delta t^2(u_{tttt})_i + \frac{\theta(2\theta^2-1)}{12}\Delta t^3(u_{ttttt})_i + \cdots \\ &= G_i - \frac{\theta(\theta-1)}{2}\Delta t^2(G_{tt})_i + \frac{\theta(\theta-1)(2\theta-1)}{2}\Delta t^3(G_{ttt})_i + \cdots. \end{aligned} \tag{6.4}$$

Using (6.3) in (6.4), we get

$$(u_{tt})_i = G_i - \theta\Delta t(G_t)_i + \frac{1-6\theta(2\theta-1)}{12}\Delta t^2(G_{tt})_i + \cdots. \tag{6.5}$$

From (6.3), we define the truncation error $e_i = (u_{tt})_i - \big(-\alpha(U_{zz})_i - \beta(U_{zzzz})_i\big)$ as

$$e_i = \theta\Delta t(u_{ttt})_i + \frac{7\beta h^3}{600}(u_{zzzzzzz})_i + \cdots. \tag{6.6}$$

Hence, the presented scheme for BE equation is $O(\Delta t + h^3)$ accurate.

7. Numerical results

In this section, we shall show the validity of our proposed numerical scheme by considering some test problems. The accuracy of the scheme is tested by the following error norms [22]

$$L_\infty = \max_{0 \leqslant i \leqslant n} |u_i - U_i| \ , \ \ L_2 = \sqrt{h \sum_{i=0}^{n} (u_i - U_i)^2},$$

where U_i and u_i represent the numerical and exact solutions at the i^{th} knot, respectively. The computational outcomes are compared with those obtained by quintic B-spline collocation method (QnBSM) [20] and quintic trigonometric B-spline collocation method (QnTBSM) [25].

Problem 7.1. Consider the following BE [20]

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, \quad x \in [a, b], \ t \in [0, T].$$

The exact solution is given by $u(x,t) = -\lambda \mathrm{sech}^2\left[\sqrt{\frac{\lambda}{6}}(x - c_1 t + \mu)\right] - (c_2 + \frac{1}{2})$. The initial/boundary conditions can be extracted from exact solution. Table 1 displays a comparison of maximum absolute error at different time steps with typical QnBSM [20] corresponding to different choices of mesh size when $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 2$, $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$, and $\Delta t = 0.05$. It can be seen that our numerical outcomes are better than QnBSM. In Figure 1, the approximate and exact solution at $t = 1, 5, 10, 15, 20$ are plotted for $-40 \leqslant x \leqslant 40$, $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$, and $\mu t = 60$. The three dimensional graphs of exact and approximate solution are displayed in Figure 2 when $-40 \leqslant x \leqslant 40$, $0 \leqslant t \leqslant 1$, $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$, and $\Delta t = 0.01$.

Table 1: Maximum absolute error for Example 7.1, when $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$.

t	h	μ	QnBSM [20]	Proposed method
0.5	1/40	30	8.2943×10^{-7}	1.0520×10^{-10}
1.0	1/60	40	7.3326×10^{-9}	1.4798×10^{-13}
1.5	1/80	50	6.4525×10^{-11}	1.9831×10^{-15}
2.0	1/100	60	5.2066×10^{-13}	1.4215×10^{-17}

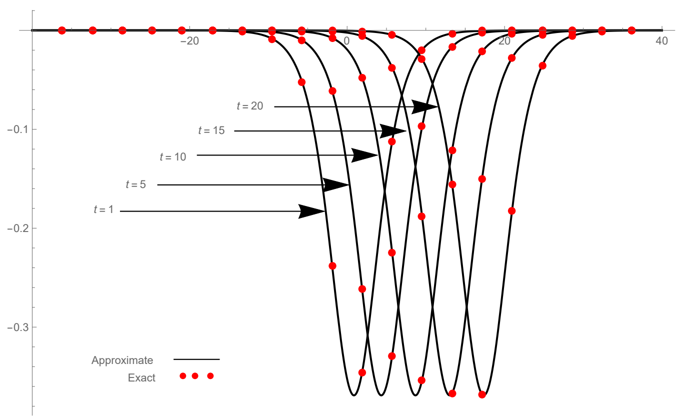
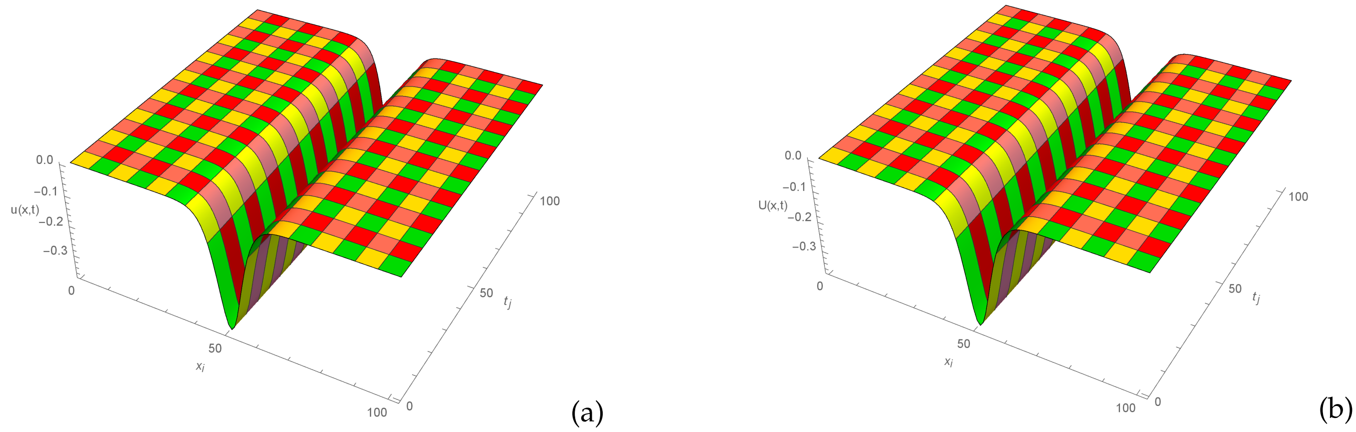


Figure 1: Numerical and exact solution for Problem 7.1 using $-40 \leqslant x \leqslant 40$, $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$, and $\mu = 60$.



(a) Exact solution

(b) Approximate solution

Figure 2: Exact and approximate solution for Problem 7.1 when $-40 \leqslant x \leqslant 40$, $0 \leqslant t \leqslant 1$, $\lambda = 0.369$, $c_1 = 0.868$, $c_2 = -0.5$, and $\Delta t = 0.01$.

Problem 7.2. Consider the following BE [25]

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0, \quad x \in [a, b], \ t \in [0, T].$$

The exact solution is given by $u(x, t) = \frac{\lambda^2-1}{2} sech^2\left[\frac{\sqrt{\lambda^2-1}}{2}(x - \lambda t)\right]$. The initial/boundary conditions can be extracted from exact solution. The computational error norms L_2 and L_∞ corresponding to different time stages are reported in Table 2 by setting $0 \leqslant x \leqslant 10$, $0 \leqslant t \leqslant 0.1$, $\lambda = 1.116$, and $\Delta t = 0.01$. The approximate results are found to be better than QnTBSM [25]. The one dimensional plots of exact and approximate solutions are shown in Figure 3 using $-10 \leqslant x \leqslant 15$, $t = 1, 2, 3, 4, 5$, $\lambda = 1.116$, and $\Delta t = 0.01$. Figure 4 exhibits the 3D plots of approximate and analytical exact solution when $-10 \leqslant x \leqslant 15$, $0 \leqslant t \leqslant 1$, $\lambda = 1.116$, and $\Delta t = 0.01$.

Table 2: Computational error norms for Example 7.2 when $\lambda = 1.116$.

t	QnTBS [25]		Proposed method	
	L_2	L_∞	L_2	L_∞
0.02	1.01×10^{-3}	5.20×10^{-4}	1.66×10^{-6}	9.73×10^{-7}
0.04	2.05×10^{-3}	1.05×10^{-3}	7.30×10^{-6}	5.55×10^{-6}
0.06	3.12×10^{-3}	1.59×10^{-3}	1.73×10^{-5}	1.38×10^{-5}
0.08	4.22×10^{-3}	2.14×10^{-3}	3.19×10^{-5}	2.59×10^{-5}
0.10	5.37×10^{-3}	2.70×10^{-3}	5.10×10^{-5}	4.18×10^{-5}

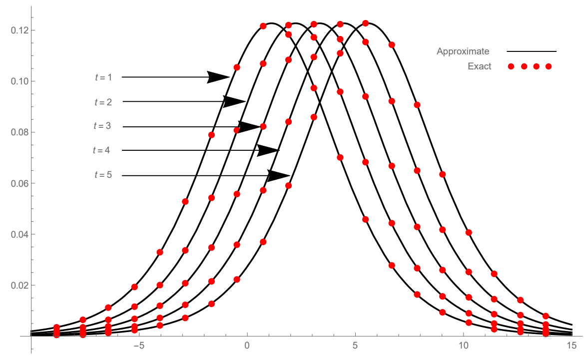
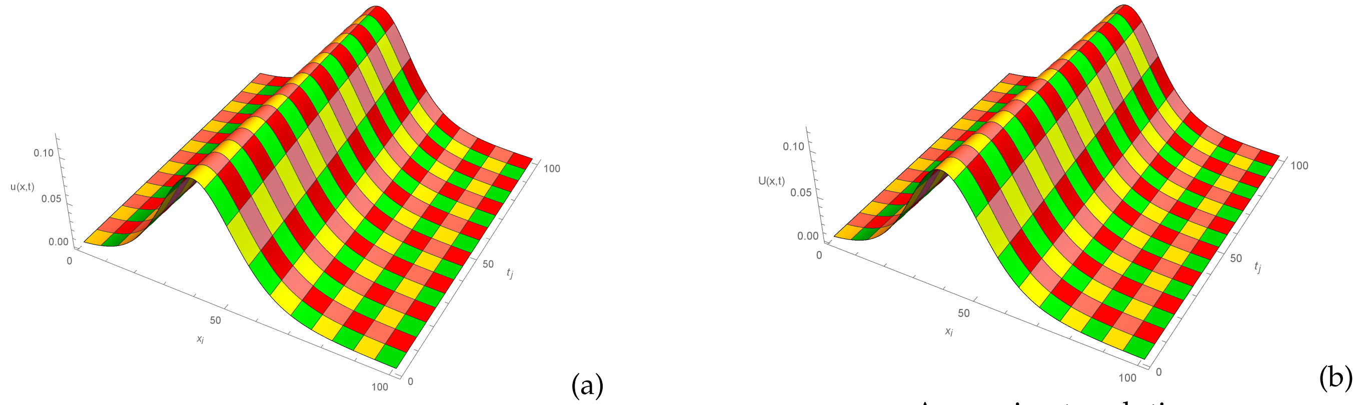


Figure 3: Numerical and exact solution for Problem 7.2 using $n = 100$, $-10 \leqslant x \leqslant 15$, $t = 1, 2, 3, 4, 5$, $\lambda = 1.116$, and $\Delta t = 0.01$.



(a) Exact solution

(b) Approximate solution

Figure 4: Exact and approximate solution for Problem 7.2 when $n = 100$, $-10 \leqslant x \leqslant 15$, $0 \leqslant t \leqslant 1$, $\lambda = 1.116$, and $\Delta t = 0.01$.

8. Conclusion

In this article, a new quintic B-spline approximation for fourth order derivative has been proposed. This new approximation is then used with typical quintic B-spline functions and theta weighted scheme for numerical approximation of Boussinesq equation. The presented numerical method is shown to be stable and $O(\Delta t + h^3)$ convergent. Some test examples have been taken from literature and the numerical outcomes are compared with those obtained by QnBSM [20] and QnTBSM [25]. It is revealed that the computational algorithm performs superior to the current variants on the topic by virtue of its better accuracy and simple implementation.

References

[1] M. Abbas, M. K. Iqbal, B. Zafar, S. B. M. Zin, *New cubic B-spline approximations for solving nonlinear Korteweg-de Vries equation*, Indian J. Sci. Tech., **12** (2019), 9 pages. 4

[2] M. A. Akinlar, A. Secer, *Wavelet-Petrov-Galerkin Method for Numerical Solution of Boussinesq Equation*, Appl. Mech. Mater., **319** (2013), 451–455. 1

[3] A. G. Bratsos, I. T. Famelis, A. M. Prospathopoulos, *A parametric finite-difference method for shallow sea waves*, Internat. J. Numer. Methods Fluids, **53** (2007), 129–147. 1

[4] A. G. Bratsos, C. Tsitouras, D. G. Natsis, *Linearized numerical schemes for the Boussinesq equation*, Appl. Numer. Anal. Comput. Math., **2** (2005), 34–53. 1

[5] C. Chávez, C. Fuentes, M. Zavala, F. Brambila, *Numerical solution of the Boussinesq equation: Application to the agricultural drainage*, Afr. J. Agricul. Rese., **6** (2011), 4210–4222. 1

[6] M. T. Darvishi, M. Najafi, A. M. Wazwaz, *Traveling wave solutions for Boussinesq-like equations with spatial and spatial-temporal dispersion*, Romanian Rep. Phys., **70** (2018), 12 pages. 1

[7] C. de Boor *A practical guide to splines*, Springer-Verlag, New York, (1978). 2

[8] D. J. Fyfe, *The use of cubic splines in the solution of two-point boundary value problems*, Comput. J., **12** (1969/70), 188–192. 6

[9] H. Grad, P. N. Hu, *Unified shock profile in a plasma*, Phys. Fluids, **10** (1967), 2596–2602. 1

[10] M. K. Iqbal, M. Abbas, I. Wasim, *New cubic B-spline approximation for solving third order Emden–Flower type equations*, Appl. Math. Comput., **331** (2018), 319–333. 3

[11] M. K. Iqbal, M. Abbas, B. Zafar, *New Quartic B-Spline Approximation for Numerical Solution of Third Order Singular Boundary Value Problems*, Punjab Univ. J. Math. (Lahore), **51** (2019), 43–59. 3

[12] M. S. Ismail, A. G. Bratsos, *A predictor-corrector scheme for the numerical solution of the Boussinesq equation*, J. Appl. Math. Comput., **13** (2003), 11–27. 1

[13] M. S. Ismail, F. Mosally, *A fourth order finite difference method for the good Boussinesq equation*, Abstr. Appl. Anal., **2014** (2014), 10 pages. 1

[14] R. S. Johnson, *A non-linear equation incorporating damping and dispersion*, J. Fluid Mech., **42** (1970), 49–60. 1

[15] R. S. Johnson, *Shallow water waves on a viscous fluid–the undular bore*, Phys. Fluids, **15** (1972), 1693–1699. 1

[16] T. S. Jung, *A new dispersion-relation preserving method for integrating the classical Boussinesq equation*, Commun. Nonlinear Sci. Numer. Simul., **43** (2017), 118–138. 1

[17] R. K. Lodhi, H. K. Mishra, *Solution of a class of fourth order singular singularly perturbed boundary value problems by quintic B-spline method*, J. Nigerian Math. Soc., **35** (2016), 257–265. 2, 2, 6

[18] D. Lu, J. Shen, Y. Cheng, *The approximate solutions of nonlinear Boussinesq equation*, J. Phys. Conference Series, **710** (2016), 8 pages. 1

[19] S. Sharifi, J. Rashidinia, *Numerical solution of hyperbolic telegraph equation by cubic B-spline collocation method*, Appl. Math. Comput., **281** (2016), 28–38. 5

[20] S. S. Siddiqi, S. Arshed, *Quintic B-spline for the numerical solution of the good Boussinesq equation*, J. Egyptian Math. Soc., **22** (2014), 209–213. 1, 4, 7, 7.1, 7, 1, 8

[21] M. Uddin, S. Haq, M. Ishaq, *RBF-Pseudospectral Method for the Numerical Solution of Good Boussinesq Equation*, Appl. Math. Sci. (Ruse), **6** (2012), 2403–2410. 1

[22] I. Wasim, M. Abbas, M. K. Iqbal, *Numerical solution of modified forms of Camassa-Holm and Degasperis-Procesl equations via quartic B-spline collocation method*, Commun. Math. Appl., **9** (2018), 393–409. 7

[23] A.-M. Wazwaz, *Multiple-soliton solutions for the Boussinesq equation*, Appl. Math. Comput., **192** (2007), 479–486. 1

[24] X.-P. Xu, F.-G. Lang, *Quintic B-spline method for function reconstruction from integral values of successive subintervals*, Numer. Algorithms, **66** (2014), 223–240. 2, 2, 6

[25] N. F. Zakaria, N. A. Hassan, N. N. A. Hamid, A. A. Majid, A. I. M. Ismail, *Solving Boussinesq equation using quintic B-spline and quintic trigonometric B-spline interpolation methods*, AIP Conference Proceedings, **1830** (2017), 11 pages. 1, 1, 7, 7.2, 7, 2, 8

[26] S. M. Zin, M. Abbas, A. A. Majid, A. I. M. Ismail, *A new trigonometric spline approach to numerical solution of generalized nonlinear Klien-Gordon equation*, PloS one, **9** (2014), 9 pages. 5