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Fuzzy Time-Delay Dynamical Systems

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Abstract
This paper investigates the first order linear fuzzy time-delay dynamical systems. We use a complex number representation of the α-level sets of the fuzzy time-delay system, and obtain the solution by applying a Runge-Kutta method. Several examples are considered to show the convergence and accuracy of the proposed method. We finally present some conclusions and new directions for further research in this area.

Keywords:Time-delay dynamical systems, fuzzy differential equations, fuzzy matrices, Runge-Kutta methods.

1. Introduction

The dynamics of many control systems may be expressed by time-delay differential equations. The delays may appear because of physical properties of equipment used in the system, signal transmission or measurement of system variables. For example, actuators, sensors and field networks which are involved in feedback loops may exhibit delays. Time-delay systems are also used to model several different mechanisms in the dynamics of epidemics. Many problems such as incubation periods, mechanics, viscoelasticity, physics, physiology, population dynamics, communication, information technologies and stability of networked controlled, maturation times, age structure, blood transfusions and interactions across spatial distances or through complicated paths have been modelled by the introduction of time-delay systems [10]. In recent decades, optimal control problems with delays and obtaining their approximate solutions are very important issues in control theory and have attracted much attention of many researchers and investigators. Let us briefly review some papers concerning

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different classes of control problems. Kharatishvili [9] was first to provide a maximum principle for optimal control problems with a constant state delay. In [8], he gave similar results for control problems with pure control delays. Bader [1] used collocation methods to solve the boundary value problem for the retarded state variable and advanced adjoint variable. He successfully solved several academic examples, but his method did not give accurate results for the more difficult CSTR reactor problem described in Soliman and Ray [16, 17]. A similar CSTR reactor problem was considered in Oh and Luus [11] and Dadebo and Luus [4], who used the differential dynamic programming method with a moderate number of stages. Optimal control problems with constant delays in state and control variables and mixed control-state inequality constraints (based on the use of Pontryagin-type minimum (maximum) principle) were considered by Göllmann et al [7]. Farahi and Barati [6] obtained extremely significantly superior results by applying a measure theory approach. Since the concept of fuzzy set and the corresponding fuzzy operations was introduced by Zadeh [20], an enormous effort has been dedicated to the development of various aspects of the theory and applications of fuzzy systems, in particular to the theory of differential equations with uncertainty. The usage of fuzzy differential equations has been a natural way to model dynamical systems under possibility uncertainty. The fuzzy dynamical systems based on fuzzy differential equations are also widely applied to fuzzy control systems and many other fields (see [19] and its references). The organization of this paper is as follows. In Section 2, the basic notations of fuzzy numbers, fuzzy derivative and fuzzy functions are briefly presented. In Section 3, linear fuzzy time-delay differential equations with fuzzy matrices are introduced, and then a complex number representation of the α-level sets is offered to relate the fuzzy dynamics and the original nonfuzzy linear system. In Section 4, the proposed new method (based on the use of a Runge-Kutta method) is described. In Section 5, the applicability of the method is illustrated by several examples. Finally, Section 6 presents concluding remarks.

2. Preliminaries

Definition 2.1. (see[18]). A fuzzy number u is completely determined by any pair $u = (\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha), \overline{u}(\alpha): [0,1] \longrightarrow R$, satisfying the three conditions:
(i) $\underline{u}(\alpha)$is a bounded, monotonic, increasing (nondecreasing) left-continuous function for all $\alpha \in (0,1]$ and right-continuous for $\alpha = 0$.
(ii) $\overline{u}(\alpha)$is a bounded, monotonic, decreasing (nonincreasing) left-continuous function for all $\alpha \in (0,1]$ and right-continuous for $\alpha = 0$.
(iii) For all $\alpha \in (0,1]$ we have: $\underline{u}(\alpha) \le \overline{u}(\alpha)$.
For every $u=(\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ and $k > 0$,

$$\underline{(u+v)}(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha), \quad (1)$$

$$\overline{(u+v)}(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha), \quad (2)$$

$$(\underline{ku})(\alpha) = k\underline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\overline{u}(\alpha). \quad (3)$$

The collection of all fuzzy numbers with addition and multiplication as defined by $(1) - (3)$ is denoted by E^1. For $0 < \alpha \le 1$, we define the α-level of fuzzy number u with $[u]^\alpha = \{x \in R | u(x) \ge \alpha\}$ and for $\alpha = 0$, the support of u is defined as $[u]^0 = \overline{\{x \in R | u(x) > 0\}}$.

Definition 2.2. The distance between two arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ is defined as follows:

$$d(u,v) = \sup_{\alpha \in [0,1]} \{max[|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|]\}.$$

It is shown [14] that (E^1, d) is a complete metric space.

Definition 2.3. The function $f: \mathbb{R} \longrightarrow E^1$ is called a fuzzy function. Now if, for an arbitrary fixed $\hat{t} \in R^1$ *and* $\varepsilon > 0$ there exists a δ>0 such that:

$$|t - \hat{t}| < \delta \longrightarrow d[f(t), f(\hat{t})] < \varepsilon,$$

then f is said to be continuous. Note that d is the metric which is defined in Definition 2.2 (In this article we simply replace $\mathbb{R}$ by $[t_0, T]$).

Definition 2.4. Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$ then w is called the H-difference of u, v and it is denoted by $u - v$.

Definition 2.5. A function $f: (a, b) \longrightarrow E^1$ is called H-differentiable at $\hat{t} \in (a, b)$ if, for $h > 0$ sufficiently small, there exist the H-differences $f(\hat{t} + h) - f(\hat{t})$, $f(\hat{t}) - f(\hat{t} - h)$, and an element $f'(\hat{t}) \in E^1$ such that:

$$0 = \lim_{h \to 0^+} d\left(\frac{f(\hat{t} + h) - f(\hat{t})}{\mathrm{h}}, f'(\hat{t})\right) = \lim_{h \to 0^+} d\left(\frac{f(\hat{t}) - f(\hat{t} - h)}{h}, f'(\hat{t})\right).$$

Then $f'(\hat{t})$ is called the fuzzy derivative of f at $\hat{t}$, (see [5]).

3. Fuzzy time-delay initial value problem

Consider the first-order fuzzy time-delay initial value differential equation given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - r), & t \in I = [t_0, T], \\ x(t) = x_0, & t \in [t_0 - r, t_0], \end{cases} \tag{4}$$

where $x(t)$ and $x(t - r) \in \{x(\tau): \tau < t\}$ are n-dimensional fuzzy functions of t, every element of matrices $A = [a_{ij}]_{n \times n}$, $a_{ij} \in F(\mathbb{R})$ and $B = [b_{ij}]_{n \times n}$, $b_{ij} \in F(\mathbb{R})$ are supposed to be fuzzy numbers where $F(\mathbb{R})$ represents the fuzzy sets defined on $\mathbb{R}$. The function $\dot{x}(t)$ is the fuzzy derivative of $x(t)$ at $t \in I$ and x_0 is a fuzzy number and the time-delay r is a known positive rational number. The α-level sets of $x(t)$ and $x(t - r)$ for $t \in [t_0, T]$ are as follows:

$$\begin{aligned} x_\alpha^k(t) &= \left[\underline{x}_\alpha^k(t), \overline{x}_\alpha^k(t)\right], \qquad (k = 1, 2, \cdots, n), \\ x_\alpha^k(t - r) &= \left[\underline{x}_\alpha^k(t - r), \overline{x}_\alpha^k(t - r)\right], \qquad (k = 1,2, \cdots, n). \end{aligned} \tag{5}$$

Considering the elements of matrices A *and* B as the crisp numbers the α-level equations of fuzzy time-delay initial value differential equation (4) using from (5) are defined as follows:

$$\begin{cases} \dot{\underline{x}}_\alpha^k(t) = min\left\{(Au)^k + (Bu_t)^k \colon\ u^j \in \left[\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)\right], u_t^j \in \left[\underline{x}_\alpha^j(t - r), \overline{x}_\alpha^j(t - r)\right]\right\}, \\ \dot{\overline{x}}_\alpha^k(t) = max\left\{(Au)^k + (Bu_t)^k \colon\ u^j \in \left[\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)\right], u_t^j \in \left[\underline{x}_\alpha^j(t - r), \overline{x}_\alpha^j(t - r)\right]\right\}, \\ \underline{x}_\alpha(t) = \underline{x}_{\alpha 0}, \qquad t \in [t_0 - r, t_0], \\ \overline{x}_\alpha(t) = \overline{x}_{\alpha 0}, \qquad t \in [t_0 - r, t_0], \end{cases} \tag{6}$$

where $(Au)^k \coloneqq \sum_{j=1}^n a_{kj}\, u^j$, $(Bu_t)^k \coloneqq \sum_{j=1}^n b_{kj}\, u_t^j$ are the kth rows of Au and Bu_t respectively. Since the equation in (4) is linear, the following rules can be used in (6):

$$\dot{\underline{x}}_\alpha^k(t) = \sum_{j=1}^n a_{kj}\, u^j + \sum_{j=1}^n b_{kj}\, u_t^j,$$

where

$$\begin{cases} u^j = \underline{x}_\alpha^j(t), & a_{kj} \geq 0, \\ u^j = \overline{x}_\alpha^j(t), & a_{kj} < 0, \end{cases}$$

and

$$\begin{cases} u_t^j = \underline{x}_\alpha^j(t-r), & b_{kj} \geq 0, \\ u_t^j = \overline{x}_\alpha^j(t-r), & b_{kj} < 0, \end{cases}$$

also

$$\dot{\overline{x}}_\alpha^k(t) = \sum_{j=1}^{n} a_{kj}\, v^j + \sum_{j=1}^{n} b_{kj}\, v_t^j,$$

where

$$\begin{cases} v^j = \overline{x}_\alpha^j(t), & a_{kj} \geq 0, \\ v^j = \underline{x}_\alpha^j(t), & a_{kj} < 0, \end{cases}$$

and

$$\begin{cases} v_t^j = \overline{x}_\alpha^j(t-r), & b_{kj} \geq 0, \\ v_t^j = \underline{x}_\alpha^j(t-r), & b_{kj} < 0. \end{cases}$$

As indicated in [13], the same relations can be used in a more compact way by moving to the field of complex numbers.

Define new complex variables as follows:

$$x_\alpha^k := \underline{x}_\alpha^k(t) + i\overline{x}_\alpha^k(t) \qquad (k = 1,2,\cdots,n),$$
$$x_{t\alpha}^k := \underline{x}_\alpha^k(t-r) + i\overline{x}_\alpha^k(t-r) \quad (k = 1,2,\cdots,n),$$

where $i := \sqrt{-1}$. Let $x_\alpha(t) = \underline{x}_\alpha(t) + i\overline{x}_\alpha(t)$ be the solution of the fuzzy time-delay dynamical system (4), in which the elements of matrices A and B are fuzzy numbers, then (4) can be rewritten with $x_\alpha(t) = \underline{x}_\alpha(t) + i\overline{x}_\alpha(t)$ as its solution is in the following form:

$$\begin{cases} \dot{\underline{x}}_\alpha(t) + i\dot{\overline{x}}_\alpha(t) = A_\alpha\left(\underline{x}_\alpha(t) + i\overline{x}_\alpha(t)\right) + B_\alpha\left(\underline{x}_\alpha(t-r) + i\overline{x}_\alpha(t-r)\right), \\ x_\alpha(t) = \underline{x}_{\alpha 0} + i\overline{x}_{\alpha 0}, \qquad t \in [t_0 - r, t_0], \quad 0 \leq \alpha \leq 1. \end{cases} \tag{7}$$

Let $(a_{ij})_\alpha = \left[(a_{ij})_\alpha^-, (a_{ij})_\alpha^+\right]$, $A_\alpha = [A_\alpha^-, A_\alpha^+]$, where $A_\alpha^- = [(a_{ij})_\alpha^-]_{n\times n}$, $A_\alpha^+ = [(a_{ij})_\alpha^+]_{n\times n}$ and $(b_{ij})_\alpha = \left[(b_{ij})_\alpha^-, (b_{ij})_\alpha^+\right]$, $B_\alpha = [B_\alpha^-, B_\alpha^+]$, where $B_\alpha^- = [(b_{ij})_\alpha^-]_{n\times n}$, $B_\alpha^+ = [(b_{ij})_\alpha^+]_{n\times n}$.

Then we have the following theorem:

Theorem 3.1. Let $A(\mu,\alpha) = [a_{ij}(\mu,\alpha)]_{n\times n} = (1-\mu)A_\alpha^- + \mu A_\alpha^+$, $B(\mu,\alpha) = [b_{ij}(\mu,\alpha)]_{n\times n} = (1-\mu)B_\alpha^- + \mu B_\alpha^+$, in which $\mu \in [0,1]$. Then $\underline{x}_\alpha(t) + i\overline{x}_\alpha(t)$ is the solution of problem (7), if and only if $\underline{x}_\alpha(t) + i\overline{x}_\alpha(t)$ is also the solution of the following problem

$$\begin{cases} \dot{\underline{x}}_\alpha(t) + i\dot{\overline{x}}_\alpha(t) = \bigcup_{\mu=0}^{1} C(\mu,\alpha)\left(\underline{x}_\alpha(t) + i\overline{x}_\alpha(t)\right) + \bigcup_{\mu=0}^{1} D(\mu,\alpha)\left(\underline{x}_\alpha(t-r) + i\overline{x}_\alpha(t-r)\right), \\ x_\alpha(t) = \underline{x}_{\alpha 0} + i\overline{x}_{\alpha 0}, \qquad t \in [t_0 - r, t_0], \quad 0 \leq \alpha \leq 1, \end{cases} \tag{8}$$

where the elements of the matrices C and D are determined from those of $A(\mu,\alpha)$ and $B(\mu,\alpha)$ as follows:

$$c_{ij} = \begin{cases} e a_{ij}(\mu,\alpha), & a_{ij}(\mu,\alpha) \geq 0 \\ g a_{ij}(\mu,\alpha), & a_{ij}(\mu,\alpha) < 0 \end{cases}$$

and

$$d_{ij} = \begin{cases} e b_{ij}(\mu,\alpha), & b_{ij}(\mu,\alpha) \geq 0 \\ g b_{ij}(\mu,\alpha), & b_{ij}(\mu,\alpha) < 0 \end{cases}$$

in which e is just the identity operation and g corresponds to a flip about the diagonal in the complex plane. i. e., $\forall\, z + wi \in \mathbb{C}$,

$$e: z + iw \longrightarrow z + iw,$$

$$g: z + iw \longrightarrow w + iz.$$

Proof. See [2].

4. A Runge-Kutta Method

Substituting all the fuzzy numbers of the matrices A and B in Section 3 by crisp numbers, the new system which contains $2n$ crisp delay differential equations is in the following form:

$$\begin{cases} \dot{X}(t) = EX(t) + FX(t-r), & t \in I = [t_0, T], \\ X(t) = X_0, & t \in [t_0 - r, t_0], \end{cases} \tag{9}$$

where for each $t \in I$, $2n$-dimensional nonfuzzy functions $X(t)$, $X(t-r)$, $\dot{X}(t)$ are defined as $X(t) = X = [\underline{x}_\alpha^1, \overline{x}_\alpha^1, \cdots \underline{x}_\alpha^n, \overline{x}_\alpha^n]^T$, $X(t-r) = X_t = [\underline{x}_{t\alpha}^1, \overline{x}_{t\alpha}^1, \cdots \underline{x}_{t\alpha}^n, \overline{x}_{t\alpha}^n]^T$, $\dot{X}(t) = \dot{X} = [\underline{\dot{x}}_\alpha^1, \overline{\dot{x}}_\alpha^1, \cdots, \underline{\dot{x}}_\alpha^n, \overline{\dot{x}}_\alpha^n]^T$, respectively.

Also $X_0 = [\underline{x}_{\alpha 0}^1, \overline{x}_{\alpha 0}^1, \cdots \underline{x}_{\alpha 0}^n, \overline{x}_{\alpha 0}^n]^T \in \mathbb{R}^{2n}$, $E = [e_{ij}]_{2n \times 2n}$, $e_{ij} \in \mathbb{R}$ and $F = [f_{ij}]_{2n \times 2n}$, $f_{ij} \in \mathbb{R}$ are crisp matrices which are obtained from fuzzy matrices A and B in Section 3, respectively.

We replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points

$$t_0 < t_1 < t_2 < \cdots < t_N = T, \quad h = \frac{T - t_0}{N}, \quad t_i = t_0 + ih, \quad i = 1,2, \ldots, N.$$

Thus the classical 4-step Runge-Kutta method ($RK4$) for solving the system of time-delay differential equations (9) is summarized as follows:

$$K_0 = h\left(EX_i + FX_{i - \frac{Nr}{T - t_0}}\right),$$

$$K_1 = h\left(E(X_i + \frac{K_0}{2}) + F(X_{i - \frac{Nr}{T - t_0}} + \frac{K_0}{2})\right),$$

$$K_2 = h\left(E(X_i + \frac{K_1}{2}) + F(X_{i - \frac{Nr}{T - t_0}} + \frac{K_1}{2})\right),$$

$$K_3 = h\left(E(X_i + K_2) + F(X_{i - \frac{Nr}{T - t_0}} + K_2)\right),$$

$$X_{i+1} \approx X_i + \frac{1}{6}(K_0 + 2K_1 + 2K_2 + K_3), \quad i = 0, 1, \ldots, N-1,$$

with value $X(t) = X_0$ for all $t \leq t_0$.

For the convergence of the proposed method see [12].

Remark 4.1. In this method we select N such that $\frac{Nr}{T - t_0}$ to be a natural number.

5. Numerical Examples

To show the behavior and properties of this new method, two examples will be considered in this section.

Example 5.1. Consider a model of the El Niño Southern Oscillation (ENSO) phenomenon. The differential equation corresponding to this climate modeling is:

$$\begin{cases} \dot{x}(t) = \beta_1 x(t) + \beta_2 x(t-1), & t \in I = [0,2]\ , \\ x(t) = x_0, & t \in [-1,0\ ], \end{cases}$$

where β_1 , β_2 , x_0 are fuzzy numbers about 1, about -1 and about 1 respectively. Suppose they can be defined as follows:

$$\beta_1(s) = \begin{cases} 0\ , & s < 0, \\ -(s-1)^2 + 1\ , & 0 \le s \le 2, \\ 0\ , & s > 2, \end{cases}$$

and

$$\beta_2(s) = \begin{cases} 0\ , & s < -2, \\ -(s+1)^2 + 1, & -2 \le s \le 0, \\ 0\ , & s > 0, \end{cases}$$

and

$$x_0(s) = \begin{cases} 0\ , & s < 0, \\ -(s-1)^2 + 1\ , & 0 \le s \le 2, \\ 0\ , & s > 2. \end{cases}$$

Thus

$$[\beta_1]_\alpha = \left[1 - \sqrt{1-\alpha},\ 1 + \sqrt{1-\alpha}\right],$$

$$[\beta_2]_\alpha = [-1 - \sqrt{1-\alpha},\ -1 + \sqrt{1-\alpha}],$$

$$[x_0]_\alpha = [1 - \sqrt{1-\alpha},\ 1 + \sqrt{1-\alpha}].$$

Let $a_1 = (1-\mu)\left(1 - \sqrt{1-\alpha}\right) + \mu\left(1 + \sqrt{1-\alpha}\right), a_2 = (1-\mu)\left(-1 - \sqrt{1-\alpha}\right) + \mu\left(-1 + \sqrt{1-\alpha}\right)$, where $\mu \in [0,1]$. By using the complex number representation of the α-level sets we have

$$\dot{\underline{x}}_\alpha + i\dot{\overline{x}}_\alpha = e(a_1)\left(\underline{x}_\alpha + i\overline{x}_\alpha\right) + g(a_2)\left(\underline{x}_{t\alpha} + i\overline{x}_{t\alpha}\right).$$

Now we can state (9) as follows

$$\begin{bmatrix} \dot{\underline{x}}_\alpha \\ \dot{\overline{x}}_\alpha \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha \\ \overline{x}_\alpha \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{t\alpha} \\ \overline{x}_{t\alpha} \end{bmatrix},$$

with values

$$\begin{bmatrix} \underline{x}_\alpha \\ \overline{x}_\alpha \end{bmatrix} = \begin{bmatrix} \underline{x}_{\alpha 0} \\ \overline{x}_{\alpha 0} \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{1-\alpha} \\ 1 + \sqrt{1-\alpha} \end{bmatrix},$$

for all $t \le 0$ over the time interval $I = [0, 2]$.

Figure 1 depicts the solution of Example 5.1 for $\alpha = 0.9$, $N = 10000$. The crisp shape of this example has originally been studied in [3].

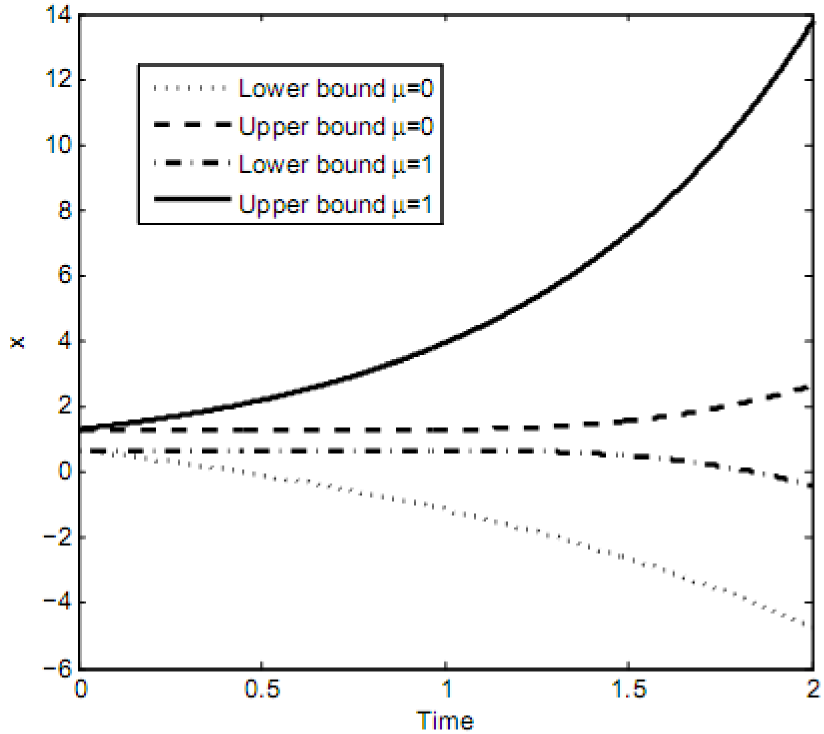

Figure 1. Lower and upper bound solutions of Example 5.1 for $\mu = 0, 1$ using the proposed method.

Example 5.2. Consider the fuzzy time-delay initial value problem

$$\begin{cases} \dot{x}_1(t) = \lambda x_1(t-1)\,, \\ \dot{x}_2(t) = \lambda x_1(t-1) + \lambda x_2(t-1), \\ \dot{x}_3(t) = \lambda x_2(t)\,, \end{cases}$$

with values $x_i(t) = x_0$, $i = 1, 2, 3$ for all $t \le 0$ over the time interval $I = [0, 5]$. So in comparison with (4) we have

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix}, B = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let λ, x_0 both to be fuzzy numbers about 1 and they can be done by setting, for example,

$$\lambda(s) = \begin{cases} 0\,, & s < 0, \\ 2s - s^2\,, & 0 \le s \le 2, \\ 0\,, & s > 2, \end{cases}$$

and

$$x_0(s) = \begin{cases} 0\,, & s < 0, \\ -(s-1)^2 + 1\,, & 0 \le s \le 2, \\ 0\,, & s > 2. \end{cases}$$

Thus

$$[\lambda]_\alpha = [1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}],$$

$$[x_0]_\alpha = [1-\sqrt{1-\alpha}, 1+\sqrt{1-\alpha}].$$

Let $a = (1-\mu)\left(1-\sqrt{1-\alpha}\right) + \mu\left(1+\sqrt{1-\alpha}\right)$, where $\mu \in [0,1]$. By using the complex number representation of the α-level sets we have

$$\begin{bmatrix} \dot{\underline{x}}_\alpha^1 + i\dot{\overline{x}}_\alpha^1 \\ \dot{\underline{x}}_\alpha^2 + i\dot{\overline{x}}_\alpha^2 \\ \dot{\underline{x}}_\alpha^3 + i\dot{\overline{x}}_\alpha^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e(a) & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha^1 + i\overline{x}_\alpha^1 \\ \underline{x}_\alpha^2 + i\overline{x}_\alpha^2 \\ \underline{x}_\alpha^3 + i\overline{x}_\alpha^3 \end{bmatrix} + \begin{bmatrix} e(a) & 0 & 0 \\ e(a) & e(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{t\alpha}^1 + i\overline{x}_{t\alpha}^1 \\ \underline{x}_{t\alpha}^2 + i\overline{x}_{t\alpha}^2 \\ \underline{x}_{t\alpha}^3 + i\overline{x}_{t\alpha}^3 \end{bmatrix}.$$

Now we can state (9) as follows

$$\begin{bmatrix} \dot{\underline{x}}_\alpha^1 \\ \dot{\overline{x}}_\alpha^1 \\ \dot{\underline{x}}_\alpha^2 \\ \dot{\overline{x}}_\alpha^2 \\ \dot{\underline{x}}_\alpha^3 \\ \dot{\overline{x}}_\alpha^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha^1 \\ \overline{x}_\alpha^1 \\ \underline{x}_\alpha^2 \\ \overline{x}_\alpha^2 \\ \underline{x}_\alpha^3 \\ \overline{x}_\alpha^3 \end{bmatrix} + \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ a & 0 & a & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{t\alpha}^1 \\ \overline{x}_{t\alpha}^1 \\ \underline{x}_{t\alpha}^2 \\ \overline{x}_{t\alpha}^2 \\ \underline{x}_{t\alpha}^3 \\ \overline{x}_{t\alpha}^3 \end{bmatrix},$$

with values

$$\begin{bmatrix} \underline{x}_\alpha^1 \\ \overline{x}_\alpha^1 \\ \underline{x}_\alpha^2 \\ \overline{x}_\alpha^2 \\ \underline{x}_\alpha^3 \\ \overline{x}_\alpha^3 \end{bmatrix} = \begin{bmatrix} \underline{x}_{\alpha 0}^1 \\ \overline{x}_{\alpha 0}^1 \\ \underline{x}_{\alpha 0}^2 \\ \overline{x}_{\alpha 0}^2 \\ \underline{x}_{\alpha 0}^3 \\ \overline{x}_{\alpha 0}^3 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{1-\alpha} \\ 1+\sqrt{1-\alpha} \\ 1-\sqrt{1-\alpha} \\ 1+\sqrt{1-\alpha} \\ 1-\sqrt{1-\alpha} \\ 1+\sqrt{1-\alpha} \end{bmatrix},$$

for all $t \leq 0$. Figure 2 depicts the solution of Example 5.2 for $\alpha = 0.9$, $N = 10000$. The crisp shape of this example has originally been studied in [15].

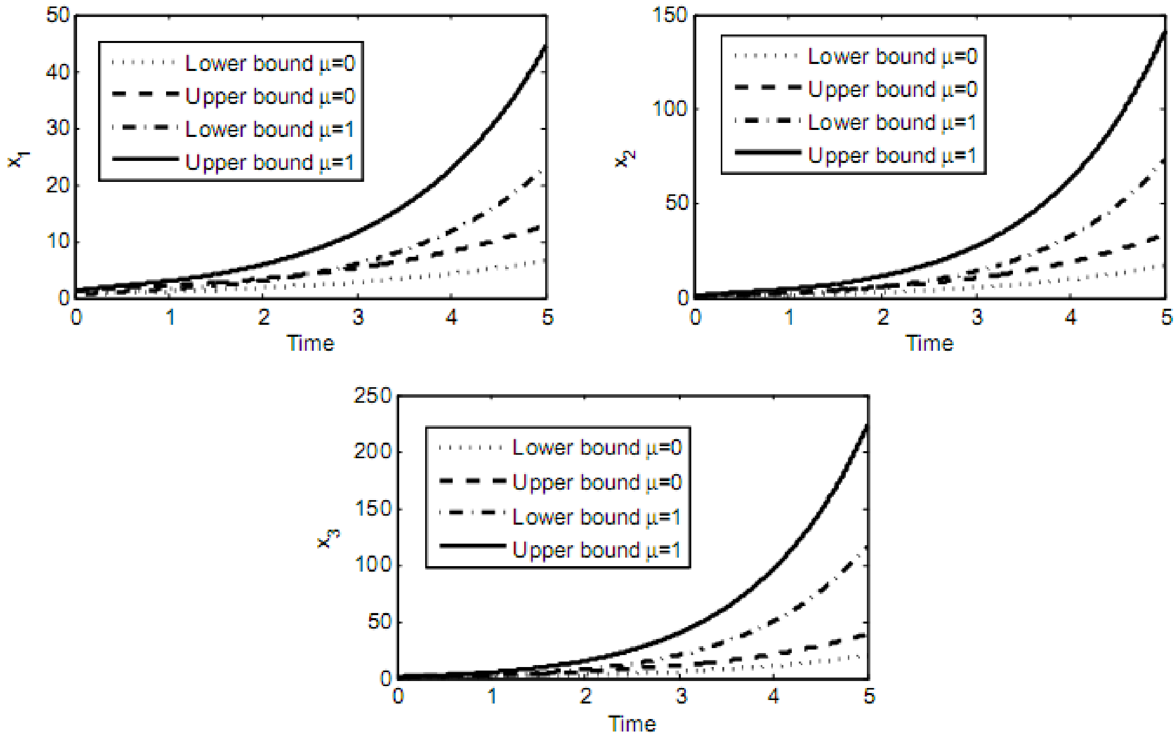

Figure 2. Lower and upper bound solutions of Example 5.2 for $\mu = 0, 1$ using the proposed method.

6. Conclusion

Using a new representation of the α-level sets of the fuzzy time-delay system, we successfully studied the first order linear time-delay dynamical systems with fuzzy matrices. Since this representation is perfectly adapted to the combination of the fuzzy time-delay differential equations, the solution of the linear fuzzy time-delay equation can be easily inherited from the solution of the classical time-delay differential equations. For further research, we will investigate the properties of the fuzzy time-delay dynamical systems. Several issues for fuzzy time-delay dynamical systems, which could not adequately be addressed in this paper, require further work. Fuzzy time-delay dynamical systems with multiple time lags should be studied in more details. Also further research is in progress to apply and extend this new approach to solve n-order fuzzy time-delay differential equations. This issue is subject to current research.

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