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# Analytical solution of the viscous flow over a stretching sheet by multi-step optimal homotopy asymptotic method

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## Abstract

In this article the governing equations of viscous flow over a stretching sheet are reduced to ordinary boundary value problem by using a similarity transformation. The new analytical approach Multi-step Optimal Homotopy Asymptotic Method (MOHAM) is formulated and used for the boundary value problem. The numerical comparison of Homotopty Perturbation Method (HPM), exact solution, DTM, and numerical results (Runge Kutta Method) revealed that the new technique is powerful method for solving boundary layer equations. Also the solution is plotted for various values of β.

**Keywords:** MOHAM, boundary layer problem, Navier Stokes equations, DTM.

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## 1. Introduction

Many engineering and industrial problems often arise due to stretching sheets. Such flows engrossed special attention due to its application is engineering and science [2, 6, 21]. The mechanical properties due to stretching and cold drawing rates were investigated by Sakiadis [19]. The suction/injection to the surface which produce evaporation/condensation were considered by Gupta [7]. The axisymetric surface of three dimensional flow is studied by Wang [26]. The closed form solution for the flow of an incompressible stretching flow over a plate was found by Crane [5]. The HPM and non iterative solution is used by Ariel [3] for the solution of the stretching problem. The boundary layer models can be reduced to system of nonlinear ordinary differential equations (ODEs). For the solution of ODEs numerical and analytical methods are used. The analytical methods have more advantage over numerical methods such as such as rapid convergence, no assumption of initial guess, discritization or linearization. The perturbation Methods (PMs) [4] were also used for the solution of nonlinear ODEs. The PMs involve a small parameter and it's in appropriate choice can affect the solution. The analytical methods were introduced that does not require the assumption of small parameter. The HPM was first introduced by He.

[9, 10] and Homotopy Analysis Method (HAM) introduced by Laio [12, 13]. Some analytical techniques on the problem related to stretching sheets have been used by Rashidi et al. and Ellahi [6, 8, 14, 17, 18, 20, 27]. These two methods combined the homotopy and perturbation. These methods also required the initial guess. We have modified the OHAM [11, 15, 22–25] for large domain.

## 2. Basic mathematical theory of MOHAM

Consider the following differential equation

$$L\left(f\left(\eta\right)\right) + \Gamma\left(\eta\right) + N\left(f\left(\eta\right)\right) = 0, \quad B\left(f, \frac{df}{d\eta}\right) = 0,$$

where L is the linear operator, $\Gamma$ is a known function, N is a nonlinear differential operator and B is a boundary operator and f is unknown function.

According to MOHAM the homotopy $\tau\left(\eta, r\right) : \Theta \times [0, 1] \to R$ satisfies

$$(1 - r)\left[L\left(\Theta\right) + \Gamma\left(\eta\right)\right] = U\left(r\right)\left[L\left(\Theta\right) + \Gamma\left(\eta\right) + N\left(\Theta\right)\right], \quad B\left(\Theta, \frac{\partial\Theta}{\partial\eta}\right) = 0, \tag{2.1}$$

where $r \in [0, 1]$ is an embedding parameter, $\Theta$ is an unknown function, $U_i\left(r\right)$ is the auxiliary function, when r varies from 0 to 1, the solution varies from $f_0(\eta)$ to $f\left(\eta\right)$. $r = 0 \Rightarrow \Theta(0) = f_0\left(\eta\right)$ and $r = 1 \Rightarrow \Theta(1) = f\left(\eta\right)$.

Choose auxiliary function $U\left(r\right)$ in the form

$$U\left(r\right) = rD_{1,j} + r^2 D_{2,j} + r^3 D_{3,j} + \cdots + r^m D_{m,j},$$

where $D_{1,j}, D_{2,j}, D_{3,j}, \ldots, D_{m,j}$ are optimal constants. Expanding $\Theta\left(\eta, r, D_i\right)$ by Taylor's series we have

$$\Theta\left(\eta, r, D_{i,j}\right) = f_0\left(\eta\right) + \sum_{k=1}^{\infty} f_k\left(\eta, D_{i,j}\right) r^k, \quad i, j \in N. \tag{2.2}$$

Using Eq. (2.2) into Eq. (2.1) and equating the same powers of r, we get

$$L\left(f_0(\eta)\right) + \Gamma(\eta) = 0, \qquad B\left(f_0, \frac{df_0}{d\eta}\right) = 0,$$

$$L\left(f_1(\mu)\right) = D_{1,j} N_0\left(f_0(\eta)\right), \qquad B\left(f_1, \frac{df_1}{d\eta}\right) = 0,$$

$$L\left(f_k(\eta)\right) - L\left(f_{k-1}(\eta)\right) = D_{i,j} N_0\left(f_0(\eta)\right) + \sum_{i=1}^{k-1} D_{i,j}\left[L\left(f_{k-i}(\eta)\right) + N_{k-i}\left(f_0(\eta), f_1(\eta), \ldots, f_{k-i}(\eta)\right)\right],$$

$$B\left(f_k, \frac{df_k}{d\eta}\right) = 0,$$

where $N_{k-i}\left(f_0(t), f_1(t), \ldots, f_{k-i}(t)\right)$ is the coefficient of $r^{k-i}$ in the expansion series (2.2).

The approximated solution is

$$\tilde{f}\left(\eta, D_{i,j}\right) = f_0(\eta) + \sum_{j=1}^{m} f_j\left(\eta, D_{i,j}\right).$$

The residual is given by

$$R\left(\eta, D_{i,j}\right) = L\tilde{f}\left(\eta, D_{i,j}\right) + \Gamma(\eta) + N\tilde{f}\left(\eta, D_{i,j}\right).$$

If $R = 0$ then $\tilde{f}$ will be exact solution of the problem and it doesn't happen, especially in nonlinear problems.

The method of least square is used for finding

$$L\left(D_{i,j}\right) = \int_{z_j}^{z_{j+h}} R^2\left(\eta, D_{i,j}\right)\, d\eta,$$

where $h$ is the length of subinterval and $\left[z_j, z_{j+1}\right]$

$$\frac{\partial L}{\partial D_{i,j}} = 0, \ i, j = 1, 2, \ldots, m.$$

Therefore, the approximate analytic solution will be

$$\tilde{\Theta}\left(\eta\right) = \left\{ \begin{array}{ll} \tilde{f}_1\left(\eta\right), & z_0 \leqslant t \leqslant z_1 \\ \tilde{f}_2\left(\eta\right), & z_1 \leqslant t \leqslant z_2, \\ \vdots & \vdots \\ \tilde{f}_N\left(\eta\right), & z_{N-1} \leqslant t \leqslant T, \end{array} \right\}.$$

The analytic solution of the problem for large interval will be obtained successfully.

## 3. The differential transform method

Transformation of the $k^{\text{th}}$ derivative of function in one variable, defined as

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0},$$

is the original function and is the transformed function. Differential inverse transform of it is defined as

$$f(t) = \Sigma^\infty{}_{k=0} F(k)(t-t_0)^k, \quad f(t) \approx \Sigma^\infty{}_{k=0} F(k)(t-t_0)^k,$$

we obtain

$$f(t) \approx \sum_{k=0}^{i}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0}\left(\frac{t-t_0}{k!}\right)^k. \tag{3.1}$$

Eq. (3.1) implies that the concept of differential transformation is derived from the Taylor series expansion. From the definitions, it is easy to obtain the mathematical operations according to Table 1.

Table 1: Various differential transform operators.

| | |
|---|---|
| If $f(x) = g(x) \pm h(x)$ | then $F(k) = G(k) + H(k)$ |
| If $f(x) = cg(x)$ | then $F(k) = cG(k)$ |
| If $f(x) = \frac{d^n g(x)}{dx^n}$ | then $F(k) = \frac{(k+n)!}{k!}G(k+n)$ |
| If $f(x) = g(x)h(x)$ | then $F(k) = \sum_{i=0}^{k} G(l) + H(K-l)$ |
| If $f(x) = x^n$ | then $F(k) = \delta(k-n)$ |
| If $f(x) = \int g(t)dt$ | then $F(k) = \frac{G(k-1)}{k}$, where $k \geqslant 1$ |
| If $f(x) = g(x)h(x)i(x)$ | then $F(k) = \sum_{s=0}^{k}\sum_{m=0}^{k-s} G(s)H(m)I(k-s-m)$ |

## 4. Mathematical formulation of the flow problem

Consider the flow of an incompressible viscous fluid over a stretching sheet at $y = 0$. The governing

equations are

$$(\bar{V}.\Delta\bar{V}) = \mu\Delta^2\bar{V}, \quad \Delta.\bar{V} = 0,$$

where $\bar{V} = (u(x,0), v(x,0), 0)$ is velocity field and $\mu$ is dynamic viscosity. The boundary conditions for the flow due to stretching sheet are [27]

$$u(x,0) = cx^n, \ v(x,0) = 0, \ u(x,\infty) = 0.$$

In order to reduce the governing equation into boundary value problem using the following similarity transformation we have [27].

$$u(x,y) = ux^k f'(\eta), \quad \eta = y\sqrt{\frac{c(k+1)}{2v}}x^{\frac{k-1}{2}}y, \quad v(\eta) = -\sqrt{\frac{c(k+1)}{2}}x^{\frac{k-1}{2}}\left[f(\eta) + \frac{(k-1)}{(k+1)}\eta f'(\eta)\right].$$

We obtain

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} - \beta\left(\frac{df}{d\eta}\right)^2 = 0, \quad f(0) = 0, \ f'(0) = 1, \ f'(\infty) = 0, \tag{4.1}$$

where $\beta = \frac{2n}{1+n}$. Using the formulation mentioned in Section 3, we will obtain the zeroth, first, and second order problems

$$f_0'''(\eta) = 0, \ f_0(0) = 0, f_0'(0) = 1, \ f_0'(\infty) = 0, \tag{4.2}$$

$$f_1'''(\eta) = f_0''' + C_{11}\left(f_0''' - \beta(f_0')^2 + f_0f_0''\right), \ f_1(0) = 0, f_1'(0) = 0, \ f_1'(\infty) = 0, \tag{4.3}$$

$$f_2'''(\eta) = (1 + C_{11})f_1''' + C_{11}\left(f_0f_1'' + f_1f_0''\right) - 2\beta C_{11}f_0'f_1' + C_{12}\left(f_0f_0'' - \beta\left(f_0'\right)^2\right), \tag{4.4}$$

$$f_2(0) = 0, f_2'(0) = 0, f_2'(\infty) = 0,$$

respectively. The remaining terms can be calculated in the similar fashion.

The solutions of Eqs. (4.2)-(4.4) are respectively given as

$$f_0 = \frac{1}{2}\left(2\eta - \eta^2\right), \tag{4.5}$$

$$f_1 = \frac{1}{240}\left(10(1+2\beta)\eta^2 - 20\beta\eta^3 - 8(1-2\beta)\eta^4 + (1-2\beta)\eta^5\right)C_{11}, \tag{4.6}$$

$$f_2 = \frac{1}{40320}\begin{pmatrix}(1520(1+2\beta)\eta^2 - 4020\eta^3\beta - 680\eta^4(1-2\beta) + 136\eta^5(1-2\beta))C_{11} \\ \begin{pmatrix}(1550 + 3072\beta + 260\beta^2)\eta^2 - 4720\beta\eta^3 - (1070 - 2000\beta - 240\beta^2)\eta^4 \\ + (152 - 1044\beta + 1000\beta^2)\eta^5 - (160 - 600\beta - 260\beta^2)\eta^6 \\ + (48 - 156\beta + 100\beta^2)\eta^7 - (5 - 16\beta - 10\beta^2)\eta^8\end{pmatrix}C_{11}^2 \\ (2000(1+2\beta)\eta^2 - 6000\eta^3\beta - 1600\eta^4(1-2\beta) + 236\eta^5(1-2\beta))C_{12}\end{pmatrix}. \tag{4.7}$$

Adding Eqs. (4.5)-(4.7), and using the optimal constants

$$C_{11} = -0.8832102956129863 \text{ and } C_{12} = -0.006984790139877385$$

obtained by method of least square, we obtain

$$\tilde{f}_1 = x\begin{bmatrix}(1 + 0.00193467x^6(-2.12321 + \beta)(-0.376788 + \beta) \\ -0.000193467x^7(-2.82288 + \beta)(-0.177124 + \beta) \\ -0.00309548x^5(-3.57518 + \beta)(-0.1748416 + \beta) + 0.0853305x^2\beta \\ +0.0193467x^4(-0.853852 + \beta)(0.360845 + \beta) \\ -0.0154774x^2(-0.621026 + \beta)(2.25074 + \beta) \\ +0.00580420x(-13.3039 + \beta)(7.00037 + \beta)\end{bmatrix},$$

Repeating the same procedure for the next subinterval, we obtain the approximate solution for $C_{21} = -0.08832102956121245$, $C_{22} = -0.006984790131248976$,

$$
\tilde{f}_2 = x^2 \begin{bmatrix} 0.080227x\beta + 0.0011608(-68.5132 + \beta)(0.263572 + \beta) \\ -0.000773869x^2(-0.411559 + \beta)(74.7846 + \beta) \\ -0.000154774x^3(-0.300028 + \beta)(84.8932 + \beta) \\ +x^4(-0.000608 + 0.00139297\beta - 0.00193467\beta^2) \\ +x^6(-0.00011608 + 0.000386935\beta - 0.000193467\beta^2) \\ +x^5(0.00154774 - 0.00011608\beta + 0.0011608\beta^2) \end{bmatrix}.
$$

The resulting solution is obtained as $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$.

$$
\tilde{f} = x \begin{bmatrix} (1 + 0.00309548x^6(-1.1 + \beta)(-0.5 + \beta) - 0.000386935x^7(-1.1 + \beta)(-0.5 + \beta) \\ -0.0108342x^5(-0.844949 + \beta)(-0.355051 + \beta) + 0.165557x^2\beta \\ +0.0195015x^4(-0.62727 + \beta)(-0.80955 + \beta) - 0.0162513x^2(-0.5 + \beta)(5.59368 + \beta) \\ +0.00696483x(-20.5508 + \beta)(5.59368 + \beta) \end{bmatrix}.
$$

Solution obtained by DTM is

$$(k + 1)(k + 2)(k + 3)F(k + 3)$$

$$= \sum_{r=0}^{k}(r + 1)(k - r + 1)F(r + 1)F(k - r + 1) - \sum_{r=0}^{k}(k + 1)(k + 2)F(k + 2)(k - r + 1)F(r),$$

$$F(k + 3) = \frac{\left[\sum_{r=0}^{k}(r + 1)(k - r + 1)F(r + 1)F(k - r + 1) - \sum_{r=0}^{k}(k + 1)(k + 2)F(k + 2)(k - r + 1)F(r)\right]}{(k + 1)(k + 2)(k + 3)}.$$

The obtained DRM solution is

$$
F = x + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{135} + \frac{227x^7}{161280} - \frac{1769x^8}{2721600} + \frac{60449321x^9}{267544166400}
$$
$$
- \frac{310522852811x^{10}}{533310167232} + \frac{64713747257871x^{11}}{26245956357379325952} - \frac{1686196551370591609738131551x^{12}}{2402791478907114697732952}
$$
$$
+ \frac{65992893102829616914793608x^{13}}{29307445944190854537006142728358437} - \frac{4432513405606213333658026865168899865981x^{14}}{56040536223027859925533945644951821302298414580}
$$
$$
+ \frac{31545063035795668059267834758547110338757388930563x^{15}}{138749647227017770493766566449112374542418855928778106} + \cdots.
$$

Table 2: Comparison of OHAM, MOHAM, DTM, and absolute errors of $f(\eta)$ for $\beta = 1$.

| X | OHAM | MOHAM | DTM | E* | E** |
|---|------|-------|-----|----|-----|
| 0 | 0 | 0 | 0 | 0.000000000 | 0.000000000 |
| 0.1 | 0.0975488 | 0.0978159 | 0.0978159 | $1.09282 \times 10^{-19}$ | $1.87548 \times 10^{-3}$ |
| 0.2 | 0.1924588 | 0.191786 | 0.191786 | $2.12458 \times 10^{-19}$ | $1.02135 \times 10^{-3}$ |
| 0.3 | 0.2845879 | 0.282691 | 0.282691 | $3.012548 \times 10^{-19}$ | $2.15487 \times 10^{-3}$ |
| 0.4 | 0.3721445 | 0.37131 | 0.37131 | $1.87548 \times 10^{-19}$ | $1.54877 \times 10^{-3}$ |
| 0.5 | 0.4545789 | 0.458421 | 0.458421 | $1.21458 \times 10^{-19}$ | $2.8457 \times 10^{-3}$ |
| 0.6 | 0.5445785 | 0.544797 | 0.544797 | $1025489 \times 10^{-19}$ | $1.45879 \times 10^{-3}$ |
| 0.7 | 0.6384785 | 0.631213 | 0.631213 | $1.87548 \times 10^{-20}$ | $2.1548 \times 10^{-3}$ |
| 0.8 | 0.7187954 | 0.718437 | 0.718437 | $1.125487 \times 10^{-20}$ | $1.24587 \times 10^{-3}$ |
| 0.9 | 0.8087958 | 0.807238 | 0.807238 | $1.45876 \times 10^{-20}$ | $2.15482 \times 10^{-4}$ |
| 1.0 | 0.8987962 | 0.898382 | 0.898382 | $2.12458 \times 10^{-21}$ | $2.15487 \times 10^{-4}$ |
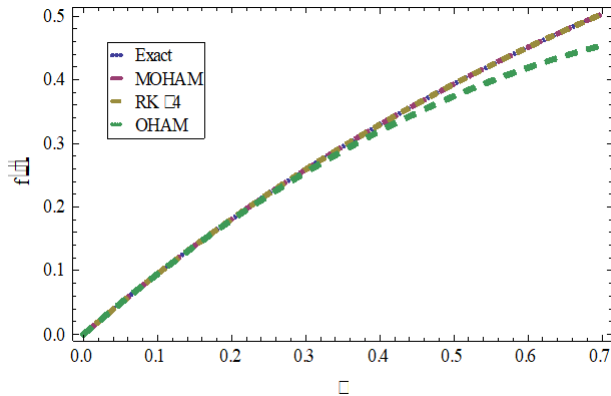
Figure 1: Comparison of solutions f (η) for β = 1.
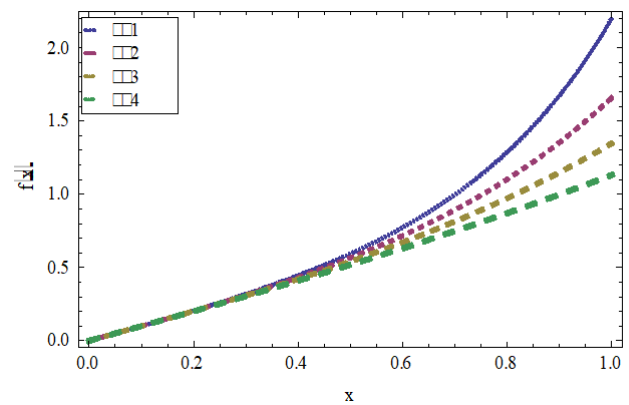


Figure 2: Variation of f (η) for different values of β.

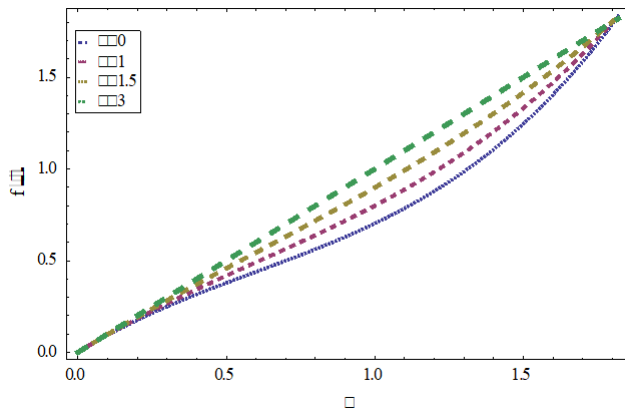

Figure 3: Variation of f′ (η) for different values of β.

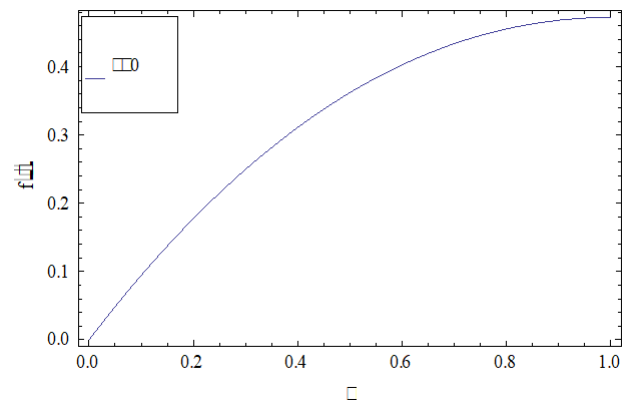

Figure 4: Plot of f (η) with respect to η for β = 0.

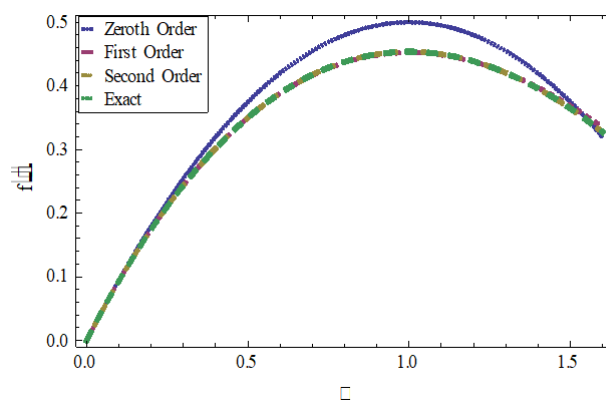In case of β = 0 in Eq. (4.1), we obtain the well known Blasius equation.



Figure 5: Plot of order of approximation f (η) with respect to η for β = 1.

## 5. Results discussions and conclusion

Fig. 1 and Table 1 show the comparison of results of MOHAM with exact, numerical results obtained by Runge Kutta-4 Method, OHAM, and DTM. It is found that the method presented works very well and

provides the same values to exact solution. MOHAM provides better results than OHAM. The Fig. 2 shows the variation of the function $f(\eta)$ against $\eta$ for different values of $\beta$ with MOHAM. It is concluded from Figs. 2 and 3 that the boundary layer flow increases by increasing $\beta$. The Fig. 4 shows the variation of $f(\eta)$ with respect to $\eta$ for the Blasius equation which is identical to results in literature [27]. The convergence of the proposed method is given in Fig. 5. It is concluded that the accuracy of the method increased by increasing the order of approximations. The accuracy of MOHAM is proved by comparing with other results. We conclude that MOHAM is a powerful, simple, involves less computational work, and fast convergent for the ODEs problems with large domain.

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