



Some operations on hesitant fuzzy soft sets over UP-algebras



Phakawat Mosrijai, Aiyared Iampan*

Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand.

Abstract

This paper aim is to apply the notions of the (restricted) union, the (extended) intersection, the AND, and the OR of any hesitant fuzzy soft set to UP-algebras. Moreover, we discuss results of the intersection and the union of hesitant fuzzy sets on UP-algebras and also apply the notions of the hesitant union and the hesitant intersection of hesitant fuzzy sets to UP-algebras.

Keywords: UP-algebra, hesitant fuzzy soft set, anti-hesitant fuzzy soft set, hesitant union, hesitant intersection.

2010 MSC: 03G25.

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1. Introduction

The branch of the logical algebra, UP-algebras were introduced by Iampan [5] in 2017, and it is known that the class of KU-algebras [21] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [28] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [9], Kaijae et al. [8] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [31], Sripaeng et al. [30] introduced the notion anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the notion of \mathcal{N} -fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [29], Senapati et al. [26, 27] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [22] introduced the notion of proper UP-filters in UP-algebras, etc..

A soft set over a universe set is a parametrized family of subsets of the universe set. Molodtsov [14] introduced the concept of soft sets over a universe set in 1999. The soft set theory was a new mathematical tool to handle uncertainty but the classical soft sets are not appropriate to deal with some imprecisions. The soft set model has been combined with other mathematical models. For instance, fuzzy soft sets [10] are based on a combination of fuzzy sets and soft sets, intuitionistic fuzzy soft sets [11, 13] are combined by intuitionistic fuzzy sets and soft sets, interval-valued fuzzy soft sets [34], and so on. By the way, Maji

*Corresponding author

Email addresses: phakawat.mo@gmail.com (Phakawat Mosrijai), aiyared.ia@up.ac.th (Aiyared Iampan)

doi: [10.22436/jmcs.020.02.06](https://doi.org/10.22436/jmcs.020.02.06)

Received: 2019-03-03 Revised: 2019-09-07 Accepted: 2019-09-16

et al. [12] defined equality of two soft sets, subset and superset of a soft set, complement of a soft set, null soft set, and absolute soft set in 2003. Moreover, they also introduced the concepts of the AND operation and the OR operation on two soft sets, and the union and the intersection of two soft sets. Ali et al. [1] latterly introduced the others such as the restricted intersection, the restricted union, the restricted difference, and the extended intersection of two soft sets.

A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The concept of a hesitant fuzzy set on a set was first considered by Torra [32] in 2010. The hesitant fuzzy set, which can be perfectly described in terms of the opinions of decision makers is a very useful tool to deal with uncertainty. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics, for example, Jun [7] defined the hesitant union, and the hesitant intersection of two hesitant fuzzy sets, etc.. In UP-algebras, Mosrijai et al. [19] extended the concept of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [25] considered level subsets of a hesitant fuzzy set on UP-algebras in 2017. The concept of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [20] afterwards. Mosrijai and Iampan [15] introduced the notion of anti-type of hesitant fuzzy sets on UP-algebras in 2018.

The concept of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets was introduced by Babitha and John [3] in 2013. Wang et al. [33] applied the hesitant fuzzy soft set theory to a decision making problem in 2014 and they also defined the complement, the “AND”, the “OR”, the union, and the intersection operations on hesitant fuzzy soft set. In UP-algebras, Mosrijai and Iampan [16] extended the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras, and they also introduced the notion of hesitant fuzzy soft sets over UP-algebras by mean of anti-type (see [17]).

In this paper, we study about some operations that can be applied to theories of hesitant fuzzy sets on UP-algebras or theories of hesitant fuzzy soft sets over UP-algebras. We introduce the notions of the hesitant union and the hesitant intersection of any hesitant fuzzy set, and examine some results on UP-algebras. Afterward, we introduce the notions of the intersection, the union, the restricted intersection, the extended intersection, the AND and the OR of any hesitant fuzzy soft set, and investigate some results over UP-algebras.

2. Basic results on UP-algebras

Before we begin our study, we will give the definition of a UP-algebra.

Definition 2.1 ([5]). An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, \cdot is a binary operation on A , and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

$$(UP-1) \quad (\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0);$$

$$(UP-2) \quad (\forall x \in A)(0 \cdot x = x);$$

$$(UP-3) \quad (\forall x \in A)(x \cdot 0 = 0); \text{ and}$$

$$(UP-4) \quad (\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y).$$

From [5], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.2 ([24]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation \cdot on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to Ω* .

In particular, $(\mathcal{P}_\emptyset(X), \cdot, \emptyset)$ is the power UP-algebra of type 1.

Example 2.3 ([24]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to Ω* .

In particular, $(\mathcal{P}^X(X), *, X)$ is the power UP-algebra of type 2.

For more examples of UP-algebras, see [2, 6, 23, 24].

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.4 ([5, 6]). *In a UP-algebra A , the following properties hold:*

- (1) $(\forall x \in A)(x \cdot x = 0)$;
- (2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$;
- (3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$;
- (4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$;
- (5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$;
- (6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$;
- (7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$;
- (8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$;
- (9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$;
- (10) $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot z) = 0)$;
- (11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$;
- (12) $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (x \cdot (y \cdot z)) = 0)$; and
- (13) $(\forall a, x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot (a \cdot z)) = 0)$.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation \leq on A [5] as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Definition 2.5 ([4, 5, 28]). A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

- (1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$;
- (2) a *UP-filter* of A if
 - (i) the constant 0 of A is in S ; and
 - (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$;
- (3) a *UP-ideal* of A if
 - (i) the constant 0 of A is in S ; and
 - (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$;
- (4) a *strongly UP-ideal* of A if
 - (i) the constant 0 of A is in S ; and

$$(ii) (\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S).$$

Guntasow et al. [4] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Basic Results on Hesitant Fuzzy (Soft) Sets

Definition 3.1 ([32]). Let X be a reference set. A *hesitant fuzzy set* on X is defined in term of a function h_H that when applied to X return a subset of $[0, 1]$, that is, $h_H: X \rightarrow \mathcal{P}([0, 1])$. A hesitant fuzzy set h_H can also be viewed as the following mathematical representation:

$$H := \{(x, h_H(x)) \mid x \in X\},$$

where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the elements $x \in X$ to the set H .

Definition 3.2 ([19]). A hesitant fuzzy set H on A is called

- (1) a *hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(h_H(x \cdot y) \supseteq h_H(x) \cap h_H(y))$;
- (2) a *hesitant fuzzy UP-filter* of A if
 - (i) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$; and
 - (ii) $(\forall x, y \in A)(h_H(y) \supseteq h_H(x \cdot y) \cap h_H(x))$;
- (3) a *hesitant fuzzy UP-ideal* of A if
 - (i) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$; and
 - (ii) $(\forall x, y, z \in A)(h_H(x \cdot z) \supseteq h_H(x \cdot (y \cdot z)) \cap h_H(y))$;
- (4) a *hesitant fuzzy strongly UP-ideal* of A if
 - (i) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$; and
 - (ii) $(\forall x, y, z \in A)(h_H(x) \supseteq h_H((z \cdot y) \cdot (z \cdot x)) \cap h_H(y))$.

Mosrijai et al. [19] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

Theorem 3.3 ([19]). A hesitant fuzzy set H on A is a *hesitant fuzzy strongly UP-ideal* of A if and only if it is a *constant hesitant fuzzy set* on A .

Definition 3.4 ([15]). A hesitant fuzzy set H on A is called

- (1) an *anti-hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(h_H(x \cdot y) \subseteq h_H(x) \cup h_H(y))$;
- (2) an *anti-hesitant fuzzy UP-filter* of A if
 - (i) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$; and
 - (ii) $(\forall x, y \in A)(h_H(y) \subseteq h_H(x \cdot y) \cup h_H(x))$;
- (3) an *anti-hesitant fuzzy UP-ideal* of A if

- (i) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$; and
- (ii) $(\forall x, y, z \in A)(h_H(x \cdot z) \subseteq h_H(x \cdot (y \cdot z)) \cup h_H(y))$;

(4) an *anti-hesitant fuzzy strongly UP-ideal* of A if

- (i) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$; and
- (ii) $(\forall x, y, z \in A)(h_H(x) \subseteq h_H((z \cdot y) \cdot (z \cdot x)) \cup h_H(y))$.

Mosrijai and Iampan [15] proved that the notion of anti-hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of anti-hesitant fuzzy UP-filters, the notion of anti-hesitant fuzzy UP-filters of UP-algebras is a generalization of anti-hesitant fuzzy UP-ideals, and the notion of anti-hesitant fuzzy UP-ideals of UP-algebras is a generalization of anti-hesitant fuzzy strongly UP-ideals.

Theorem 3.5 ([15]). *A hesitant fuzzy set H on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A .*

Definition 3.6 ([3]). Let X be a reference set (or an initial universe set) and P be a set of parameters. Let $HFS(X)$ be the set of all hesitant fuzzy sets on X and Y be a nonempty subset of P . A pair (\tilde{H}, Y) is called a *hesitant fuzzy soft set* over X , where \tilde{H} is a mapping given by

$$\tilde{H}: Y \rightarrow HFS(X), p \mapsto \tilde{H}[p].$$

Definition 3.7 ([16]). Let Y be a nonempty subset of P . A hesitant fuzzy soft set (\tilde{H}, Y) over A is called a *hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal)* based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-subalgebra (resp., p-hesitant fuzzy soft UP-filter, p-hesitant fuzzy soft UP-ideal, p-hesitant fuzzy soft strongly UP-ideal)*) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-subalgebra (resp., hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of A . If (\tilde{H}, Y) is a *p-hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal)* of A for all $p \in Y$, we state that (\tilde{H}, Y) is a *hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal)* of A .

Definition 3.8 ([18]). Let Y be a nonempty subset of P . A hesitant fuzzy soft set (\tilde{H}, Y) over A is called an *anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal)* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-subalgebra (resp., p-anti-hesitant fuzzy soft UP-filter, p-anti-hesitant fuzzy soft UP-ideal, p-anti-hesitant fuzzy soft strongly UP-ideal)*) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-subalgebra of A . If (\tilde{H}, Y) is a *p-anti-hesitant fuzzy soft UP-subalgebra* of A for all $p \in Y$, we state that (\tilde{H}, Y) is an *anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal)* of A .

4. Operations on hesitant fuzzy sets

Torra [32] defined many operations on hesitant fuzzy sets in 2010. For instance, the *intersection* operation \cap_T , the *union* operation \cup_T and so on.

Definition 4.1 ([32]). Let H and F be hesitant fuzzy sets on a reference set X . Then the following operations are defined:

- (1) $H \cap_T F = \{(x, \{h \in (h_H(x) \cup h_F(x)) \mid h \leq \min\{\sup h_H(x), \sup h_F(x)\}\}) \mid x \in X\}$, and
- (2) $H \cup_T F = \{(x, \{h \in (h_H(x) \cup h_F(x)) \mid h \geq \max\{\inf h_H(x), \inf h_F(x)\}\}) \mid x \in X\}$.

We can write the *intersection* and the *union* of hesitant fuzzy sets with different form by

$$H \cap_T F = \{(x, \bigcup_{\gamma_1 \in h_H(x), \gamma_2 \in h_F(x)} \{\min\{\gamma_1, \gamma_2\}\}) \mid x \in X\}$$

and

$$H \cup_T F = \{(x, \bigcup_{\gamma_1 \in h_H(x), \gamma_2 \in h_F(x)} \{\max\{\gamma_1, \gamma_2\}\}) \mid x \in X\}.$$

First, we will consider about results of the intersection and the union of hesitant fuzzy sets on UP-algebras.

The following example show that the intersection of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A .

Example 4.2. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0, 0.5, 1\}, & \text{if } x = 0, \\ \{0\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.5, 1\}, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \{0, 5\}, & \text{if } x = 2, \\ \{0.5, 1\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0, 0.5, 1\}, & \text{if } x = 0, \\ \{0\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{0.5, 1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cap_T H_2}(2 \cdot 3) = h_{H_1 \cap_T H_2}(1) = \{0\} \not\supseteq \{0.5\} = \{0.5\} \cap \{0.5, 1\} = h_{H_1 \cap_T H_2}(2) \cap h_{H_1 \cap_T H_2}(3).$$

The following example shows that the intersection of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A .

Example 4.3. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0, 1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0, 0.5\}, & \text{if } x = 0, \\ \{0, 0.5\}, & \text{if } x = 1, \\ \{0, 5\}, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0, 0.5\}, & \text{if } x = 0, \\ \{0, 0.5\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not a hesitant fuzzy UP-filter of A because

$$h_{H_1 \cap_T H_2}(3) = \{0\} \not\supseteq \{0.5\} = \{0, 0.5\} \cap \{0.5\} = h_{H_1 \cap_T H_2}(0) \cap h_{H_1 \cap_T H_2}(2) = h_{H_1 \cap_T H_2}(2 \cdot 3) \cap h_{H_1 \cap_T H_2}(2).$$

The following example shows that the intersection of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A .

Example 4.4. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0, 0.1\} & \text{if } x = 0, \\ \{0, 0.1\} & \text{if } x = 1, \\ \{0.1\} & \text{if } x = 2, \\ \{0.1\} & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0, 0.1\} & \text{if } x = 0, \\ \{0.1\} & \text{if } x = 1, \\ \{0, 0, 1\} & \text{if } x = 2, \\ \{0.1\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0, 0.1\}, & \text{if } x = 0, \\ \{0, 0.1\}, & \text{if } x = 1, \\ \{0, 0.1\}, & \text{if } x = 2, \\ \{0.1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not a hesitant fuzzy UP-ideal of A because

$$\begin{aligned} h_{H_1 \cap_T H_2}(0 \cdot 3) &= h_{H_1 \cap_T H_2}(3) = \{0.1\} \not\supseteq \{0, 0.1\} = \{0, 0.1\} \cap \{0, 0.1\} = h_{H_1 \cap_T H_2}(2) \cap h_{H_1 \cap_T H_2}(2) \\ &= h_{H_1 \cap_T H_2}(0 \cdot (1 \cdot 3)) \cap h_{H_1 \cap_T H_2}(1). \end{aligned}$$

Theorem 4.5. *The intersection of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideals of A .*

Proof. Assume that H and G are hesitant fuzzy strongly UP-ideals of A . By Theorem 3.3, we have H and G are constant hesitant fuzzy sets of A . Thus $H \cap_T G$ is a constant hesitant fuzzy set of A . By Theorem 3.3, we can conclude that $H \cap_T G$ is a hesitant fuzzy strongly UP-ideal of A . □

The following example shows that the intersection of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A .

Example 4.6. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.7\}, & \text{if } x = 0, \\ [0.7, 0.8], & \text{if } x = 1, \\ [0.7, 0.9], & \text{if } x = 2, \\ [0.6, 0.9], & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0.7\}, & \text{if } x = 0, \\ \{0.7, 0.8\}, & \text{if } x = 1, \\ \{0.7, 0.8\}, & \text{if } x = 2, \\ \{0.7\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0.7\}, & \text{if } x = 0, \\ [0.7, 0.8], & \text{if } x = 1, \\ [0.7, 0.9], & \text{if } x = 2, \\ [0.6, 0.7], & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cap_T H_2}(1 \cdot 3) = h_{H_1 \cap_T H_2}(2) = [0.7, 0.9] \not\subseteq [0.6, 0.8] = [0.7, 0.8] \cup [0.6, 0.7] = h_{H_1 \cap_T H_2}(1) \cup h_{H_1 \cap_T H_2}(3).$$

The following example shows that the intersection of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A .

Example 4.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0\}, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0.2\}, & \text{if } x = 0, \\ \{0, 0.2\}, & \text{if } x = 1, \\ \{0.2, 0.8, 1\}, & \text{if } x = 2, \\ \{0.2, 1\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0.2\}, & \text{if } x = 0, \\ \{0, 0.2\}, & \text{if } x = 1, \\ \{0\}, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$h_{H_1 \cap_T H_2}(0) = \{0.2\} \not\subseteq \{0\} = h_{H_1 \cap_T H_2}(2).$$

The following example shows that the intersection of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A .

Example 4.8. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0.8, 1\}, & \text{if } x = 2, \\ \{0.8, 1\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{1\}, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \{0.8, 1\}, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$h_{H_1 \cap_T H_2}(0) = \{1\} \not\subseteq \{0.5\} = h_{H_1 \cap_T H_2}(1).$$

Theorem 4.9. *The intersection of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideals of A .*

Proof. Assume that H and G are anti-hesitant fuzzy strongly UP-ideals of A . By Theorem 3.5, we have H and G are constant hesitant fuzzy sets of A . Thus $H \cap_T G$ is a constant hesitant fuzzy set of A . By Theorem 3.5, we can conclude that $H \cap_T G$ is an anti-hesitant fuzzy strongly UP-ideal of A . \square

The following example shows that the union of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A .

Example 4.10. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0, 0.5, 1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0, 0.5\}, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \{0, 5\}, & \text{if } x = 2, \\ \{0, 0.5\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cup_T H_2(x) = \begin{cases} \{0, 0.5, 1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{0, 0.5\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cup_T H_2}(2 \cdot 3) = h_{H_1 \cup_T H_2}(1) = \{1\} \not\supseteq \{0.5\} = \{0.5\} \cap \{0, 0.5\} = h_{H_1 \cup_T H_2}(2) \cap h_{H_1 \cup_T H_2}(3).$$

The following example shows that the union of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A .

Example 4.11. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0, 1\}, & \text{if } x = 0, \\ \{0\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.5, 1\}, & \text{if } x = 0, \\ \{0.5, 1\}, & \text{if } x = 1, \\ \{0, 5\}, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cup_T H_2(x) = \begin{cases} \{0.5, 1\}, & \text{if } x = 0, \\ \{0.5, 1\}, & \text{if } x = 1, \\ \{0.5\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not a hesitant fuzzy UP-filter of A because

$$h_{H_1 \cup_T H_2}(3) = \{1\} \not\supseteq \{0.5\} = \{0.5, 1\} \cap \{0.5\} = h_{H_1 \cup_T H_2}(0) \cap h_{H_1 \cup_T H_2}(2) = h_{H_1 \cup_T H_2}(2 \cdot 3) \cap h_{H_1 \cup_T H_2}(2).$$

The following example shows that the union of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A .

Example 4.12. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{0.9, 1\}, & \text{if } x = 1, \\ \{0.9\}, & \text{if } x = 2, \\ \{0.9\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{0.9\}, & \text{if } x = 1, \\ \{0, 9, 1\}, & \text{if } x = 2, \\ \{0.9\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{0.9, 1\}, & \text{if } x = 1, \\ \{0.9, 1\}, & \text{if } x = 2, \\ \{0.9\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not a hesitant fuzzy UP-ideal of A because

$$\begin{aligned} h_{H_1 \cap_T H_2}(0 \cdot 3) &= h_{H_1 \cap_T H_2}(3) = \{0.9\} \not\subseteq \{0.9, 1\} = \{0.9, 1\} \cap \{0.9, 1\} \\ &= h_{H_1 \cap_T H_2}(2) \cap h_{H_1 \cap_T H_2}(2) = h_{H_1 \cap_T H_2}(0 \cdot (1 \cdot 3)) \cap h_{H_1 \cap_T H_2}(1). \end{aligned}$$

Theorem 4.13. *The union of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideals of A .*

Proof. Assume that H and G are hesitant fuzzy strongly UP-ideals of A . By Theorem 3.3, we have H and G are constant hesitant fuzzy sets of A . Thus $H \cup_T G$ is a constant hesitant fuzzy set of A . By Theorem 3.3, we can conclude that $H \cup_T G$ is a hesitant fuzzy strongly UP-ideal of A . \square

The following example shows that the union of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A .

Example 4.14. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ \{0.2, 0.3\}, & \text{if } x = 1, \\ \{0.1, 0.3\}, & \text{if } x = 2, \\ \{0.1, 0.4\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ \{0.2, 0.3\}, & \text{if } x = 1, \\ \{0.2, 0.3\}, & \text{if } x = 2, \\ \{0.3\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cup_T H_2(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ \{0.2, 0.3\}, & \text{if } x = 1, \\ \{0.1, 0.3\}, & \text{if } x = 2, \\ \{0.3, 0.4\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cup_T H_2}(1 \cdot 3) = h_{H_1 \cup_T H_2}(2) = \{0.1, 0.3\} \not\subseteq \{0.2, 0.4\} = \{0.2, 0.3\} \cup \{0.3, 0.4\} = h_{H_1 \cup_T H_2}(1) \cup h_{H_1 \cup_T H_2}(3).$$

The following example shows that the union of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A .

Example 4.15. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ \{0\}, & \text{if } x = 1, \\ \{1\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0.8\}, & \text{if } x = 0, \\ \{0.8, 1\}, & \text{if } x = 1, \\ \{0, 0.2, 0.8\}, & \text{if } x = 2, \\ \{0, 0.8\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0.8\}, & \text{if } x = 0, \\ \{0.8, 1\}, & \text{if } x = 1, \\ \{1\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$h_{H_1 \cup_T H_2}(0) = \{0.8\} \not\subseteq \{1\} = h_{H_1 \cup_T H_2}(2).$$

The following example shows that the union of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A .

Example 4.16. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0\}, & \text{if } x = 1, \\ \{0, 0.2\}, & \text{if } x = 2, \\ \{0, 0.2\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A . We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cup_T H_2(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0.5\}, & \text{if } x = 1, \\ \{0, 0.2\}, & \text{if } x = 2, \\ \{0.5\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$h_{H_1 \cup_T H_2}(0) = \{1\} \not\subseteq \{0.5\} = h_{H_1 \cup_T H_2}(1).$$

Theorem 4.17. *The union of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideals of A .*

Proof. Assume that H and G are anti-hesitant fuzzy strongly UP-ideals of A . By Theorem 3.5, we have H and G are constant hesitant fuzzy sets of A . Thus $H \cup_T G$ is a constant hesitant fuzzy set of A . By Theorem 3.5, we can conclude that $H \cup_T G$ is an anti-hesitant fuzzy strongly UP-ideal of A . \square

We see that the intersection and the union can not be applied to several types of hesitant fuzzy sets on UP-algebras except hesitant fuzzy strongly UP-ideals and anti-hesitant fuzzy strongly UP-ideals. That make we will consider the others.

The concepts of the hesitant union, and the hesitant intersection of two hesitant fuzzy sets were introduced by Jun [7] in 2015. We will introduce the notions of the hesitant union and the hesitant intersection of any hesitant fuzzy sets and apply to UP-algebras.

Definition 4.18. Let $\{H_i \mid i \in I\}$ be a family of hesitant fuzzy sets on a reference set X . We define the hesitant intersection operation \cap on $\{H_i \mid i \in I\}$ by

$$\bigcap_{i \in I} H_i = \{(x, \bigcap_{i \in I} h_{H_i}(x)) \mid x \in X\},$$

which is called the *hesitant intersection* of hesitant fuzzy sets. In particular, $H_i \cap H_k = \{(x, h_{H_i}(x) \cap h_{H_k}(x)) \mid x \in X\}$ for all $i, k \in I$.

Theorem 4.19. *The hesitant intersection of hesitant fuzzy UP-subalgebras of A is a hesitant fuzzy UP-subalgebra of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of hesitant fuzzy UP-subalgebras of A . Let $x, y \in A$. Then $h_{H_i}(x \cdot y) \supseteq h_{H_i}(x) \cap h_{H_i}(y)$ for all $i \in I$. Thus

$$h_{\bigcap_{i \in I} H_i}(x \cdot y) = \bigcap_{i \in I} h_{H_i}(x \cdot y) \supseteq \bigcap_{i \in I} (h_{H_i}(x) \cap h_{H_i}(y)) = \bigcap_{i \in I} h_{H_i}(x) \cap \bigcap_{i \in I} h_{H_i}(y) = h_{\bigcap_{i \in I} H_i}(x) \cap h_{\bigcap_{i \in I} H_i}(y).$$

Hence, $\bigcap_{i \in I} H_i$ is a hesitant fuzzy UP-subalgebra of A . □

Theorem 4.20. *The hesitant intersection of hesitant fuzzy UP-filters of A is a hesitant fuzzy UP-filter of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of hesitant fuzzy UP-filters of A . Let $x, y \in A$. Then $h_{H_i}(0) \supseteq h_{H_i}(x)$ and $h_{H_i}(y) \supseteq h_{H_i}(x \cdot y) \cap h_{H_i}(x)$ for all $i \in I$. Thus $h_{\bigcap_{i \in I} H_i}(0) = \bigcap_{i \in I} h_{H_i}(0) \supseteq \bigcap_{i \in I} h_{H_i}(x) = h_{\bigcap_{i \in I} H_i}(x)$

and

$$h_{\bigcap_{i \in I} H_i}(y) = \bigcap_{i \in I} h_{H_i}(y) \supseteq \bigcap_{i \in I} (h_{H_i}(x \cdot y) \cap h_{H_i}(x)) = \bigcap_{i \in I} h_{H_i}(x \cdot y) \cap \bigcap_{i \in I} h_{H_i}(x) = h_{\bigcap_{i \in I} H_i}(x \cdot y) \cap h_{\bigcap_{i \in I} H_i}(x).$$

Hence, $\bigcap_{i \in I} H_i$ is a hesitant fuzzy UP-filter of A . □

Theorem 4.21. *The hesitant intersection of hesitant fuzzy UP-ideals of A is a hesitant fuzzy UP-ideal of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of hesitant fuzzy UP-ideals of A . Let $x, y \in A$. Then $h_{H_i}(0) \supseteq h_{H_i}(x)$ and $h_{H_i}(x \cdot z) \supseteq h_{H_i}(x \cdot (y \cdot z)) \cap h_{H_i}(y)$ for all $i \in I$. Thus $h_{\bigcap_{i \in I} H_i}(0) = \bigcap_{i \in I} h_{H_i}(0) \supseteq \bigcap_{i \in I} h_{H_i}(x) =$

$h_{\bigcap_{i \in I} H_i}(x)$ and

$$\begin{aligned} h_{\bigcap_{i \in I} H_i}(x \cdot z) &= \bigcap_{i \in I} h_{H_i}(x \cdot z) \supseteq \bigcap_{i \in I} (h_{H_i}(x \cdot (y \cdot z)) \cap h_{H_i}(y)) = \bigcap_{i \in I} h_{H_i}(x \cdot (y \cdot z)) \cap \bigcap_{i \in I} h_{H_i}(y) \\ &= h_{\bigcap_{i \in I} H_i}(x \cdot (y \cdot z)) \cap h_{\bigcap_{i \in I} H_i}(y). \end{aligned}$$

Hence, $\bigcap_{i \in I} H_i$ is a hesitant fuzzy UP-ideal of A . □

Theorem 4.22. *The hesitant intersection of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideal of A.*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of hesitant fuzzy strongly UP-ideals of A. Then H_i is a hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 3.3, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcap_{i \in I} H_i$ is a constant hesitant fuzzy set on A. By Theorem 3.3, we know that $\bigcap_{i \in I} H_i$ is a hesitant fuzzy strongly UP-ideal of A. \square

The following example shows that the hesitant intersection of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A.

Example 4.23. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ [0.3, 0.4], & \text{if } x = 1, \\ [0.3, 0.5], & \text{if } x = 2, \\ [0, 0.5], & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ \{0.3, 0.4\}, & \text{if } x = 1, \\ \{0.3, 0.4\}, & \text{if } x = 2, \\ \{0.3\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A. We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$h_{H_1 \cap H_2}(x) = \begin{cases} \{0.3\}, & \text{if } x = 0, \\ \{0.3\}, & \text{if } x = 1, \\ \{0.3, 0.4\}, & \text{if } x = 2, \\ \{0.3\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap H_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cap H_2}(1 \cdot 3) = h_{H_1 \cap H_2}(2) = \{0.3, 0.4\} \not\subseteq \{0.3\} = \{0.3\} \cup \{0.3\} = h_{H_1 \cap H_2}(1) \cup h_{H_1 \cap H_2}(3).$$

The following example shows that the hesitant intersection of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A.

Example 4.24. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.6\}, & \text{if } x = 0, \\ \{0.6, 0.7\}, & \text{if } x = 1, \\ \{0.5, 0.6\}, & \text{if } x = 2, \\ \{0.5, 0.6\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.6\}, & \text{if } x = 0, \\ \{0.5, 0.6, 0.7\}, & \text{if } x = 1, \\ \{0.4, 0.5, 0.6\}, & \text{if } x = 2, \\ \{0.4, 0.6\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A . We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$h_{H_1 \cap H_2}(x) = \begin{cases} \{0.6\}, & \text{if } x = 0, \\ \{0.6, 0.7\}, & \text{if } x = 1, \\ \{0.5, 0.6\}, & \text{if } x = 2, \\ \{0.6\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap H_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$h_{H_1 \cap H_2}(2) = \{0.5, 0.6\} \not\subseteq \{0.6, 0.7\} = \{0.6\} \cup \{0.6, 0.7\} = h_{H_1 \cap H_2}(3) \cup h_{H_1 \cap H_2}(1) = h_{H_1 \cap H_2}(1 \cdot 2) \cup h_{H_1 \cap H_2}(1).$$

The following example shows that the hesitant intersection of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A .

Example 4.25. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.9\}, & \text{if } x = 0, \\ \{0.9\}, & \text{if } x = 1, \\ \{0.9, 1\}, & \text{if } x = 2, \\ \{0.9, 1\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.9\}, & \text{if } x = 0, \\ \{0.9, 1\}, & \text{if } x = 1, \\ \{0.9\}, & \text{if } x = 2, \\ \{0.9, 1\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A . We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$h_{H_1 \cap H_2}(x) = \begin{cases} \{0.9\}, & \text{if } x = 0, \\ \{0.9\}, & \text{if } x = 1, \\ \{0.9\}, & \text{if } x = 2, \\ \{0.9, 1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap H_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$\begin{aligned} h_{H_1 \cap H_2}(0 \cdot 3) &= h_{H_1 \cap H_2}(3) = \{0.9, 1\} \not\subseteq \{0.9\} = \{0.9\} \cup \{0.9\} \\ &= h_{H_1 \cap H_2}(1) \cup h_{H_1 \cap H_2}(2) = h_{H_1 \cap H_2}(0 \cdot (2 \cdot 3)) \cup h_{H_1 \cap H_2}(2). \end{aligned}$$

Theorem 4.26. *The hesitant intersection of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideal of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy strongly UP-ideals of A . Then H_i is an anti-hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 3.5, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcap_{i \in I} H_i$ is a constant hesitant fuzzy set on A . By Theorem 3.5, we know

that $\bigcap_{i \in I} H_i$ is an anti-hesitant fuzzy strongly UP-ideal of A . □

Definition 4.27. Let $\{H_i \mid i \in I\}$ be a family of hesitant fuzzy sets on a reference set X . We define the hesitant union operation \cup on $\{H_i \mid i \in I\}$ by

$$\bigcup_{i \in I} H_i = \{(x, \bigcup_{i \in I} h_{H_i}(x)) \mid x \in X\},$$

which is called the *hesitant union* of hesitant fuzzy sets. In particular, $H_i \cup H_k = \{(x, h_{H_i}(x) \cup h_{H_k}(x)) \mid x \in X\}$ for all $i, k \in I$.

The following example shows that the hesitant union of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A .

Example 4.28. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} [0.6, 0.9], & \text{if } x = 0, \\ \{0.8\}, & \text{if } x = 1, \\ \{0.7\}, & \text{if } x = 2, \\ \{0.6\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} [0.6, 0.8], & \text{if } x = 0, \\ \{0.7\}, & \text{if } x = 1, \\ \{0.7\}, & \text{if } x = 2, \\ [0.6, 0.8], & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A . We thus obtain the hesitant union of H_1 and H_2 as follows:

$$h_{H_1 \cup H_2}(x) = \begin{cases} [0.6, 0.9], & \text{if } x = 0, \\ \{0.7, 0.8\}, & \text{if } x = 1, \\ \{0.7\}, & \text{if } x = 2, \\ [0.6, 0.8], & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup H_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$h_{H_1 \cup H_2}(1 \cdot 3) = h_{H_1 \cap H_2}(2) = \{0.7\} \not\subseteq \{0.7, 0.8\} = \{0.7, 0.8\} \cap [0.6, 0.8] = h_{H_1 \cup H_2}(1) \cap h_{H_1 \cup H_2}(3).$$

The following example shows that the hesitant union of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A .

Example 4.29. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0\} \cup [0.8, 1], & \text{if } x = 0, \\ \{0\} \cup [0.8, 1], & \text{if } x = 1, \\ \{0.8\}, & \text{if } x = 2, \\ \{0.8\}, & \text{if } x = 3, \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} [0, 0.8], & \text{if } x = 0, \\ \{0.8\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x = 2, \\ \{0\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A . We thus obtain the hesitant union of H_1 and H_2 as follows:

$$h_{H_1 \cup H_2}(x) = \begin{cases} [0, 1], & \text{if } x = 0, \\ \{0\} \cup [0.8, 1], & \text{if } x = 1, \\ \{0.8\}, & \text{if } x = 2, \\ \{0, 0.8\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup H_2$ is not a hesitant fuzzy UP-filter of A because

$$h_{H_1 \cup H_2}(2) = \{0.8\} \not\subseteq \{0, 0.8\} = \{0, 0.8\} \cap (\{0\} \cup [0.8, 1]) = h_{H_1 \cup H_2}(3) \cap h_{H_1 \cup H_2}(1) = h_{H_1 \cup H_2}(1 \cdot 2) \cap h_{H_1 \cup H_2}(1).$$

The following example shows that the hesitant union of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A .

Example 4.30. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation \cdot defined by the following Cayley table:

\cdot	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{0.9, 1\}, & \text{if } x = 1, \\ \{1\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3, \end{cases} \quad \text{and} \quad h_{H_2}(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0.9, 1\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A . We thus obtain the hesitant union of H_1 and H_2 as follows:

$$h_{H_1 \cup H_2}(x) = \begin{cases} \{0.9, 1\}, & \text{if } x = 0, \\ \{0.9, 1\}, & \text{if } x = 1, \\ \{0.9, 1\}, & \text{if } x = 2, \\ \{1\}, & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup H_2$ is not a hesitant fuzzy UP-ideal of A because

$$\begin{aligned} h_{H_1 \cup H_2}(0 \cdot 3) &= h_{H_1 \cup H_2}(3) = \{1\} \not\subseteq \{0.9, 1\} = \{0.9, 1\} \cap \{0.9, 1\} \\ &= h_{H_1 \cup H_2}(1) \cap h_{H_1 \cup H_2}(2) = h_{H_1 \cup H_2}(0 \cdot (2 \cdot 3)) \cap h_{H_1 \cup H_2}(2). \end{aligned}$$

Theorem 4.31. *The hesitant union of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideal of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of hesitant fuzzy strongly UP-ideals of A . Then H_i is a hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 3.3, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcup_{i \in I} H_i$ is a constant hesitant fuzzy set on A . By Theorem 3.3, we know that $\bigcup_{i \in I} H_i$ is a hesitant fuzzy strongly UP-ideal of A . □

Theorem 4.32. *The hesitant union of anti-hesitant fuzzy UP-subalgebras of A is an anti-hesitant fuzzy UP-subalgebra of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-subalgebras of A . Let $x, y \in A$. Then $h_{H_i}(x \cdot y) \subseteq h_{H_i}(x) \cup h_{H_i}(y)$ for all $i \in I$. Thus

$$h_{\bigcup_{i \in I} H_i}(x \cdot y) = \bigcup_{i \in I} h_{H_i}(x \cdot y) \subseteq \bigcup_{i \in I} (h_{H_i}(x) \cup h_{H_i}(y)) = \bigcup_{i \in I} h_{H_i}(x) \cup \bigcup_{i \in I} h_{H_i}(y) = h_{\bigcup_{i \in I} H_i}(x) \cup h_{\bigcup_{i \in I} H_i}(y).$$

Hence, $\bigcup_{i \in I} H_i$ is an anti-hesitant fuzzy UP-subalgebra of A . □

Theorem 4.33. *The hesitant union of anti-hesitant fuzzy UP-filters of A is an anti-hesitant fuzzy UP-filter of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-filters of A . Let $x, y \in A$. Then $h_{H_i}(0) \subseteq h_{H_i}(x)$ and $h_{H_i}(y) \subseteq h_{H_i}(x \cdot y) \cup h_{H_i}(x)$ for all $i \in I$. Thus $h_{\bigcup_{i \in I} H_i}(0) = \bigcup_{i \in I} h_{H_i}(0) \subseteq \bigcup_{i \in I} h_{H_i}(x) = h_{\bigcup_{i \in I} H_i}(x)$ and

$$h_{\bigcup_{i \in I} H_i}(y) = \bigcup_{i \in I} h_{H_i}(y) \subseteq \bigcup_{i \in I} (h_{H_i}(x \cdot y) \cup h_{H_i}(x)) = \bigcup_{i \in I} h_{H_i}(x \cdot y) \cup \bigcup_{i \in I} h_{H_i}(x) = h_{\bigcup_{i \in I} H_i}(x \cdot y) \cup h_{\bigcup_{i \in I} H_i}(x).$$

Hence, $\bigcup_{i \in I} H_i$ is an anti-hesitant fuzzy UP-filter of A . □

Theorem 4.34. *The hesitant union of anti-hesitant fuzzy UP-ideals of A is an anti-hesitant fuzzy UP-ideal of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-ideals of A . Let $x, y \in A$. Then $h_{H_i}(0) \subseteq h_{H_i}(x)$ and $h_{H_i}(x \cdot z) \subseteq h_{H_i}(x \cdot (y \cdot z)) \cup h_{H_i}(y)$ for all $i \in I$. Thus $h_{\bigcup_{i \in I} H_i}(0) = \bigcup_{i \in I} h_{H_i}(0) \subseteq$

$$\bigcup_{i \in I} h_{H_i}(x) = h_{\bigcup_{i \in I} H_i}(x) \text{ and}$$

$$\begin{aligned} h_{\bigcup_{i \in I} H_i}(x \cdot z) &= \bigcup_{i \in I} h_{H_i}(x \cdot z) \subseteq \bigcup_{i \in I} (h_{H_i}(x \cdot (y \cdot z)) \cup h_{H_i}(y)) \\ &= \bigcup_{i \in I} h_{H_i}(x \cdot (y \cdot z)) \cup \bigcup_{i \in I} h_{H_i}(y) = h_{\bigcup_{i \in I} H_i}(x \cdot (y \cdot z)) \cup h_{\bigcup_{i \in I} H_i}(y). \end{aligned}$$

Hence, $\bigcup_{i \in I} H_i$ is an anti-hesitant fuzzy UP-ideal of A . □

Theorem 4.35. *The hesitant union of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideal of A .*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy strongly UP-ideals of A . Then H_i is an anti-hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 3.5, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcup_{i \in I} H_i$ is a constant hesitant fuzzy set on A . By Theorem 3.5, we know

that $\bigcup_{i \in I} H_i$ is an anti-hesitant fuzzy strongly UP-ideal of A . □

5. Operations on hesitant fuzzy soft sets

The concepts of the “AND” operation and the “OR” operation on two hesitant fuzzy soft sets and the union and the intersection of two hesitant fuzzy soft sets were introduced by Wang et al. [33] in 2014.

Definition 5.1 ([33]). Let (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) be two hesitant fuzzy soft sets over a reference set X . The AND operation on (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = Y_1 \times Y_2$, and
- (2) $\tilde{H}[p_1, p_2] = \tilde{H}_1[p_1] \cap \tilde{H}_2[p_2]$ for any $(p_1, p_2) \in Y$.

We write $(\tilde{H}_1, Y_1) \tilde{\wedge} (\tilde{H}_2, Y_2) = (\tilde{H}, Y)$.

Definition 5.2 ([33]). Let (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) be two hesitant fuzzy soft sets over a reference set X . The OR operation on (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = Y_1 \times Y_2$, and
- (2) $\tilde{H}[p_1, p_2] = \tilde{H}_1[p_1] \cup \tilde{H}_2[p_2]$ for any $(p_1, p_2) \in Y$.

We write $(\tilde{H}_1, Y_1) \tilde{\vee} (\tilde{H}_2, Y_2) = (\tilde{H}, Y)$.

Definition 5.3 ([33]). Let (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) be two hesitant fuzzy soft sets over a reference set X . The union of (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = Y_1 \cup Y_2$, and
- (2) for any $p \in Y$,

$$\tilde{H}[p] = \begin{cases} \tilde{H}_1[p], & \text{if } p \in Y_1 \setminus Y_2, \\ \tilde{H}_2[p], & \text{if } p \in Y_2 \setminus Y_1, \\ \tilde{H}_1[p] \cup \tilde{H}_2[p], & \text{if } p \in Y_1 \cap Y_2. \end{cases}$$

We write $(\tilde{H}_1, Y_1) \tilde{\cup} (\tilde{H}_2, Y_2) = (\tilde{H}, Y)$.

Definition 5.4 ([33]). Let (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) be two hesitant fuzzy soft sets over a reference set X . The intersection of (\tilde{H}_1, Y_1) and (\tilde{H}_2, Y_2) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = Y_1 \cap Y_2 \neq \emptyset$, and
- (2) $\tilde{H}[p] = \tilde{H}_1[p] \cap \tilde{H}_2[p]$ for all $p \in Y$.

We write $(\tilde{H}_1, Y_1) \tilde{\cap}_r (\tilde{H}_2, Y_2) = (\tilde{H}, Y)$.

We will introduce the notions of the union, the intersection, the restricted union, the extended intersection, the AND, and the OR of any hesitant fuzzy soft sets and apply to UP-algebras.

Definition 5.5. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The *restricted union* of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = \bigcap_{i \in I} Y_i \neq \emptyset$, and
- (2) $\tilde{H}[p] = \bigcup_{i \in I} \tilde{H}_i[p]$ for all $p \in Y$.

We write $\bigcup_r(\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.6. *The restricted union of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A .*

Proof. Let (\tilde{H}_i, Y_i) be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\tilde{H}_i[p_i]$ is an anti-hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that (\tilde{H}, Y) is the restricted union of (\tilde{H}_i, Y_i) for all $i \in I$ and let $p \in Y$. Then $p \in Y = \bigcap_{i \in I} Y_i \neq \emptyset$ and $\tilde{H}[p] = \bigcup_{i \in I} \tilde{H}_i[p]$. Thus $p \in Y_i$

for all $i \in I$ and so $\tilde{H}_i[p]$ is a hesitant fuzzy UP-subalgebras of A for all $i \in I$. By Theorem 4.32, we have $\tilde{H}[p] = \bigcup_{i \in I} \tilde{H}_i[p]$ is a hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a p-hesitant fuzzy soft

UP-subalgebra of A . Since p is arbitrary, we know that (\tilde{H}, Y) is a hesitant fuzzy soft UP-subalgebra of A . □

In the same way as Theorem 5.6, we can use Theorems 4.33, 4.34, 4.35, 4.31, respectively, to prove the following theorems.

Theorem 5.7. *The restricted union of anti-hesitant fuzzy soft UP-filters of A is an anti-hesitant fuzzy soft UP-filter of A .*

Theorem 5.8. *The restricted union of anti-hesitant fuzzy soft UP-ideals of A is an anti-hesitant fuzzy soft UP-ideal of A .*

Theorem 5.9. *The restricted union of anti-hesitant fuzzy soft strongly UP-ideals of A is an anti-hesitant fuzzy soft strongly UP-ideal of A .*

Theorem 5.10. *The restricted union of hesitant fuzzy soft strongly UP-ideals of A is a hesitant fuzzy soft strongly UP-ideal of A .*

By Example 4.28 (resp., Example 4.29, Example 4.30), we can imply that the restricted union of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 5.11. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The union of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

$$(1) Y = \bigcup_{i \in I} Y_i, \text{ and}$$

$$(2) \text{ for any } p \in Y \text{ with } p \in \bigcap_{j \in J} Y_j \setminus \bigcup_{k \in I \setminus J} Y_k, \text{ where } \emptyset \neq J \subseteq I, \tilde{H}[p] = \bigcup_{j \in J} \tilde{H}_j[p].$$

We write $\bigcup_{i \in I} (\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.12. *The union of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A .*

Proof. Let (\tilde{H}_i, Y_i) be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\tilde{H}_i[p_i]$ is an anti-hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that (\tilde{H}, Y) is the union of (\tilde{H}_i, Y_i) for all $i \in I$. Then $Y = \bigcup_{i \in I} Y_i$. Let $p \in Y$.

If $p \in \bigcap_{i \in I} Y_i \neq \emptyset$, then it follows from Theorem 5.6 that (\tilde{H}, Y) is a p -anti-hesitant fuzzy soft UP-subalgebra of A .

If $p \in \bigcap_{j \in J} Y_j \setminus \bigcup_{k \in I \setminus J} Y_k$, where $\emptyset \neq J \subseteq I$, then $\tilde{H}[p] = \bigcup_{j \in J} \tilde{H}_j[p]$ and $p \in Y_j$ for all $j \in J$ but $p \notin Y_k$ for some $k \in I \setminus J$. Thus $\tilde{H}_j[p]$ is an anti-hesitant fuzzy UP-subalgebra of A for all $j \in J$. By Theorem 4.32, we have $\tilde{H}[p] = \bigcup_{j \in J} \tilde{H}_j[p]$ is an anti-hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a p -anti-hesitant

fuzzy soft UP-subalgebra of A . Since p is arbitrary, we know that (\tilde{H}, Y) is an anti-hesitant fuzzy soft UP-subalgebra of A . \square

In the same way as Theorem 5.12, we can use Theorem 5.7 (resp., Theorems 5.8, 5.9, 5.10) and Theorem 4.33 (resp., Theorems 4.34, 4.35, 4.31) to prove that the union of anti-hesitant fuzzy soft UP-filters (resp., anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly

UP-ideals) of A is an anti-hesitant fuzzy soft UP-filter (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A . By the way, we also confirm that the union of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 5.13. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The *intersection* of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = \bigcap_{i \in I} Y_i \neq \emptyset$, and
- (2) $\tilde{H}[p] = \bigcap_{i \in I} \tilde{H}_i[p]$ for all $p \in Y$.

We write $\bigcap_{i \in I} (\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.14. *The intersection of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A .*

Proof. Let (\tilde{H}_i, Y_i) be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\tilde{H}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that (\tilde{H}, Y) is the intersection of (\tilde{H}_i, Y_i) for all $i \in I$ and let $p \in Y$. Then $p \in Y = \bigcap_{i \in I} Y_i \neq \emptyset$ and $\tilde{H}[p] = \bigcap_{i \in I} \tilde{H}_i[p]$. Thus $p \in Y_i$ for all $i \in I$ and so $\tilde{H}_i[p]$ is a hesitant fuzzy UP-subalgebras of A for all $i \in I$. By Theorem 4.19, we have $\tilde{H}[p] = \bigcap_{i \in I} \tilde{H}_i[p]$ is a hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a p -hesitant fuzzy soft UP-subalgebra of A . Since p is arbitrary, we know that (\tilde{H}, Y) is a hesitant fuzzy soft UP-subalgebra of A . \square

In the same way as Theorem 5.14, we can use Theorems 4.20, 4.21, 4.22, and 4.26, respectively, to prove the following theorems.

Theorem 5.15. *The intersection of hesitant fuzzy soft UP-filters of A is a hesitant fuzzy soft UP-filter of A .*

Theorem 5.16. *The intersection of hesitant fuzzy soft UP-ideals of A is a hesitant fuzzy soft UP-ideal of A .*

Theorem 5.17. *The intersection of hesitant fuzzy soft strongly UP-ideals of A is a hesitant fuzzy soft strongly UP-ideal of A .*

Theorem 5.18. *The intersection of anti-hesitant fuzzy soft strongly UP-ideals of A is an anti-hesitant fuzzy soft strongly UP-ideal of A .*

By Example 4.23 (resp., Example 4.24, 4.25), we can imply that the intersection of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals) of A is not an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal) of A in general.

Definition 5.19. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The *extended intersection* of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = \bigcup_{i \in I} Y_i$, and

(2) for any $p \in Y$ with $p \in \bigcap_{j \in J} Y_j \setminus \bigcup_{k \in I \setminus J} Y_k$, where $\emptyset \neq J \subseteq I$, $\tilde{H}[p] = \bigcap_{j \in J} \tilde{H}_j[p]$.

We write $\bigcap_{i \in I} e(\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.20. *The extended intersection of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A .*

Proof. Let (\tilde{H}_i, Y_i) be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\tilde{H}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that (\tilde{H}, Y) is the extended intersection of (\tilde{H}_i, Y_i) for all $i \in I$. Then $Y = \bigcup_{i \in I} Y_i$. Let $p \in Y$.

If $p \in \bigcap_{i \in I} Y_i \neq \emptyset$, then it follows from Theorem 5.16 that (\tilde{H}, Y) is a p -hesitant fuzzy soft UP-subalgebra of A .

If $p \in \bigcap_{j \in J} Y_j \setminus \bigcup_{k \in I \setminus J} Y_k$, where $\emptyset \neq J \subseteq I$, then $\tilde{H}[p] = \bigcap_{j \in J} \tilde{H}_j[p]$ and $p \in Y_j$ for all $j \in J$ but $p \notin Y_k$ for some $k \in I \setminus J$. Thus $\tilde{H}_j[p]$ is a hesitant fuzzy UP-subalgebra of A for all $j \in J$. By Theorem 4.19, we have $\tilde{H}[p] = \bigcap_{j \in J} \tilde{H}_j[p]$ is a hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a p -hesitant fuzzy soft

UP-subalgebra of A . Since p is arbitrary, we know that (\tilde{H}, Y) is a hesitant fuzzy soft UP-subalgebra of A . \square

In the same way as Theorem 5.20, we can use Theorem 5.15 (resp., Theorems 5.16, 5.17, 5.18) and Theorem 4.20 (resp., Theorems 4.21, 4.22, 4.23) to prove that the extended intersection of hesitant fuzzy soft UP-filters (resp., hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-filter (resp., a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal) of A . By the way, we also confirm that the extended intersection of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 5.21. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The AND of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = \prod_{i \in I} Y_i$, and
- (2) $\tilde{H}[(p_i)_{i \in I}] = \bigcap_{i \in I} \tilde{H}_i[p_i]$ for all $(p_i)_{i \in I} \in Y$.

We write $\bigwedge_{i \in I} (\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.22. *The AND of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A .*

Proof. Let (\tilde{H}_i, Y_i) be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$ and let $p_i \in Y_i$ for all $i \in I$. Then $\tilde{H}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A . Assume that (\tilde{H}, Y) is the AND of (\tilde{H}_i, Y_i) for all $i \in I$. Then $Y = \prod_{i \in I} Y_i$ and $\tilde{H}[(p_i)_{i \in I}] = \bigcap_{i \in I} \tilde{H}_i[p_i]$. By Theorem 4.19, we have $\tilde{H}[(p_i)_{i \in I}] = \bigcap_{i \in I} \tilde{H}_i[p_i]$ is a

hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a $(p_i)_{i \in I}$ -hesitant fuzzy soft UP-subalgebra of A . Since $(p_i)_{i \in I}$ is arbitrary, we know that (\tilde{H}, Y) is a hesitant fuzzy soft UP-subalgebra of A . \square

In the same way as Theorem 5.22, we can use Theorem 4.20 (resp., Theorems 4.21 and 4.22) to prove that the AND of hesitant fuzzy soft UP-filters (resp., hesitant fuzzy soft UP-ideals and hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-filter (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A .

Definition 5.23. Let $\{(\tilde{H}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X . The OR of (\tilde{H}_i, Y_i) is defined to be the hesitant fuzzy soft sets (\tilde{H}, Y) satisfying the following properties:

- (1) $Y = \prod_{i \in I} Y_i$, and
- (2) $\tilde{H}[(p_i)_{i \in I}] = \bigcup_{i \in I} \tilde{H}_i[p_i]$ for all $(p_i)_{i \in I} \in Y$.

We write $\bigvee_{i \in I} (\tilde{H}_i, Y_i) = (\tilde{H}, Y)$.

Theorem 5.24. The OR of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A .

Proof. Let (\tilde{H}_i, Y_i) be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$ and let $p_i \in Y_i$ for all $i \in I$. Then $\tilde{H}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A . Assume that (\tilde{H}, Y) is the OR of (\tilde{H}_i, Y_i) for all $i \in I$. Then $Y = \prod_{i \in I} Y_i$ and $\tilde{H}[(p_i)_{i \in I}] = \bigcup_{i \in I} \tilde{H}_i[p_i]$. By Theorem 4.32, we have $\tilde{H}[(p_i)_{i \in I}]$ is an anti-hesitant fuzzy UP-subalgebra of A . Therefore, (\tilde{H}, Y) is a $(p_i)_{i \in I}$ -anti-hesitant fuzzy soft UP-subalgebra of A . Since p is arbitrary, we know that (\tilde{H}, Y) is an anti-hesitant fuzzy soft UP-subalgebra of A . \square

In the same way as Theorem 5.24, we can use Theorem 4.33 (resp., Theorems 4.34 and 4.35) to prove that the OR of anti-hesitant fuzzy soft UP-filters (resp., anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-filter (resp., anti-hesitant fuzzy soft UP-ideal and anti-hesitant fuzzy soft strongly UP-ideal) of A .

Acknowledgment

This work was supported by the Unit of Excellence, University of Phayao. The authors would also like to thank the anonymous referee for giving many helpful suggestion on the revision of present paper.

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