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A Qi formula for translated r-Dowling numbers

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Abstract

Another form of an explicit formula for translated r-Dowling numbers is derived using Faa di Bruno's formula and certain identity of Bell polynomials of the second kind. This formula is expressed in terms of the translated r-Whitney numbers of the second kind and the ordinary Lah numbers, which is analogous to Qi formula. As a consequence, a relation between translated r-Dowling numbers and the sums of row entries of the product of two matrices containing the translated r-Whitney numbers of the second kind and the ordinary Lah numbers is established.

Keywords: Qi formula, Translated r-Dowling numbers, Bell polynomials, Lah numbers, translated r-Whitney numbers, Faa di Bruno's formula, r-Whitney-Lah numbers.

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1. Introduction

The Bell numbers, denoted by B_n, were defined in [8] as the sum of Stirling numbers of the second kind

$$B_n := \sum_{k=0}^{n} S(n,k).$$

Several properties and application were obtained for these numbers including generating functions, recursive formulas, explicit formula, and expression in terms of a moment of the Poisson random variable. Recently, Feng Qi [22] established a different form of explicit formula for the ordinary Bell numbers, which is expressed in terms of the classical Stirling numbers of the second kind and ordinary Lah numbers. This formula was known as *Qi formula*, and was derived in two ways: one by using the inverse relation for the classical Stirling numbers and the other one by using Faa di Bruno's formula and certain identity for Bell polynomials of the second kind. This formula would possibly help the researchers in giving another way to describe the structure of Bell numbers. This attracted the attention of the present authors to investigate the explicit formula for some generalizations of Bell numbers.

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Motivated by the work of Broder [5], Belbachir and Bousbaa [2] defined combinatorially certain generalization of Stirling and Whitney numbers by introducing new parameter α. These numbers were called the translated r-Whitney numbers of the first and second kind and were denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)}$, respectively. More precisely, these numbers were defined by

$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)}$:= number of permutations of n elements with k cycles such that the first r elements are n distinct cycles and the elements of each cycle can mutate in α ways, except the dominant one;

$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)}$:= number of partitions of n elements with k parts such that the first r elements are in distinct parts and the elements of each part can mutate in α ways, except the dominant one.

In the same paper, the translated r-Whitney-Lah numbers were defined by

$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)}$:= number of ways to distribute the set $\{1, 2, \ldots, n\}$ into k ordered lists such that the first r elements are in distinct lists and the elements of each list can mutate in α ways, except the dominant one.

These numbers are certain variation of r-Whitney numbers and r-Whitney-Lah numbers in [7, 13, 20]. Belbachir and Bousbaa [2] obtained some properties of these numbers using combinatorial argument. These include:

1. *triangular recurrence relation*

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r^{(\alpha)} + \alpha(n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(\alpha)}, \\
\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} &= \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_r^{(\alpha)} + \alpha k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_r^{(\alpha)}, \\
\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} &= \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r^{(\alpha)} + \alpha(n-1+k)\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(\alpha)};
\end{aligned} \tag{1.1}$$

2. *expression of translated r-Whitney-Lah numbers in terms of translated r-Whitney numbers of the first and second kind*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r^{(\alpha)} \begin{Bmatrix} j \\ k \end{Bmatrix}_r^{(\alpha)}; \tag{1.2}$$

3. *horizontal generating function*

$$(x + r|\alpha)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} (x+r)^k, \tag{1.3}$$

$$(x + r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} (x - r| - \alpha)_k, \tag{1.4}$$

$$(x + r|\alpha)_n = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} (x - r| - \alpha)_k, \tag{1.5}$$

where $(x|\alpha)_n = x(x+\alpha)(x+2\alpha)\ldots(x+(n-1)\alpha)$, the rising factorial of x of degree n with increment α.

However, among the properties established by Belbachir and Bousbaa, there was no mention about the numbers parallel to Bell numbers, particularly, a formula analogous to Qi formula. Now, if we define the translated r-Dowling numbers, denoted by $D_n(r,\alpha)$, to be the sum of translated r-Whitney numbers of the second kind

$$D_n(r,\alpha)=\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_r^{(\alpha)},$$

then $D_n(r,\alpha)$ may be interpreted as the total number of ways to partition n elements such that the first r elements are in distinct partitions and the elements of each partition can mutate in α ways, except the dominant one. These numbers can be considered as a generalization and variation of Bell numbers and other Bell-type numbers [7, 12, 21]. It would then be interesting to derive another form of explicit formula for $D_n(r,\alpha)$ so one can provide another way to describe the structure of these numbers.

This study aimed to establish an explicit formula analogous to Qi formula, for the translated r-Dowling numbers using the methods of Qi [22]. This formula was derived by obtaining first the orthogonality and inverse relations of the translated r-Whitney numbers of the first and second kind and the exponential generating functions of the translated r-Whitney numbers of the second kind.

2. Orthogonality and inverse relations

In this section, some properties of translated r-Whitney and translated r-Whitney-Lah numbers, which are necessary for the derivation of Qi formula, will be established. Throughout this paper, the notation $\langle x|\alpha\rangle_n$ will be used to represent the falling factorial of x of degree n with increment α defined by

$$\langle x|\alpha\rangle_n = x(x-\alpha)(x-2\alpha)\cdots(x-(n-1)\alpha).$$

Belbachir and Bousbaa [2] did not mention the proof of the horizontal generating functions in (1.3)-(1.5). One can easily observe that if the recurrence relation in (1.1) is true, then the horizontal generating function in (1.4) must be

$$(x-r)^n=\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k. \tag{2.1}$$

We can prove this by induction on n. That is, for $n=0$, we have

$$(x-r)^0=1=\left\{ {0 \atop 0} \right\}_r^{(\alpha)}(x-r|-\alpha)_0=\sum_{k=0}^{0}\left\{ {n \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k.$$

Assume that the horizontal generating function in (2.1) is true for some $n\geqslant 0$. Then, using (1.1), we have

$$\begin{aligned}
(x-r)^{n+1}&=\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k\\
&=\sum_{k=0}^{n}(x-r-\alpha k+\alpha k)\left\{ {n \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k\\
&=\sum_{k=0}^{n+1}\left\{ {n \atop k-1} \right\}_r^{(\alpha)}(x-r|-\alpha)_k+\sum_{k=0}^{n+1}+\alpha k\left\{ {n \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k\\
&=\sum_{k=0}^{n+1}\left\{\left\{ {n \atop k-1} \right\}_r^{(\alpha)}+\alpha k\left\{ {n \atop k} \right\}_r^{(\alpha)}\right\}(x-r|-\alpha)_k\\
&=\sum_{k=0}^{n+1}\left\{ {n+1 \atop k} \right\}_r^{(\alpha)}(x-r|-\alpha)_k.
\end{aligned}$$

This proves the horizontal generating function in (2.1). Note that equations (1.3) and (2.1) can be written as

$$(x-r)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{(\alpha)} \langle x - r|\alpha\rangle_k, \qquad (-x+r|\alpha)_n = \sum \left[{n \atop k} \right]_r^{(\alpha)} (-x+r)^k.$$

Since

$$\begin{aligned}(-x+r|\alpha)_n &= (-x+r)(-x+r+\alpha)(-x+r+2\alpha)\cdots(-x+r+(n-1)\\ &= (-1)^n(x-r)(x-r-\alpha)(x-r-2\alpha)\cdots(x-r-(n-1)\alpha) = (-1)^n\langle x-r|\alpha\rangle_n,\end{aligned}$$

we have

$$\langle x-r|\alpha\rangle_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} (x-r)^k.$$

With

$$(x-r)^k = \sum_{j=0}^{k} \left\{ {k \atop j} \right\}_r^{(\alpha)} \langle x-r|\alpha\rangle_j,$$

we obtain

$$\begin{aligned}\langle x-r|\alpha\rangle_n &= \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} \sum_{j=0}^{k} \left\{ {k \atop j} \right\}_r^{(\alpha)} \langle x-r|\alpha\rangle_j \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} \left\{ {k \atop j} \right\}_r^{(\alpha)} \langle x-r|\alpha\rangle_j = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} \left\{ {k \atop j} \right\}_r \right] \langle x-r|\alpha\rangle_j.\end{aligned}$$

It follows that

$$\sum_{k=j}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} \left\{ {k \atop j} \right\}_r = \begin{cases} 0, & j \neq n \\ 1, & j = n \end{cases} = \delta_{n,j}. \tag{2.2}$$

Similarly, we can easily derive

$$\sum_{k=j}^{n} (-1)^{k-j} \left\{ {n \atop k} \right\}_r^{(\alpha)} \left[{k \atop j} \right]_r^{(\alpha)} = \begin{cases} 0, & n \neq j \\ 1, & j = n \end{cases} = \delta_{n,j}. \tag{2.3}$$

Equations (2.2) and (2.3) are called the orthogonality relations for translated r-Whitney numbers of the first and second kind. These imply the following inverse relations

$$\begin{aligned} f_n &= \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} g_k \Longleftrightarrow g_h = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{(\alpha)} f_k, \\ f_k &= \sum_{n=k}^{\infty} (-1)^{n-k} \left[{n \atop k} \right]_r^{(\alpha)} g_n \Longleftrightarrow g_k = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_r^{(\alpha)} f_n. \end{aligned} \tag{2.4}$$

To derive the orthogonality relation for translated r-Whitney-Lah numbers, we rewrite first equation (1.5) as

$$\begin{aligned}(-x+r|\alpha)_n &= \sum \left\lfloor {n \atop k} \right\rfloor_r^{(\alpha)} (-x-r|-\alpha)_k, \\ \langle x-r|\alpha\rangle_n &= \sum_{k=0}^{n} (-1)^n \left\lfloor {n \atop k} \right\rfloor_r^{(\alpha)} \langle -x-r|\alpha\rangle_k,\end{aligned}$$

$$\langle -x-r|\alpha\rangle_n = \sum_{k=0}^{n} (-1)^n \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} \langle x-r|\alpha\rangle_k.$$

Using the notation $\widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} = (-1)^n \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)}$, we have

$$\begin{aligned} \langle -x-r|\alpha\rangle_n = \sum_{k=0}^{n} \widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} \langle x-r|\alpha\rangle_k &= \sum_{k=0}^{n} \widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} \sum_{j=0}^{k} \widetilde{\left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rfloor}_r^{(\alpha)} \langle -x-r|\alpha\rangle_j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} \widetilde{\left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rfloor}_r^{(\alpha)} \langle -x-r|\alpha\rangle_j \\ &= \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} \widetilde{\left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rfloor}_r^{(\alpha)} \right] \langle -x-r|\alpha\rangle_j. \end{aligned}$$

Therefore

$$\sum_{k=j}^{n} \widetilde{\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor}_r^{(\alpha)} \left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rfloor_r^{(\alpha)} = \begin{cases} 0, & j \neq n \\ 1, & j = n \end{cases} = \delta_{n,j}.$$

This is the orthogonality relation for translated r-Whitney-Lah numbers.

3. Qi Formula for $D_n(r, \alpha)$: first form

In this section, a new explicit formula for translated r-Dowling numbers expressed in terms of translated r-Whitney-Lah numbers and translated r-Whitney numbers of the second kind is established. As a consequence, a relation in terms of matrices involving the translated r-Dowling numbers, the translated r-Whitney-Lah numbers, and the translated r-Whitney numbers of the second kind is obtained.

Note that equation (1.2) can be rewritten as follows

$$(-1)^n \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = \sum_{j=k}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_r^{(\alpha)} (-1)^j \begin{Bmatrix} j \\ k \end{Bmatrix}_r^{(\alpha)}. \tag{3.1}$$

Using the inverse relation of translated r-Whitney numbers in (2.4) with

$$f_n = \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} \text{ and } g_j = (-1)^j \begin{Bmatrix} j \\ k \end{Bmatrix}_r^{(\alpha)},$$

equation (3.1) yields

$$(-1)^n \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} = \sum_{j=0}^{n} \begin{Bmatrix} n \\ j \end{Bmatrix}_r^{(\alpha)} (-1)^j \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_r^{(\alpha)},$$

that is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} = \sum_{j=0}^{n} (-1)^{n-j} \begin{Bmatrix} n \\ j \end{Bmatrix}_r^{(\alpha)} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_r^{(\alpha)}.$$

Summing up both sides over k from 0 to n, gives the following theorem.

Theorem 3.1. *The explicit formula for translated r-Dowling numbers is given by*

$$D_n(r, \alpha) = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_r^{(\alpha)} \right\} \begin{Bmatrix} n \\ j \end{Bmatrix}_r^{(\alpha)}. \tag{3.2}$$

Now, we can rewrite the sum in (3.2) as

$$D_n(r,\alpha) = S_0 + S_1 + S_2 + \cdots + S_n,$$

where $S_j = \sum_{j=0}^{n} (-1)^{n-j} \left\{ {n \atop j} \right\}_r^{(\alpha)} \left\lfloor {j \atop k} \right\rfloor_r^{(\alpha)}$. As a consequence, we have the following theorem.

Theorem 3.2. *For $n \in \mathbb{N}$, the translated r-Dowling numbers $D_i(r,\alpha)$ are equal to the sum of the entries of the i^{th} row of the product of two matrices*

$$\left[(-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)} \right]_{n+1 \times n+1} \left[\left\lfloor {j \atop k} \right\rfloor_r^{(\alpha)} \right]_{n+1 \times n+1},$$

whose entries are respectively the translated r-Whitney numbers of the second kind and the translated r-Whitney Lah numbers.

As a direct consequence the above theorem, we have the following corollary.

Corollary 3.3. *For $0 \leqslant i, l \leqslant n$, the translated r-Whitney numbers of the second kind satisfy the following explicit formula*

$$\left\{ {i \atop l} \right\}_r^{(\alpha)} = \sum_{j=0}^{i} (-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)} \left\lfloor {j \atop l} \right\rfloor_r^{(\alpha)},$$

that is

$$\left[\left\{ {i \atop j} \right\}_r^{(\alpha)} \right]_{n+1 \times n+1} = \left[(-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)} \right]_{n+1 \times n+1} \left[\left\lfloor {i \atop j} \right\rfloor_r^{(\alpha)} \right]_{n+1 \times n+1}.$$

It can easily be shown that

$$\sum_{j=i}^{n} (-1)^{n-j} \left[{n \atop j} \right]_r^{(\alpha)} \left\{ {j \atop i} \right\}_r^{(\alpha)} = \sum_{j=i}^{n} \left\{ {n \atop j} \right\}_r^{(\alpha)} (-1)^{j-i} \left\{ {j \atop i} \right\}_r^{(\alpha)} = \delta_{ni},$$

where δ_{ni} is the Kronecker delta. This relation implies that

$$\left[\left\{ {i \atop j} \right\}_r^{(\alpha)} \right]_{n+1 \times n+1}^{-1} = \left[(-1)^{i-j} \left[{i \atop j} \right]_r^{(\alpha)} \right]_{n+1 \times n+1}.$$

Thus, we have

$$\left[(-1)^{i-j} \left[{i \atop j} \right]_r^{(\alpha)} \right]_{n+1 \times n+1} \left[(-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)} \right]_{n+1 \times n+1} \left[\left\lfloor {i \atop j} \right\rfloor_r^{(\alpha)} \right]_{n+1 \times n+1} = I_{n+1}.$$

4. Qi Formula for $D_n(r,\alpha)$: second form

In this section, another form of explicit formula for computing the translated r-Dowling numbers $D_n(r,\alpha)$ is derived using Faa di Bruno's formula and certain identity of Bell polynomials of the second kind. Before we derive the formula, let us consider first the exponential generating of the translated r-Whitney numbers of the second kind and that of the translated r-Dowling numbers.

Theorem 4.1. *The exponential generating function for translated r-Whitney numbers of the second kind is given by*

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\}_r^{(\alpha)} \frac{t^n}{n!} = \frac{\left(e^{\alpha t} - 1\right)^k}{\alpha^k k!}.$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}_r^{(\alpha)}\frac{t^n}{n!}\right\}(x-r|-\alpha)_k &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\left\{{n \atop k}\right\}_r^{(\alpha)}(x-r|-\alpha)_k\right\}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{[(x-r)t]^n}{n!} = e^{(x-r)t} = \left(1+\left(e^{\alpha t}-1\right)^{\frac{x-r}{\alpha}}\right) = \sum_{k=0}^{\infty}\frac{\left(e^{\alpha t}-1\right)^k}{\alpha^k k!}(x-r|-\alpha)_k.\end{aligned}$$

Comparing the coefficients of $(x-r|-\alpha)_k$ proves the theorem. □

This exponential generating function can be used to derive the explicit formula for $\left\{{n \atop k}\right\}_r^{(\alpha)}$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}_r^{(\alpha)}\frac{t^n}{n!} = \frac{\left(e^{\alpha t}-1\right)^k}{\alpha^k k!} &= \frac{1}{\alpha^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{j\alpha t}\\ &= \frac{1}{\alpha^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n\geqslant 0}\frac{(j\alpha t)^n}{n!} = \sum_{n\geqslant 0}\left\{\frac{\alpha^{n-k}}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients yields

$$\left\{{n \atop k}\right\}_r^{(\alpha)} = \alpha^{n-k}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n = \alpha^{n-k}S(n,k). \tag{4.1}$$

Now, let us derive the exponential generating function for $D_n(r,\alpha)$.

$$\sum_{n=0}^{\infty}D_n(r,\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{{n \atop k}\right\}_r^{(\alpha)}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}_r^{(\alpha)}\frac{t^n}{n!}\right\} = \sum_{k=0}^{\infty}\frac{\left(\frac{e^{\alpha t}-1}{\alpha}\right)^k}{k!} = e^{\frac{e^{\alpha t}-1}{\alpha}}.$$

Theorem 4.2. *The exponential generating function for translated* r*-Dowling numbers is given by*

$$\sum_{n=0}^{\infty}D_n(r,\alpha)\frac{t^n}{n!} = e^{\frac{e^{\alpha t}-1}{\alpha}}.$$

Theorem 4.3. *For* $n\in\mathbb{N}$*, the translated* r*-Dowling numbers* $D_n(r,\alpha)$ *are equal to*

$$D_n(r,\alpha) = \sum_{j=0}^{n}(-1)^{n-j}\left\{\sum_{i=0}^{j}\alpha^{j-i}L(j,i)\right\}\left\{{n \atop j}\right\}_r^{(\alpha)}. \tag{4.2}$$

Proof. Let us recall the following identity from [4, 9, 25] on the n^{th} derivative of the exponential function $e^{\pm 1/t}$ expressed in terms of the Lah numbers

$$\left(e^{\pm 1/t}\right)^{(n)} = (-1)^n e^{\pm 1/t}\sum_{k=1}^{n}(\pm 1)^k L(n,k)\frac{1}{t^{n+k}}, \tag{4.3}$$

the identity from [8] on Bell polynomials of the second kind

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}),$$

and the famous identity from [8] on Faá di Bruno formula described in terms of the Bell polynomials of the second kind

$$\frac{d^n}{dt^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \tag{4.4}$$

Replacing t by $-t$ in the generating function for the translated r-Dowling numbers $D_n(r,\alpha)$ in Theorem 4.2, yields

$$e^{1/\alpha}\sum_{n\geqslant 0} D_n(r,\alpha)\frac{(-t)^n}{n!} = e^{1/\alpha e^{\alpha t}};$$

equivalently

$$e^{1/\alpha}\sum_{n\geqslant 0} (-1)^n D_n(r,\alpha)\frac{t^n}{n!} = e^{1/\alpha e^{\alpha t}}. \tag{4.5}$$

Then taking k^{th} derivative both sides of (4.5) with respect to t yields

$$e^{1/\alpha}\sum_{n=k}^{\infty} (-1)^n D_n(r,\alpha)\frac{t^{n-k}}{(n-k)!} = \frac{d^k}{dt^k}\left(e^{1/\alpha e^{\alpha t}}\right). \tag{4.6}$$

Taking $f(u) = e^{1/u}$ and $h(t) = \alpha e^{\alpha t}$ in (4.4) and making use of (4.3) gives

$$\begin{aligned}
\frac{d^k\left(e^{1/\alpha e^{\alpha t}}\right)}{dt^k} &= \frac{d^k\big(f\circ h(t)\big)}{dt^k}\\
&= \sum_{j=1}^{k}\frac{d^j(e^{1/u})}{du^j}B_{k,j}(\alpha(\alpha e^{\alpha t}), \alpha^2(\alpha e^{\alpha t}), \dots, \alpha^{k-j+1}(\alpha e^{\alpha t}))\\
&= \sum_{j=1}^{k}(-1)^j e^{1/u}\sum_{i=1}^{j} L(j,i)\cdot\frac{1}{u^{j+i}}B_{k,j}(\alpha(\alpha e^{\alpha t}), \alpha^2(\alpha e^{\alpha t}), \dots, \alpha^{k-j+1}(\alpha e^{\alpha t}))\\
&= \sum_{j=1}^{k}(-1)^j \sum_{i=1}^{j} L(j,i)\cdot\frac{e^{\frac{1}{\alpha e^{\alpha t}}}}{(\alpha e^{\alpha t})^{j+i}}B_{k,j}(\alpha(\alpha e^{\alpha t}), \alpha^2(\alpha e^{\alpha t}), \dots, \alpha^{k-j+1}(\alpha e^{\alpha t})),
\end{aligned}$$

where $u(t) = \alpha e^{\alpha t}$. Further by virtue of

$$B_{k,j}(abx_1, ab^2x_2, \dots, ab^{k-j+1}x_{k-j+1}) = a^j b^k B_{k,j}(x_1, x_2, \dots, x_{k-j+1})$$

and

$$B_{k,j}(\overbrace{1, 1, \dots, 1}^{k-j+1}) = S(k,j)$$

listed in [8, p.135], where a and b are complex numbers, we obtain

$$\begin{aligned}
\frac{d^k\left(e^{1/\alpha e^{\alpha t}}\right)}{dt^k} &= e^{1/\alpha e^{\alpha t}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j} L(j,i)\cdot\frac{1}{(\alpha e^{\alpha t})^{j+i}}\cdot(\alpha e^{\alpha t})^j\alpha^k B_{k,j}(\overbrace{1, 1, \dots, 1}^{k-j+1})\\
&= e^{1/\alpha e^{\alpha t}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j} L(j,i)\cdot\frac{\alpha^{k-i}}{(e^{\alpha t})^i}S(k,j).
\end{aligned}$$

Thus, replacing k by n and evaluating at $t = 0$ in equation (4.6) gives

$$e^{1/\alpha}(-1)^n D_n(r,\alpha) = \sum_{j=1}^{n}(-1)^j e^{1/\alpha}\sum_{i=1}^{j} L(j,i)\cdot\alpha^{n-i}S(n,j).$$

Rearranging the above sum and using the fact that $L(0,i) = 0$ for all positive integers i, we get

$$D_n(r,\alpha) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{i=0}^{j} \alpha^{n-j} S(n,j) \alpha^{j-i} L(j,i).$$

Using the explicit formula in (4.1), we obtain (4.2). □

The following corollary is a direct consequence of Theorem 4.3.

Corollary 4.4. *For $n \in \mathbb{N}$, the translated r-Dowling numbers $D_n(r,\alpha)$ are equal to the sum of the entries of the i^{th} row of the product of two matrices*

$$\left[(-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)}\right]_{n\times n} \left[\alpha^{j-i} L(i,j)\right]_{n\times n},$$

whose entries are respectively r-Whitney numbers of the second kind and the Lah numbers.

Proof. We can rewrite the formula in Theorem 4.3 as

$$D_i(r,\alpha) = \sum_{l=0}^{i} T_{il}, \quad i = 0,1,2,\ldots,n,$$

where

$$T_{il} = \sum_{j=0}^{i} (-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)} \alpha^{j-l} L(j,l), \quad l = 0,1,2,\ldots i.$$

Clearly, T_{il} is the (i,l)-entry of the following product of two matrices

$$\left[(-1)^{i-j} \left\{ {i \atop j} \right\}_r^{(\alpha)}\right]_{n\times n} \left[\alpha^{j-i} L(i,j)\right]_{n\times n},$$

containing the translated r-Whitney numbers of the second kind and Lah numbers, respectively. □

5. Modified translated r-Dowling numbers

The right-hand side of Qi formula in Theorem 4.3 and the explicit formula in (4.1) are not dependent on the parameter r. This means that the parameter r has no bearing on the process of computing the values of $D_n(r,\alpha)$ and $\left\{ {n \atop k} \right\}_r^{(\alpha)}$. To address this weakness, we can modify the definition of translated r-Whitney numbers of the first and second kind as follows:

$$(x+r|\alpha)_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{r,1}^{(\alpha)} (x-r)^k,$$
$$(x+r)^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} (x-r|-\alpha)_k. \tag{5.1}$$

These horizontal generating functions will imply the following recurrence relation:

$$\left\langle {n+1 \atop k} \right\rangle_{r,1}^{(\alpha)} = \left\langle {n \atop k-1} \right\rangle_{r,1}^{(\alpha)} + (\alpha n + 2r) \left\langle {n \atop k} \right\rangle_{r,1}^{(\alpha)},$$
$$\left\langle {n+1 \atop k} \right\rangle_{r,2}^{(\alpha)} = \left\langle {n \atop k-1} \right\rangle_{r,2}^{(\alpha)} + (\alpha k + 2r) \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)}.$$

Theorem 5.1. *The exponential generating function for the modified translated* r*-Whitney numbers of the second kind is given by*

$$\sum_{n=0}^{\infty} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} \frac{t^n}{n!} = \frac{\left(e^{\alpha t}-1\right)^k e^{2rt}}{\alpha^k k!}.$$

Proof. Using (5.1), we have

$$\begin{aligned}\sum_{k=0}^{\infty} &\left\{ \sum_{n=k}^{\infty} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} \frac{t^n}{n!} \right\} (x-r|-\alpha)_k \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} (x-r|-\alpha)_k \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{[(x+r)t]^n}{n!} = e^{(x-r)t} e^{2rt} = \left(1+\left(e^{\alpha t}-1\right)^{\frac{x-r}{\alpha}}\right) e^{2rt} = \sum_{k=0}^{\infty} \frac{\left(e^{\alpha t}-1\right)^k e^{2rt}}{\alpha^k k!} (x-r|-\alpha)_k.\end{aligned}$$

Comparing the coefficients of $(x-r|-\alpha)_k$ proves the theorem. □

This exponential generating function can be used to derive the explicit formula for $\left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)}$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} \frac{t^n}{n!} = \frac{\left(e^{\alpha t}-1\right)^k e^{2rt}}{\alpha^k k!} &= \frac{1}{\alpha^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{(j\alpha+2r)t} \\ &= \frac{1}{\alpha^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{n \geqslant 0} \frac{((j\alpha+2r)t)^n}{n!} \\ &= \sum_{n \geqslant 0} \left\{ \frac{1}{\alpha^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (j\alpha+2r)^n \right\} \frac{t^n}{n!}\end{aligned}$$

Comparing coefficients yields

$$\left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} = \alpha^{n-k} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (j+\frac{2r}{\alpha})^n = \alpha^{n-k} \left\{ {n+r \atop k+r} \right\}_{\frac{2r}{\alpha}},$$

where $\left\{ {n+r \atop k+r} \right\}_{\frac{2r}{\alpha}}$ is the r-Stirling numbers of the second kind [5]. This can also be written as

$$\left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} = \alpha^{n-k} \sum_{j=0}^{n} \binom{n}{j} S(j,k) \left(\frac{2r}{\alpha}\right)^{n-j} = \sum_{j=0}^{n} \binom{n}{j} \alpha^{j-k} (2r)^{n-j} S(j,k). \tag{5.2}$$

Now, let us define the modified translated r-Dowling numbers, denoted by $\widehat{D}_n(r,\alpha)$, as follows:

$$\widehat{D}_n(r,\alpha) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)}.$$

Then

$$\sum_{n=0}^{\infty} \widehat{D}_n(r,\alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} \left\langle {n \atop k} \right\rangle_{r,2}^{(\alpha)} \frac{t^n}{n!} \right\} = \sum_{k=0}^{\infty} \frac{\left(\frac{e^{\alpha t}-1}{\alpha}\right)^k e^{2rt}}{k!} = e^{2rt} e^{\frac{e^{\alpha t}-1}{\alpha}}.$$

This result is formally stated in the following theorem.

Theorem 5.2. *The exponential generating function for the modified translated* r*-Dowling numbers is given by*

$$\sum_{n=0}^{\infty} \widehat{D}_n(r,\alpha)\frac{t^n}{n!} = e^{\frac{e^{\alpha t}-1}{\alpha}+2rt}.$$

Theorem 5.3. *For* $n \in \mathbb{N}$, *the modified translated* r*-Dowling numbers* $\widehat{D}_n(r,\alpha)$ *are equal to*

$$\widehat{D}_n(r,\alpha) = \sum_{i=0}^{n}(-1)^{n-j}\sum_{j=0}^{i}\left\langle {n \atop j} \right\rangle_{-2r}^{(\alpha)} \alpha^{j-i}L(j,i). \tag{5.3}$$

Proof. Replacing t by $-t$ in the generating function for the modified translated $($r-Dowling numbers $\widehat{D}_n(r,\alpha)$ in Theorem 5.2 yields

$$\sum_{n\geqslant 0} \widehat{D}_n(r,\alpha)\frac{(-t)^n}{n!} = \frac{e^{-2rt}\cdot e^{1/\alpha e^{\alpha t}}}{e^{1/\alpha}};$$

equivalently

$$e^{1/\alpha}\sum_{n\geqslant 0}(-1)^n\widehat{D}_n(r,\alpha)\frac{t^n}{n!} = e^{1/\alpha e^{\alpha t}}\cdot e^{-2rt}. \tag{5.4}$$

Then taking k^{th} derivative on both sides of (5.4) with respect to t yields

$$e^{1/\alpha}\sum_{n=k}^{\infty}(-1)^k\widehat{D}_n(r,\alpha)\frac{t^{n-k}}{(n-k)!} = \frac{d^k}{dt^k}\left(e^{1/\alpha e^{\alpha t}}\cdot e^{-2rt}\right). \tag{5.5}$$

Taking $f(u) = e^{1/u}$ and $h(t) = \alpha e^{\alpha t}$ in (4.4) and making use of (4.3) gives

$$\begin{aligned}
\frac{d^k\left(e^{1/\alpha e^{\alpha t}}\right)}{dt^k} &= \frac{d^k(f\circ h(t))}{dt^k}\\
&= \sum_{j=1}^{k}\frac{d^j(e^{1/u})}{du^j}B_{k,j}(\alpha(\alpha e^{\alpha t}),\alpha^2(\alpha e^{\alpha t}),\dots,\alpha^{k-j+1}(\alpha e^{\alpha t}))\\
&= \sum_{j=1}^{k}(-1)^j e^{1/u}\sum_{i=1}^{j}L(j,i)\cdot\frac{1}{u^{j+i}}B_{k,j}(\alpha(\alpha e^{\alpha t}),\alpha^2(\alpha e^{\alpha t}),\dots,\alpha^{k-j+1}(\alpha e^{\alpha t}))\\
&= \sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j}L(j,i)\cdot\frac{e^{1/\alpha e^{\alpha t}}}{(\alpha e^{\alpha t})^{j+i}}B_{k,j}(\alpha(\alpha e^{\alpha t}),\alpha^2(\alpha e^{\alpha t}),\dots,\alpha^{k-j+1}(\alpha e^{\alpha t})),
\end{aligned}$$

where $u(t) = \alpha e^{\alpha t}$. Further by virtue of

$$B_{k,j}(abx_1, ab^2x_2,\dots,ab^{k-j+1}x_{k-j+1}) = a^j b^k B_{k,j}(x_1,x_2,\dots,x_{k-j+1})$$

and

$$B_{k,j}(\overbrace{1,1,\dots,1}^{k-j+1}) = S(k,j)$$

listed in [8, p.135], where a and b are complex numbers, we obtain

$$\frac{d^k\left(e^{1/\alpha e^{\alpha t}}\right)}{dt^k} = e^{1/\alpha e^{\alpha t}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j}L(j,i)\frac{1}{(\alpha e^{\alpha t})^{j+i}}(\alpha e^{\alpha t})^j\alpha^k B_{k,j}(\overbrace{1,1,\dots,1}^{k-j+1})$$

$$= e^{1/\alpha e^{\alpha t}} \sum_{j=1}^{k} (-1)^j \sum_{i=1}^{j} L(j,i) \frac{\alpha^{k-i}}{(e^{\alpha t})^i} S(k,j).$$

Hence, using Leibniz formula

$$\begin{aligned} \frac{d^n}{dz^n}\left(e^{1/\alpha e^{\alpha t}} e^{-2rt}\right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dt^k} e^{1/\alpha e^{\alpha t}} \frac{d^{n-k}}{dt^{n-k}} e^{-2rt} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left\{ e^{1/\alpha e^{\alpha t}} \sum_{j=1}^{k} (-1)^j \sum_{i=1}^{j} L(j,i) \frac{\alpha^{k-i}}{(e^{\alpha t})^i} S(k,j) \right\} (-2r)^{n-k} e^{-2rt}. \end{aligned}$$

Thus, replacing k by n and evaluating at $t = 0$ in equation (5.5) gives

$$e^{1/\alpha} (-1)^n \widehat{D}_n(r,\alpha) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=1}^{k} (-1)^j e^{1/\alpha} \sum_{i=1}^{j} L(j,i) \alpha^{k-i} S(k,j) (-2r)^{n-k}.$$

Rearranging the above sum and using the fact that $L(0,i) = 0$ for all positive integers i, we get

$$\widehat{D}_n(r,\alpha) = \sum_{i=0}^{n} (-1)^{n-j} \sum_{j=0}^{i} \left\{ \sum_{k=j}^{n} \binom{n}{k} \alpha^{k-j} (-2r)^{n-k} S(k,j) \right\} \alpha^{j-i} L(j,i).$$

Applying the property of modified translated r-Whitney numbers of the second kind in equation (5.2) yields

$$\widehat{D}_n(r,\alpha) = \sum_{i=0}^{n} (-1)^{n-j} \sum_{j=0}^{i} \left\langle {n \atop j} \right\rangle_{-2r}^{(\alpha)} \alpha^{j-i} L(j,i).$$

This is exactly the formula in (5.3). □

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