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On determinants and inverses of some triband Toeplitz matrices with permuted columns

Pingyun Li[a], Zhaolin Jiang[a,*], Yanpeng Zheng[b,*]

[a]School of Mathematics and Statistics, Linyi University, Linyi 276000, China.
[b]School of Automation and Electrical Engineering, Linyi University, Linyi 276000, China.

Abstract

In this paper, we study the triband Toeplitz and Hankel matrices with permuted columns. We obtain expressions for the determinants and the inverses of the triband Toeplitz and Hankel matrices with permuted columns by the Sherman-Morrison-Woodbury formula, where the Pell numbers play an essential role.

Keywords: Determinant, inverse, Laplace theorem, Pell number, triband Toeplitz matrices with permuted columns, Sherman-Morrison-Woodbury.

2010 MSC: 15A09, 15A15, 15B05, 65F40.

1. Introduction

In this paper, we study the determinants and inverses of triband Toeplitz matrices with permuted columns and triband Hankel matrices with permuted columns, respectively. The matrix $A = (a_{i,j})_{i,j=1}^{n}$ is called a triband Toeplitz matrices with permuted columns, whose entries are as follows

$$a_{i,j} = \begin{cases} \delta_i, & 1 \leqslant i \leqslant n, j = 1, \\ \epsilon_i, & 1 \leqslant i \leqslant n, j = n, \\ -\nu, & i = j+2, 2 \leqslant j \leqslant n-1, \\ -2\nu, & i = j+1, 2 \leqslant j \leqslant n-1, \\ \nu, & 2 \leqslant i = j \leqslant n-1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\epsilon_1, \cdots \epsilon_n, \delta_1, \cdots \delta_n$ and ν are complex numbers with $\nu \neq 0$. More explicitly,

*Corresponding authors
Email address: jzh1208@sina.com (Zhaolin Jiang)
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$$A = \begin{pmatrix} \delta_1 & 0 & \cdots & \cdots & \cdots & 0 & \epsilon_1 \\ \delta_2 & \nu & 0 & & & \vdots & \epsilon_2 \\ \delta_3 & -2\nu & \nu & \ddots & & \vdots & \epsilon_3 \\ \delta_4 & -\nu & -2\nu & \nu & \ddots & \vdots & \vdots \\ \vdots & 0 & -\nu & -2\nu & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \nu & \epsilon_{n-1} \\ \delta_n & 0 & \cdots & 0 & -\nu & -2\nu & \epsilon_n \end{pmatrix}_{n\times n}. \tag{1.1}$$

Let $\hat{I}_n$ be the $n \times n$ "reverse unit matrix", which has ones along the secondary diagonal and zeros elsewhere. That is,

$$\hat{I}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Let A be defined as in (1.1). A matrix of the form $B := \hat{I}_n A$ is called a triband Hankel matrices with permuted columns of Type I. A matrix of the form $C := A\hat{I}_n$ is called a triband Hankel matrices with permuted columns of Type II. In this case, we say B and C are induced by A. That is,

$$B = \begin{pmatrix} \delta_n & 0 & \cdots & 0 & -\nu & -2\nu & \epsilon_n \\ \delta_{n-1} & \vdots & \iddots & -\nu & -2\nu & \nu & \epsilon_{n-1} \\ \vdots & 0 & \iddots & \iddots & \iddots & 0 & \epsilon_{n-2} \\ \vdots & -\nu & \iddots & \nu & \iddots & \vdots & \vdots \\ \delta_3 & -2\nu & \iddots & 0 & & \vdots & \vdots \\ \delta_2 & \nu & 0 & & & 0 & \epsilon_2 \\ \delta_1 & 0 & \cdots & \cdots & \cdots & 0 & \epsilon_1 \end{pmatrix}_{n\times n} \tag{1.2}$$

and

$$C = \begin{pmatrix} \epsilon_1 & 0 & \cdots & \cdots & \cdots & 0 & \delta_1 \\ \epsilon_2 & 0 & & & 0 & \nu & \delta_2 \\ \epsilon_3 & 0 & & \iddots & \iddots & -2\nu & \delta_3 \\ \vdots & \vdots & \iddots & \nu & \iddots & -\nu & \vdots \\ \epsilon_{n-1} & 0 & \iddots & \iddots & \iddots & 0 & \delta_{n-2} \\ \epsilon_{n-1} & \nu & -2\nu & -\nu & \iddots & \vdots & \delta_{n-1} \\ \epsilon_n & -2\nu & -\nu & 0 & \cdots & 0 & \delta_n \end{pmatrix}_{n\times n}. \tag{1.3}$$

Many special banded matrices, such as Toeplitz banded matrix, symmetric Toeplitz banded matrix, especially tridiagonal Toeplitz matrix, can be seen in [4, 5, 8, 9, 12]. The inverse of Toeplitz banded matrix has been studied by many scholars. For example, Roebuck and Barnett [14] made some good conclusions about Toeplitz matrix. Yamamoto and Ikebe [16] studied the inverse matrix of general band matrix, including the inverse matrix of special Toeplitz band matrix, while the concrete formula of the inverse matrix of Toeplitz band matrix is deduced in [15]. Besides, Bareiss [2] proposed a recursive method for solving Toeplitz equations. Both [1, 11] gave a method for calculating the inverse matrix of symmetric Toeplitz strip matrix. Especially, two special cases of two symmetric Toeplitz band matrices are studied in [7, 13].

Now we introduce a Pell number sequence [3, 10, 17–19] that plays a very important role in our main results. The Pell number P_n satisfies the following recurrence:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{where } P_0 = 0,\ P_1 = 1,\ n \geqslant 2. \tag{1.4}$$

It could be verified that

$$\sum_{i=1}^{n} iP_i = \frac{3n-2}{2}P_n + \frac{n-1}{2}P_{n-1} + \frac{1}{2}, \tag{1.5}$$

$$\sum_{i=1}^{n} P_{n+1-i}a^i = \frac{aP_n + a^2P_{n+1} - a^{n+2}}{1+2a-a^2} (a \neq 1 \pm \sqrt{2}). \tag{1.6}$$

2. Determinants and inverses

In this section, we derive explicit formulas for the determinants and inverses of triband Toeplitz matrices with permuted columns.

Theorem 2.1. *Let* $A = (a_{i,j})_{i,j=1}^{n}$ $(n \geqslant 3)$ *be given as in* (1.1)*. Then*

$$\det A = \sum_{m=1}^{n-1} P_{n-m}(\delta_1 \nu^{n-2}\epsilon_{m+1} + (-1)^{n+1}\epsilon_1\nu^{n-2}\delta_{m+1}),$$

where P_i $(i = n - m)$ *is the* i^{th} *Pell number.*

Proof. Expanding matrix $A = (a_{i,j})_{i,j=1}^{n}$ $(n \geqslant 3)$ by the first row, we get

$$\det A = \delta_1 \det A_1 + (-1)^{n+1}\epsilon_1 \det A_2, \tag{2.1}$$

where

$$A_1 = \begin{pmatrix} \nu & 0 & \cdots & \cdots & \cdots & \epsilon_2 \\ -2\nu & \nu & \ddots & & \vdots & \epsilon_3 \\ -\nu & -2\nu & \nu & \ddots & \vdots & \vdots \\ 0 & -\nu & -2\nu & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \nu & \epsilon_{n-1} \\ 0 & \cdots & 0 & -\nu & -2\nu & \epsilon_n \end{pmatrix}_{(n-1)\times(n-1)},$$

and

$$A_2 = \begin{pmatrix} \delta_2 & \nu & 0 & \cdots & \cdots & 0 \\ \delta_3 & -2\nu & \nu & \ddots & & \vdots \\ \delta_4 & -\nu & -2\nu & \nu & \ddots & \vdots \\ \vdots & 0 & -\nu & -2\nu & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \nu \\ \delta_n & 0 & \cdots & 0 & -\nu & -2\nu \end{pmatrix}_{(n-1)\times(n-1)}.$$

By performing a series of elementary transformations on A_1, we obtain the matrix

$$B_1 = \begin{pmatrix} \nu & 0 & \cdots & \cdots & \cdots & \epsilon_2 \\ 0 & \nu & \ddots & & \vdots & \epsilon_3 + 2\epsilon_2 \\ 0 & 0 & \nu & \ddots & \vdots & \epsilon_4 + 2\epsilon_3 + 5\epsilon_2 \\ 0 & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \nu & \sum_{m=1}^{n-2} P_{n-m}\epsilon_{m+1} \\ 0 & \cdots & 0 & 0 & 0 & \sum_{m=1}^{n-1} P_{n-m}\epsilon_{m+1} \end{pmatrix}_{(n-1)\times(n-1)}. \tag{2.2}$$

Calculating the determinant of matrix B_1,

$$\det B_1 = \nu^{n-2} \sum_{m=1}^{n-1} P_{n-m}\epsilon_{m+1}.$$

Since all elementary transformation preserve the determinants, we get

$$\det A_1 = \det B_1 = \nu^{n-2} \sum_{m=1}^{n-1} P_{n-m}\epsilon_{m+1}. \tag{2.3}$$

Similarly, by performing a series of elementary transformations on A_2, we obtain the matrix

$$B_2 = \begin{pmatrix} \delta_2 & \nu & 0 & \cdots & \cdots & 0 \\ \delta_3 + 2\delta_2 & 0 & \nu & \ddots & & \vdots \\ \delta_4 + 2\delta_3 + 5\delta_2 & 0 & 0 & \nu & \ddots & 0 \\ \vdots & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \nu \\ \sum_{m=1}^{n-1} P_{n-m}\delta_{m+1} & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

We also have

$$\det A_2 = \det B_2 = \nu^{n-2} \sum_{m=1}^{n-1} P_{n-m}\delta_{m+1}. \tag{2.4}$$

Based on (2.1), (2.3), and (2.4), we get

$$\det A = \sum_{m=1}^{n-1} P_{n-m}(\delta_1\nu^{n-2}\epsilon_{m+1} + (-1)^{n+1}\epsilon_1\nu^{n-2}\delta_{m+1}). \tag{2.5}$$

□

Corollary 2.2. *Let* $\delta_j = \delta + (n+1-j)d_1$ *and* $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ *in* (1.1), *where* δ, ϵ, d_1 *and* d_2 *are nonzero complex numbers. Then*

$$\begin{aligned} \det A =& \nu^{n-2}(\delta + nd_1)\left(\epsilon\frac{P_{n-1} + P_n - 1}{2} + d_2\frac{(n-2)P_n + (n-1)P_{n-1} + 1}{2}\right) \\ &+ (-1)^{n+1}\nu^{n-2}(\epsilon + nd_2)\left(\delta\frac{P_{n-1} + P_n - 1}{2} + d_1\frac{(n-2)P_n + (n-1)P_{n-1} + 1}{2}\right). \end{aligned}$$

Corollary 2.3. *Let* $\delta_j = \delta q_1^j$ *and* $\epsilon_j = \epsilon q_2^j$, $j = 1, 2, \ldots, n$ *in* (1.1)*, where* δ, ϵ, q_1 *and* q_2 *are nonzero complex numbers. Then*

$$\det A = \delta\epsilon q_1\left(\frac{q_2 P_n + q_2^2 P_{n+1} - q_2^{n+2}}{1+2q_2-q_2^2} - P_n q_2\right) - \delta\epsilon q_2\left(\frac{q_1 P_n + q_1^2 P_{n+1} - q_1^{n+2}}{1+2q_1-q_1^2} - P_n q_1\right).$$

Theorem 2.4. *Let* $A = (a_{i,j})_{i,j=1}^n (n \geqslant 3)$ *be given as in* (1.1) *and assume* A *to be nonsingular. Then* $A^{-1} = (\breve{a}_{i,j})_{i,j=1}^n$, *where*

$$\breve{a}_{i,j} = \begin{cases} \dfrac{\sum_{m=2}^{n} P_{n+1-m}\epsilon_m}{a}, & i=1, j=1, \\ -\dfrac{\sum_{m=2}^{n} P_{n+1-m}\delta_m}{a}, & i=n, j=1, \\ -\dfrac{\sum_{m=2}^{n} P_{n+1-m}(\epsilon_m \sum_{h=2}^{i} P_{i+1-h}\delta_h - \delta_m \sum_{h=2}^{i} P_{i+1-h}\epsilon_h)}{\nu a}, & 2 \leqslant i \leqslant n-1, j=1, \\ -\dfrac{P_{n+1-j}\epsilon_1}{a}, & i=1, 2 \leqslant j \leqslant n, \\ \dfrac{P_{n+1-j}\delta_1}{a}, & i=n, 2 \leqslant j \leqslant 2, \\ -\dfrac{P_{n+1-j}\sum_{h=2}^{i} P_{i+1-h}(\epsilon_h\delta_1 - \delta_h\epsilon_1)}{\nu a}, & 2 \leqslant i \leqslant n-1, 2 \leqslant j \leqslant n, i<j, \\ \dfrac{aP_{i-j+1} - P_{n+1-j}\sum_{h=2}^{i} P_{i+1-h}(\epsilon_h\delta_1 - \delta_h\epsilon_1)}{\nu a}, & 2 \leqslant i \leqslant n-1, 2 \leqslant j \leqslant n, i \geqslant j, \end{cases} \tag{2.6}$$

$$a = \sum_{m=1}^{n} P_{n+1-m}(\delta_1\epsilon_m - \epsilon_1\delta_m)$$

and P_i *is the* i^{th} *Pell number.*

Proof. We decompose A as follows

$$A = \nu\Delta + LE, \tag{2.7}$$

where

$$\Delta = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -2 & 1 & \ddots & & & 0 \\ -1 & -2 & 1 & \ddots & & \vdots \\ 0 & -1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & -2 & 1 \end{pmatrix}_{n\times n}, \qquad L = \begin{pmatrix} \delta_1 - \nu & \epsilon_1 \\ \delta_2 + 2\nu & \epsilon_2 \\ \delta_3 + \nu & \epsilon_3 \\ \delta_4 & \epsilon_4 \\ \vdots & \vdots \\ \delta_{n-1} & \epsilon_{n-1} \\ \delta_n & \epsilon_n - \nu \end{pmatrix}_{n\times 2},$$

and

$$E = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}_{2\times n}.$$

By some computation, it is not difficult to find that the inverse of Δ is

$$\Delta^{-1} = \begin{pmatrix} P_1 & 0 & \cdots & \cdots & 0 \\ P_2 & P_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ P_{n-1} & \ddots & \ddots & P_1 & 0 \\ P_n & P_{n-1} & \cdots & P_2 & P_1 \end{pmatrix}_{n\times n},$$

where P_i is the i^{th} Pell number.

Applying the Sherman-Morrison-Woodbury formula (see, e.g., [6, p.50]) to (2.7), we get

$$A^{-1} = (\nu\Delta + LE)^{-1} = \frac{1}{\nu}\Delta^{-1} - \frac{1}{\nu^2}\Delta^{-1}L(I + \frac{1}{\nu}E\Delta^{-1}L)^{-1}E\Delta^{-1}. \tag{2.8}$$

Now we compute each component on the right side of (2.8). Multiplying Δ^{-1} by E from left, we get

$$E\Delta^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ P_n & P_{n-1} & \cdots & P_2 & P_1 \end{pmatrix}_{2\times n}.$$

Consider multiplying $E\Delta^{-1}$ by L from right yields, we get

$$E\Delta^{-1}L = \begin{pmatrix} \delta_1 - \nu & \epsilon_1 \\ \sum\limits_{m=1}^{n} P_{n+1-m}\delta_m & \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m - P_1\nu \end{pmatrix}_{2\times 2}. \tag{2.9}$$

Then multiplying (2.9) by $\frac{1}{\nu}$ from the left and further adding I, we have

$$I + \frac{1}{\nu}E\Delta^{-1}L = \frac{1}{\nu}\begin{pmatrix} \delta_1 & \epsilon_1 \\ \sum\limits_{m=1}^{n} P_{n+1-m}\delta_m & \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m \end{pmatrix}_{2\times 2}. \tag{2.10}$$

Computing the inverse of the matrices on both sides of (2.10), we obtain

$$(I + \frac{1}{\nu}E\Delta^{-1}L)^{-1} = \frac{1}{\nu d}\begin{pmatrix} \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m & -\epsilon_1 \\ -\sum\limits_{m=1}^{n} P_{n+1-m}\delta_m & \delta_1 \end{pmatrix}_{2\times 2},$$

where

$$d = \frac{\delta_1 \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m - \epsilon_1 \sum\limits_{m=1}^{n} P_{n+1-m}\delta_m}{\nu^2}.$$

Multiplying Δ^{-1} by L from the right, we get

$$\Delta^{-1}L = \begin{pmatrix} \delta_1 - \nu & \epsilon_1 \\ \sum\limits_{m=1}^{2} P_{3-m}\delta_m & \sum\limits_{m=1}^{2} P_{3-m}\epsilon_m \\ \vdots & \vdots \\ \sum\limits_{m=1}^{n-1} P_{n-m}\delta_m & \sum\limits_{m=1}^{n-1} P_{n-m}\epsilon_m \\ \sum\limits_{m=1}^{n} P_{n+1-m}\delta_m & \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m - \nu \end{pmatrix}_{n\times 2}.$$

Multiplying $(I + \frac{1}{\nu}E\Delta^{-1}L)^{-1}$ by $\Delta^{-1}L$ from the left, yields

$$\Delta^{-1}L(I + \frac{1}{\nu}E\Delta^{-1}L)^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \vdots & \vdots \\ \alpha_n & \beta_n \end{pmatrix}_{n\times 2},$$

where

$$\alpha_i = \begin{cases} \nu - \dfrac{\sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m}{d}, & i = 1, \\ P_i\nu + \dfrac{\sum\limits_{h=2}^{i} P_{i+1-h}(\delta_h \sum\limits_{m=1}^{n} P_{n+1-m}\epsilon_m - \epsilon_h \sum\limits_{m=1}^{n} P_{n+1-m}\delta_m)}{\nu d}, & 2 \leqslant i \leqslant n-1, \\ \dfrac{\sum\limits_{m=1}^{n} P_{n+1-m}\delta_m}{d}, & i = n, \end{cases}$$

and

$$\beta_i = \begin{cases} \frac{\epsilon_1}{d}, & i = 1, \\ \frac{\sum\limits_{h=2}^{i} P_{i+1-h}(\epsilon_h \delta_1 - \delta_h \epsilon_1)}{vd}, & 2 \leqslant i \leqslant n-1, \\ v - \frac{\delta_1}{d}, & i = n. \end{cases}$$

Multiplying the previous formula $\left(I + \frac{1}{v} E\Delta^{-1} L\right)^{-1}$ by $\Delta^{-1}L$ from the left and by $E\Delta^{-1}$ from the right, respectively, yields

$$G = \Delta^{-1} L \left(I + \frac{1}{v} E\Delta^{-1} L\right)^{-1} E\Delta^{-1} = (g_{ij})_{i,j=1}^{n}, \tag{2.11}$$

where

$$g_{ij} = \begin{cases} v - \frac{v^2 \sum\limits_{m=2}^{n} P_{n+1-m}\epsilon_m}{a}, & i = 1, j = 1, \\ P_n v + v^2 \frac{\sum\limits_{m=2}^{n} P_{n+1-m}\delta_m}{a}, & i = n, j = 1, \\ \frac{v^2 P_{n+1-j}\epsilon_1}{a}, & i = 1, 2 \leqslant j \leqslant n, \\ P_i v + v \frac{\sum\limits_{m=2}^{n} P_{n+1-m}(\epsilon_m \sum\limits_{h=2}^{i} P_{i+1-h}\delta_h - \delta_m \sum\limits_{h=2}^{i} P_{i+1-h}\epsilon_h)}{a}, & 2 \leqslant i \leqslant n-1, j = 1, \\ v \frac{P_{n+1-j} \sum\limits_{h=2}^{i} P_{i+1-h}(\epsilon_h \delta_1 - \delta_h \epsilon_1)}{a}, & 2 \leqslant i \leqslant n-1, 2 \leqslant j \leqslant n, \\ P_{n+1-j} v - v^2 \frac{P_{n+1-j}\delta_1}{a}, & i = n, 2 \leqslant j \leqslant n, \end{cases}$$

and

$$a = \sum_{m=1}^{n} P_{n+1-m}(\delta_1 \epsilon_m - \epsilon_1 \delta_m).$$

From (2.8) and (2.11), we have

$$(\breve{a}_{i,j})_{i,j=1}^{n} = \frac{1}{v}\Delta^{-1} - \frac{1}{v^2}(g_{ij})_{i,j=1}^{n},$$

where

$$\breve{a}_{i,j} = \begin{cases} \frac{\sum\limits_{m=2}^{n} P_{n+1-m}\epsilon_m}{a}, & i = 1, j = 1, \\ -\frac{\sum\limits_{m=2}^{n} P_{n+1-m}\delta_m}{a}, & i = n, j = 1, \\ -\frac{\sum\limits_{m=2}^{n} P_{n+1-m}(\epsilon_m \sum\limits_{h=2}^{i} P_{i+1-h}\delta_h - \delta_m \sum\limits_{h=2}^{i} P_{i+1-h}\epsilon_h)}{va}, & 2 \leqslant i \leqslant n-1, j = 1, \\ -\frac{P_{n+1-j}\epsilon_1}{a}, & i = 1, 2 \leqslant j \leqslant n, \\ \frac{P_{n+1-j}\delta_1}{a}, & i = n, 2 \leqslant j \leqslant n, \\ -\frac{P_{n+1-j} \sum\limits_{h=2}^{i} P_{i+1-h}(\epsilon_h \delta_1 - \delta_h \epsilon_1)}{va}, & 2 \leqslant i \leqslant n-1, 2 \leqslant j \leqslant n, i < j, \\ \frac{aP_{i-j+1} - P_{n+1-j} \sum\limits_{h=2}^{i} P_{i+1-h}(\epsilon_h \delta_1 - \delta_h \epsilon_1)}{va}, & 2 \leqslant i \leqslant n-1, 2 \leqslant j \leqslant n, i \geqslant j. \end{cases}$$

□

Corollary 2.5. *Let* $\delta_j = \delta + (n+1-j)d_1$ *and* $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1,2,\dots,n$ *in* (1.1), *where* δ, ϵ, d_1 *and* d_2 *are nonzero complex numbers. Then* $A^{-1} = (\dot{a}_{i,j})_{i,j=1}^{n}$, *where*

$$\dot{a}_{i,j} = \begin{cases} \frac{P_n\epsilon+(n-2)P_nd_2+P_{n-1}\epsilon+(n-1)P_{n-1}d_2+d_2-\epsilon}{2c}, & i=1, j=1,\\ -\frac{P_n\delta+(n-2)P_nd_1+P_{n-1}\delta+(n-1)P_{n-1}d_1+d_1-\delta}{2c}, & i=n, j=1,\\ \frac{(\delta d_2-\epsilon d_1)b}{4\nu c}, & 2\leqslant i\leqslant n-1, j=1,\\ -\frac{P_{n+1-j}(\epsilon+nd_2)}{c}, & i=1, 2\leqslant j\leqslant n,\\ \frac{P_{n+1-j}(\delta+nd_1)}{c}, & i=n, 2\leqslant j\leqslant n,\\ -P_{n+1-j}(\epsilon d_1-\delta d_2)\frac{P_{i+1}-i-1}{2\nu c}, & 2\leqslant i\leqslant n-1, 2\leqslant j\leqslant n, i<j,\\ \frac{P_{i-j+1}}{\nu} - P_{n+1-j}(\epsilon d_1-\delta d_2)\frac{P_{i+1}-i-1}{2\nu c}, & 2\leqslant i\leqslant n-1, 2\leqslant j\leqslant n, i\geqslant j, \end{cases} \tag{2.12}$$

$$\begin{aligned} b &= P_{i-1}P_n - nP_{i-1} - P_iP_{n-1} - (n-1)P_i + iP_{n-1} + (i-1)P_n + n - i,\\ c &= (\delta d_2 - \epsilon d_1)\frac{-2P_n - P_{n-1} + n + 1}{2}, \end{aligned}$$

and P_i *is the* i^{th} *Pell number.*

Proof. We can obtain this conclusion by using (1.4), (1.5), and Theorem 2.4. □

Corollary 2.6. *Let* $\delta_j = \delta q_1^j$ *and* $\epsilon_j = \epsilon q_2^j$, $j = 1,2,\dots,n$ *in* (1.1), *where* δ, ϵ, q_1 *and* q_2 *are nonzero complex numbers. Then* $A^{-1} = (\ddot{a}_{i,j})_{i,j=1}^{n}$, *where*

$$\ddot{a}_{i,j} = \begin{cases} \frac{(q_2^2P_{n-1}-q_2^{n+2}+P_nq_2^3)\epsilon}{d+2q_2d-q_2^2d}, & i=1, j=1,\\ -\frac{(q_1^2P_{n-1}-q_1^{n+2}+P_nq_1^3)\delta}{d+2q_1d-q_1^2d}, & i=n, j=1,\\ \frac{\epsilon\delta e}{\nu d(1+2q_1-q_1^2)(1+2q^2-q_2^2)}, & 2\leqslant i\leqslant n-1, j=1,\\ -\frac{P_{n+1-j}\epsilon q_2}{d}, & i=1, 2\leqslant j\leqslant n,\\ \frac{P_{n+1-j}\delta q_1}{d}, & i=n, 2\leqslant j\leqslant n,\\ -P_{n+1-j}\delta\epsilon(\frac{P_{i-1}+q_2^2q_1+P_iq_2^3q_1-q_2^{i+2}q_1}{\nu d+2\nu dq_2-\nu dq_2^2} - \\ \frac{P_{i-1}+q_1^2q_2+P_iq_1^3q_2-q_1^{i+2}q_2}{\nu d+2\nu dq_1-\nu dq_1^2}), & 2\leqslant i\leqslant n-1, 2\leqslant j\leqslant n, i<j,\\ \frac{P_{i-j+1}}{d} - P_{n+1-j}\delta\epsilon(\frac{P_{i-1}+q_2^2q_1+P_iq_2^3q_1-q_2^{i+2}q_1}{\nu d+2\nu dq_2-\nu dq_2^2} - \\ \frac{P_{i-1}+q_1^2q_2+P_iq_1^3q_2-q_1^{i+2}q_2}{\nu d+2\nu dq_1-\nu dq_1^2}), & 2\leqslant i\leqslant n-1, 2\leqslant j\leqslant n, i\geqslant j, \end{cases} \tag{2.13}$$

$$\begin{aligned} d &= \delta\epsilon(\frac{q_2q_1P_n + q_2^2q_1P_{n+1} - q_1q_2^{n+2}}{1+2q_2-q_2^2} - \frac{q_2q_1P_n + q_1^2q_2P_{n+1} - q_2q_1^{n+2}}{1+2q_1-q_1^2}),\\ e &= P_{i-1}(q_1^2q_2^{n+2} - q_2^2q_1^{n+2}) + P_{n-1}(q_1^{i+2}q_2^2 - q_2^{i+2}q_1^2) + P_n(q_1^{i+2}q_2^3 - q_2^{i+2}q_1^3)\\ &\quad + P_i(q_1^3q_2^{n+2} - q_2^3q_1^{n+2}) - (P_{i-1}P_n - P_iP_{n+1})(q_1^2q_2^3 - q_2^2q_1^3) - (q_1^{i+2}q_2^{n+2} - q_2^{i+2}q_1^{n+2}), \end{aligned}$$

and P_i *is the* i^{th} *Pell number.*

Proof. We can obtain this conclusion by using (1.5), (1.6), and Theorem 2.4. □

The next four theorems are parallel results of type I matrix B and Type II matrix C.

Theorem 2.7. *Let* B *be given as in* (1.2). *Then we have*

$$\det B = (-1)^{\frac{n(n-1)}{2}} \sum_{m=1}^{n-1} P_{n-m}(\delta_1\nu^{n-2}\epsilon_{m+1} + (-1)^{n+1}\epsilon_1\nu^{n-2}\delta_{m+1}),$$

where P_i *is the* i^{th} *Pell number.*

Proof. Since $\det B = \det \hat{I}_n \det A$, where A is a triband Toeplitz matrices with permuted columns as given in (1.1), $\hat{I}_n$ is an inverse unit matrix of order n. We obtain this conclusion by using Theorem 2.1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Corollary 2.8. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.2), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then*

$$\begin{aligned}\det B =& v^{n-2}(\delta + nd_1)(\epsilon\frac{P_{n-1} + P_n - 1}{2} + d_2\frac{(n-2)P_n + (n-1)P_{n-1} + 1}{2})\\ &+ (-1)^{n+1}v^{n-2}(\epsilon + nd_2)(\delta\frac{P_{n-1} + P_n - 1}{2} + d_1\frac{(n-2)P_n + (n-1)P_{n-1} + 1}{2}),\end{aligned}$$

where P_i is the i^{th} Pell number.

Corollary 2.9. *Let $\delta_j = \delta q_1^j$ and $\epsilon_j = \epsilon q_2^j$, $j = 1, 2, \ldots, n$ in (1.2), where δ, ϵ, q_1 and q_2 are nonzero complex numbers. Then*

$$\det B = \delta\epsilon q_1(\frac{q_2P_n + q_2^2P_{n+1} - q_2^{n+2}}{1 + 2q_2 - q_2^2} - P_nq_2) - \delta\epsilon q_2(\frac{q_1P_n + q_1^2P_{n+1} - q_1^{n+2}}{1 + 2q_1 - q_1^2} - P_nq_1),$$

where P_i is the i^{th} Pell number.

Theorem 2.10. *Let B be given as in (1.2). Then we have*

$$B^{-1} = (\breve{a}_{i,n+1-j})_{i,j=1}^n,$$

where $\breve{a}_{i,j}$ is the same as (2.6).

Proof. We can obtain this conclusion by using $B^{-1} = A^{-1}\hat{I}_n^{-1} = A^{-1}\hat{I}_n$ and Theorem 2.4, where A is a nonsingular form such as (1.1) given a triband Toeplitz matrices with permuted columns. $\hat{I}_n$ is a reverse unit matrix of order n. □

Corollary 2.11. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.2), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then we have*

$$B^{-1} = (\dot{a}_{i,n+1-j})_{i,j=1}^n,$$

where $\dot{a}_{i,j}$ is the same as (2.12).

Corollary 2.12. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.2), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then we have*

$$B^{-1} = (\ddot{a}_{i,n+1-j})_{i,j=1}^n,$$

where $\ddot{a}_{i,j}$ is the same as (2.13).

Theorem 2.13. *Let C be given as in (1.3). Then we have*

$$\det C = (-1)^{\frac{n(n-1)}{2}}\sum_{m=1}^{n-1} P_{n-m}(\delta_1 v^{n-2}\epsilon_{m+1} + (-1)^{n+1}\epsilon_1 v^{n-2}\delta_{m+1}),$$

where P_i is the i^{th} Pell number.

Proof. Since $\det C = \det A \det \hat{I}_n$, where A is a triband Toeplitz matrices with permuted columns as given in formula (1.1), $\hat{I}_n$ is a reverse unit matrix of order n. We obtain this conclusion by using Theorem 2.1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Corollary 2.14. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.3), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then*

$$\begin{aligned}\det C =& v^{n-2}(\delta + nd_1)(\epsilon\frac{P_{n-1}+P_n-1}{2} + d_2\frac{(n-2)P_n+(n-1)P_{n-1}+1}{2}) \\ &+ (-1)^{n+1}v^{n-2}(\epsilon + nd_2)(\delta\frac{P_{n-1}+P_n-1}{2} + d_1\frac{(n-2)P_n+(n-1)P_{n-1}+1}{2}),\end{aligned}$$

where P_i is the i^{th} Pell number.

Corollary 2.15. *Let $\delta_j = \delta q_1^j$ and $\epsilon_j = \epsilon q_2^j$, $j = 1, 2, \ldots, n$ in (1.3), where δ, ϵ, q_1 and q_2 are nonzero complex numbers. Then*

$$\det C = \delta\epsilon q_1(\frac{q_2P_n + q_2^2P_{n+1} - q_2^{n+2}}{1+2q_2-q_2^2} - P_nq_2) - \delta\epsilon q_2(\frac{q_1P_n + q_1^2P_{n+1} - q_1^{n+2}}{1+2q_1-q_1^2} - P_nq_1),$$

where P_i is the i^{th} Pell number.

Theorem 2.16. *Let C be given as in (1.3). Then we have*

$$C^{-1} = (\breve{a}_{n+1-i,j})_{i,j=1}^n,$$

where $\breve{a}_{i,j}$ is the same as (2.6).

Proof. We can obtain this conclusion by using $C^{-1} = \hat{I}_n^{-1}A^{-1} = \hat{I}_nA^{-1}$ and Theorem 2.4, where A is a nonsingular form such as (1.1) given a triband Toeplitz matrices with permuted columns. $\hat{I}_n$ is a reverse unit matrix of order n. □

Corollary 2.17. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.3), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then we have*

$$C^{-1} = (\dot{a}_{n+1-i,j})_{i,j=1}^n,$$

where $\dot{a}_{i,j}$ is the same as (2.12).

Corollary 2.18. *Let $\delta_j = \delta + (n+1-j)d_1$ and $\epsilon_j = \epsilon + (n+1-j)d_2$, $j = 1, 2, \ldots, n$ in (1.3), where δ, ϵ, d_1 and d_2 are nonzero complex numbers. Then we have*

$$C^{-1} = (\ddot{a}_{n+1-i,j})_{i,j=1}^n,$$

where $\ddot{a}_{i,j}$ is the same as (2.13).

3. Algorithm and numerical example

In this section, we give an algorithms for finding the determinant and inverse of triband Toeplitz matrices with permuted columns based on Theorems 2.1 and 2.4.

Algorithm 3.1. Calculate the inverse of triband Toeplitz matrices with permuted columns in Theorem 2.4.

Step 1: Input $\delta_1 \cdots \delta_n, v, \epsilon_1, \cdots \epsilon_n$, order n and generate Pell numbers by (1.4).

Step 2: By the formula (2.5), and output the determinant $\det A$.

Step 3: By the formula (2.6), compute respectively the entries $\breve{a}_{i,j}$.

Step 4: Output the inverse $A^{-1} = (\breve{a}_{i,j})_{i,j=1}^n$.

Example 3.2. Consider a 4×4 triband Toeplitz matrices with permuted columns:

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & -2 & 1 & 2 \\ 4 & -1 & -2 & 1 \end{pmatrix}.$$

Show the steps required to calculate the inverse of A in detail according to Algorithm 3.1.

Step 1: Input $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4, n = 4, \nu = 1, \epsilon_1 = 4, \epsilon_2 = 3, \epsilon_3 = 2, \epsilon_4 = 1$.

Step 2: Based on the formula (2.5), and output the determinant $\det A = -60$.

Step 3: Calculate $\breve{a}_{1,1} = -\frac{1}{3}, \breve{a}_{1,2} = \frac{1}{3}, \breve{a}_{1,3} = \frac{2}{15}, \breve{a}_{1,4} = \frac{1}{15}, \breve{a}_{2,1} = -\frac{1}{3}, \breve{a}_{2,2} = \frac{7}{12}, \breve{a}_{2,3} = -\frac{1}{6}, \breve{a}_{2,4} = -\frac{1}{12}, \breve{a}_{3,1} = -\frac{1}{3}, \breve{a}_{3,2} = \frac{1}{3}, \breve{a}_{3,3} = \frac{1}{3}, \breve{a}_{3,4} = -\frac{1}{3}, \breve{a}_{4,1} = \frac{1}{3}, \breve{a}_{4,2} = -\frac{1}{12}, \breve{a}_{4,3} = -\frac{1}{30}$, and $\breve{a}_{4,4} = -\frac{1}{60}$.

Step 4: Based on the above step, output A^{-1} as

$$A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{2}{15} & \frac{1}{15} \\ -\frac{1}{3} & \frac{7}{12} & -\frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{12} & -\frac{1}{30} & -\frac{1}{60} \end{pmatrix}.$$

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