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A fixed point approach to the stability of a general quartic functional equation

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Abstract

In this paper, we study the generalized Hyers-Ulam stability of the quartic functional equation

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=0,$$

by applying the fixed point method.

Keywords: Fixed point method, fixed point, stability, generalized Hyers-Ulam stability, quartic functional equation.

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1. Introduction

The stability problem of the functional equations stems from Ulam's well known question about the stability of group homomorphisms (see [16]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist an $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers [9] affirmatively answered the Ulam's question for the additive functional equation under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers proved that each solution of inequality $\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$, can be approximated by an exact solution (an additive function). In this case, we say that the Cauchy additive functional equation, $f(x+y) = f(x)+f(y)$, satisfies the Hyers-Ulam stability or it is stable in the sense of Hyers and Ulam.

Since then, the stability of various functional equations has been extensively studied by a number of mathematicians (e.g., see [1–3, 8, 10–12, 15] and the references therein).

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Every solution of the Cauchy additive functional equation

$$f(x+y)-f(x)-f(y)=0,$$

is called an additive mapping, and each solution of the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0,$$

is called a quadratic mapping.

In this paper, we will deal with a special type of quartic functional equation

$$Df(x,y)=0, \tag{1.1}$$

where we define

$$Df(x,y):=f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y),$$

for all (x,y) in the domain of f. It is not difficult to verify that the mapping $f(x)=ax^4+bx^3+cx^2+dx+e$ is a solution to this equation when a,b,c,d,e are real constants. We remind that a mapping f is called a quartic mapping provided that there exist real numbers a,b,c,d,e so that $f(x)=ax^4+bx^3+cx^2+dx+e$. For the case of $a=1$ and $b=-1$, Lee [14] proved the stability of (1.1) for restricted domains in Banach spaces.

In this paper, we will prove that every solution of functional equation (1.1) with $f(0)=0$ is a quartic mapping and we will introduce a strictly contractive mapping which allows us to use the fixed point theory in the sense of Cădariu and Radu [4–6]. And then we will adopt the fixed point method for proving the stability of the functional equation (1.1). The key point is that starting from the mapping f satisfying (1.1) approximately, we construct the exact solution F of (1.1) explicitly by using either the formula

$$F(x)=\lim_{n\to\infty}\left(\sum_{i=0}^{n} {}_nC_i\frac{(-1)^{n-i}30^i}{81^n}f_o(3^{2n-i}x)+\sum_{i=0}^{n} {}_nC_i\frac{(-1)^{n-i}90^i}{729^n}f_e(3^{2n-i}x)\right),$$

or

$$F(x)=\lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i\left(30^i(-81)^{n-i}f_o\left(\frac{x}{3^{2n-i}}\right)+90^i(-729)^{n-i}f_e\left(\frac{x}{3^{2n-i}}\right)\right).$$

In the last step we will approximate f with F.

2. Main results

We will use Margolis and Diaz's theorem in fixed point theory. Recently, this theorem has been widely used to prove the stability of various functional equations.

Theorem 2.1 ([7]). *Assume that (S,d) is a complete generalized metric space which means that the metric d may assume infinite values. Moreover, assume that $J:S\to S$ is a strictly contractive mapping with the Lipschitz constant $0<L<1$. Then, for each given element $x\in S$, either*

$$d\big(J^n x, J^{n+1}x\big)=\infty,\quad \forall\, n\in\mathbb{N}\cup\{0\},$$

or there exists a $k\in\mathbb{N}\cup\{0\}$ such that

(1) $d(J^n x, J^{n+1}x)<\infty$, *for all* $n\geqslant k$;

(2) *the sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in* $T:=\{y\in S\,|\,d(J^k x,y)<\infty\}$;

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$, *for all* $y \in T$.

Throughout this paper, let V and W be real vector spaces and let Y be a real Banach space. For a given mapping $f : V \to W$, we use the following abbreviations

$$f_e(x) := \frac{1}{2}\big(f(x) + f(-x)\big) \quad \text{and} \quad f_o(x) := \frac{1}{2}\big(f(x) - f(-x)\big),$$

for all $x \in V$.

In the following theorem, we prove the generalized Hyers-Ulam stability of the functional equation (1.1) by using the fixed point method (Theorem 2.1).

Theorem 2.2. *Assume that a mapping* $\varphi : V \times V \to [0, \infty)$ *satisfies the condition*

$$\varphi(3x, 3y) \leqslant \big(\sqrt{306} - 15\big) L \varphi(x, y), \tag{2.1}$$

for all $x, y \in V$ *and for some constant* $0 < L < 1$ *and let*

$$\Phi(x) = \frac{1}{729}\big(10\varphi_e(0, 3x) + 42\varphi_e(0, 2x) + 480\varphi_e(0, x) + 180\varphi_e(x, x)\big),$$

where

$$\varphi_e(x, y) = \frac{1}{2}\big(\varphi(x, y) + \varphi(-x, -y)\big).$$

If a mapping $f : V \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|Df(x, y)\| \leqslant \varphi(x, y), \tag{2.2}$$

for all $x, y \in V$, *then there exists a unique solution* $F : V \to Y$ *of* (1.1) *such that*

$$\|f(x) - F(x)\| \leqslant \frac{1}{1-L}\Phi(x), \tag{2.3}$$

for all $x \in V$. *In particular,* F *is represented by*

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left(\frac{(-1)^{n-i} 30^i}{81^n} f_o(3^{2n-i}x) + \frac{(-1)^{n-i} 90^i}{729^n} f_e(3^{2n-i}x) \right), \tag{2.4}$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce the generalized metric d in S defined by

$$d(g, h) = \inf\big\{K \geqslant 0 \mid \|g(x) - h(x)\| \leqslant K\Phi(x), \text{ for all } x \in V\big\}.$$

It is easy to verify that (S, d) is a complete generalized metric space (see [6, Theorem 2.5] or the proof of [13, Theorem 3.1]).

Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) = \frac{180}{729} g(3x) - \frac{90}{729} g(-3x) - \frac{5}{729} g(9x) + \frac{4}{729} g(-9x), \tag{2.5}$$

for all $x \in V$. Applying mathematical induction, we can prove the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \left(\frac{(-1)^{n-i} 30^i}{81^n} g_o(3^{2n-i}x) + \frac{(-1)^{n-i} 90^i}{729^n} g_e(3^{2n-i}x) \right), \tag{2.6}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in V$.

Indeed it follows from (2.6) that

$$J^0 g(x) = g(x) = g_o(x) + g_e(x),$$

and using the oddness and the evenness of g_o and g_e of (2.6) we get

$$\begin{aligned}
J^{n+1}g(x) &= J\big(J^n g(x)\big)\\
&= \frac{135}{729}J^n g(3x) - \frac{135}{729}J^n g(-3x) + \frac{45}{729}J^n g(3x) + \frac{45}{729}J^n g(-3x)\\
&\quad - \frac{9}{1458}J^n g(9x) + \frac{9}{1458}J^n g(-9x) - \frac{1}{1458}J^n g(9x) - \frac{1}{1458}J^n g(-9x)\\
&= \frac{270}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}30^i}{81^n} g_o\big(3^{2n+1-i}x\big)\\
&\quad + \frac{90}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}90^i}{729^n} g_e\big(3^{2n+1-i}x\big)\\
&\quad - \frac{9}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}30^i}{81^n} g_o\big(3^{2n+2-i}x\big)\\
&\quad - \frac{1}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}90^i}{729^n} g_e\big(3^{2n+2-i}x\big)\\
&= \frac{270}{729}\sum_{i=1}^{n+1} {}_nC_{i-1} \frac{(-1)^{n+1-i}30^{i-1}}{81^n} g_o\big(3^{2n+2-i}x\big)\\
&\quad + \frac{90}{729}\sum_{i=1}^{n+1} {}_nC_{i-1} \frac{(-1)^{n+1-i}90^{i-1}}{729^n} g_e\big(3^{2n+2-i}x\big)\\
&\quad - \frac{9}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}30^i}{81^n} g_o\big(3^{2n+2-i}x\big)\\
&\quad - \frac{1}{729}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}90^i}{729^n} g_e\big(3^{2n+2-i}x\big).
\end{aligned}$$

Furthermore, by using the well known formula ${}_nC_{i-1} + {}_nC_i = {}_{n+1}C_i$, we obtain

$$\begin{aligned}
J^{n+1}g(x) &= \sum_{i=1}^{n+1} {}_nC_{i-1} \frac{(-1)^{n+1-i}30^i}{81^{n+1}} g_o\big(3^{2n+2-i}x\big)\\
&\quad + \sum_{i=1}^{n+1} {}_nC_{i-1} \frac{(-1)^{n+1-i}90^i}{729^{n+1}} g_e\big(3^{2n+2-i}x\big)\\
&\quad - \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}30^i}{81^{n+1}} g_o\big(3^{2n+2-i}x\big)\\
&\quad - \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}90^i}{729^{n+1}} g_e\big(3^{2n+2-i}x\big)\\
&= \sum_{i=0}^{n+1} {}_{n+1}C_i \left(\frac{(-1)^{n+1-i}30^i}{81^{n+1}} g_o\big(3^{2n+2-i}x\big) + \frac{(-1)^{n+1-i}90^i}{729^{n+1}} g_e\big(3^{2n+2-i}x\big)\right),
\end{aligned}$$

which implies the validity of (2.6) for all $n \in \mathbb{N} \cup \{0\}$.

Assume $g, h \in S$ and suppose $K \in [0, \infty]$ is an arbitrary constant satisfying $d(g, h) \leqslant K$. From the definitions of d and Φ and by (2.1), we note that $\Phi(-x) = \Phi(x)$ and $\Phi(3x) \leqslant (\sqrt{306} - 15)L\Phi(x)$ for all $x \in V$. Hence, we further get

$$\begin{aligned}\|Jg(x) - Jh(x)\| &\leqslant \frac{5}{729}\|g(9x) - h(9x)\| + \frac{4}{729}\|g(-9x) - h(-9x)\| \\ &\quad + \frac{180}{729}\|g(3x) - h(3x)\| + \frac{90}{729}\|g(-3x) - h(-3x)\| \\ &\leqslant K\left(\frac{9}{729}\Phi(9x) + \frac{270}{729}\Phi(3x)\right) \\ &\leqslant K\left(\frac{\sqrt{306} - 15}{81}L\Phi(3x) + \frac{30}{81}\Phi(3x)\right) \\ &\leqslant K\frac{(\sqrt{306} - 15)^2 + 30(\sqrt{306} - 15)}{81}L\Phi(x) \\ &\leqslant LK\Phi(x),\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leqslant Ld(g, h),$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Moreover, it follows from (1.1) that

$$\begin{aligned}Df_e(x, y) =& \frac{1}{2}f(x + 3y) + \frac{1}{2}f(-x - 3y) - \frac{5}{2}f(x + 2y) - \frac{5}{2}f(-x - 2y) \\ &+ 5f(x + y) + 5f(-x - y) - 5f(x) - 5f(-x) \\ &+ \frac{5}{2}f(x - y) + \frac{5}{2}f(-x + y) - \frac{1}{2}f(x - 2y) - \frac{1}{2}f(-x + 2y),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}Df_o(x, y) =& \frac{1}{2}f(x + 3y) - \frac{1}{2}f(-x - 3y) - \frac{5}{2}f(x + 2y) + \frac{5}{2}f(-x - 2y) \\ &+ 5f(x + y) - 5f(-x - y) - 5f(x) + 5f(-x) \\ &+ \frac{5}{2}f(x - y) - \frac{5}{2}f(-x + y) - \frac{1}{2}f(x - 2y) + \frac{1}{2}f(-x + 2y),\end{aligned} \tag{2.8}$$

for all $x, y \in V$. On account of (2.5), and by a long and tedious calculation, we obtain

$$\begin{aligned}&\frac{1}{729}\big(Df_e(0, 3x) + 6Df_e(0, 2x) + 75Df_e(0, x) + 36Df_e(x, x)\big) \\ &\quad + \frac{1}{81}\big(Df_o(0, 3x) + 4Df_o(0, 2x) + 45Df_o(0, x) + 16Df_o(x, x)\big) \\ &\quad\quad = f(x) - Jf(x).\end{aligned} \tag{2.9}$$

Hence, by (2.2) and since $\|Df_o(x, y)\| \leqslant \varphi_e(x, y)$ and $\|Df_e(x, y)\| \leqslant \varphi_e(x, y)$, we see that

$$\begin{aligned}\|f(x) - Jf(x)\| &= \frac{1}{729}\|Df_e(0, 3x) + 6Df_e(0, 2x) + 75Df_e(0, x) + 36Df_e(x, x)\| \\ &\quad + \frac{1}{81}\|Df_o(0, 3x) + 4Df_o(0, 2x) + 45Df_o(0, x) + 16Df_o(x, x)\| \\ &\leqslant \frac{10\varphi_e(0, 3x) + 42\varphi_e(0, 2x) + 480\varphi_e(0, x) + 180\varphi_e(x, x)}{729} \\ &= \Phi(x),\end{aligned}$$

for all $x \in V$. It implies that $d(f, Jf) \leqslant 1 < \infty$ from the definition of d.

Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S \,|\, d(f, g) < \infty\}$, which is represented by (2.4) for all $x \in V$. Further, it follows from Theorem 2.1 (4) that

$$d(f, F) \leqslant \frac{1}{1-L} d(f, Jf) \leqslant \frac{1}{1-L},$$

and this inequality implies the validity of (2.3). By the definition of F, together with (2.1), (2.2), and (2.6), we have

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty} \|DJ^n f(x,y)\| \\
&= \lim_{n\to\infty} \Bigg\| \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}30^i}{81^n} Df_o(3^{2n-i}x, 3^{2n-i}y) \\
&\qquad + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}90^i}{729^n} Df_e(3^{2n-i}x, 3^{2n-i}y) \Bigg\| \\
&\leqslant \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left(\frac{30^i}{81^n} + \frac{90^i}{729^n} \right) \varphi_e(3^{2n-i}x, 3^{2n-i}y) \\
&\leqslant 2 \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{30^i}{81^n} \varphi_e(3^{2n-i}x, 3^{2n-i}y) \\
&\leqslant 2 \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{30^i(\sqrt{306}-15)^{n-i}L^{n-i}}{81^n} \varphi_e(3^n x, 3^n y) \\
&= 2 \lim_{n\to\infty} \frac{(30 + (\sqrt{306}-15)L)^n}{81^n} \varphi_e(3^n x, 3^n y) \\
&\leqslant 2 \lim_{n\to\infty} \frac{(30 + (\sqrt{306}-15))^n}{81^n} \varphi_e(3^n x, 3^n y) \\
&\leqslant 2 \lim_{n\to\infty} \frac{(\sqrt{306}+15)^n(\sqrt{306}-15)^n L^n}{81^n} \varphi_e(x, y) \\
&\leqslant \lim_{n\to\infty} 2L^n \varphi_e(x,y) \\
&= 0,
\end{aligned} \tag{2.10}$$

for all $x, y \in V$, i.e., F is a solution of the functional equation (1.1).

Finally, in view of (2.9) and (2.10), if F is a solution of the functional equation (1.1), then the equality

$$\begin{aligned}
F(x) - JF(x) &= \frac{1}{729}\big(DF_e(0,3x) + 6DF_e(0,2x) + 75DF_e(0,x) + 36DF_e(x,x)\big) \\
&\quad + \frac{1}{81}\big(DF_o(0,3x) + 4DF_o(0,2x) + 45DF_o(0,x) + 16DF_o(x,x)\big) \\
&= 0,
\end{aligned}$$

implies that F is a fixed point of J. □

Roughly speaking, the previous theorem dealt with the generalized Hyers-Ulam stability of the quartic functional equation (1.1) for the case of $\varphi(3x, 3y) < 3\varphi(x, y)$.

In the following theorem, we now deal with one of cases for $\varphi(3x, 3y) > 3\varphi(x, y)$.

Theorem 2.3. *For a given mapping $f : V \to Y$ with $f(0) = 0$, suppose there exists a mapping $\varphi : V^2 \to [0, \infty)$ such that inequality (2.2) holds for all $x, y \in V$. If there exists a constant $0 < L < 1$ such that*

$$L\varphi(3x, 3y) \geqslant \frac{10}{\sqrt{261}-16} \varphi(x, y), \tag{2.11}$$

for all $x, y \in V$*, then there exists a unique solution* $F : V \to Y$ *of* (1.1)*, for which the inequality*

$$\|f(x) - F(x)\| \leqslant \frac{1}{1-L}\Psi(x), \tag{2.12}$$

holds for all $x \in V$*, where*

$$\Psi(x) = 2\varphi_e\left(0, \frac{x}{3}\right) + 10\varphi_e\left(0, \frac{2x}{9}\right) + 120\varphi_e\left(0, \frac{x}{9}\right) + 52\varphi_e\left(\frac{x}{9}, \frac{x}{9}\right).$$

In particular, F *is represented by*

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left(30^{n-i}(-81)^i f_o\left(\frac{x}{3^{2n-i}}\right) + 90^{n-i}(-729)^i f_e\left(\frac{x}{3^{2n-i}}\right)\right), \tag{2.13}$$

for all $x \in V$*.*

Proof. Let S be the set given in the proof of Theorem 2.2. Similarly as in the proof of Theorem 2.2, we define a generalized metric d in S by

$$d(g, h) = \inf\left\{K \geqslant 0 \mid \|g(x) - h(x)\| \leqslant K\Psi(x), \text{ for all } x \in V\right\}.$$

It is not difficult to verify that (S, d) is a complete generalized metric space (or see the proof of [13, Theorem 3.1]).

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) = 60g\left(\frac{x}{3}\right) + 30g\left(\frac{-x}{3}\right) - 405g\left(\frac{x}{9}\right) - 324g\left(\frac{-x}{9}\right),$$

for all $x \in V$.

As we did in (2.6), applying mathematical induction, we can prove the following equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \left(30^i(-81)^{n-i} g_o\left(\frac{x}{3^{2n-i}}\right) + 90^i(-729)^{n-i} g_e\left(\frac{x}{3^{2n-i}}\right)\right),$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in V$.

Assume $g, h \in S$ and suppose $K \in [0, \infty]$ is an arbitrary constant satisfying $d(g, h) \leqslant K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| \leqslant & 60\left\|g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right)\right\| + 30\left\|g\left(\frac{-x}{3}\right) - h\left(\frac{-x}{3}\right)\right\| \\
& + 405\left\|g\left(\frac{x}{9}\right) - h\left(\frac{x}{9}\right)\right\| + 324\left\|g\left(\frac{-x}{9}\right) - h\left(\frac{-x}{9}\right)\right\| \\
\leqslant & 729K\Psi\left(\frac{x}{9}\right) + 90K\Psi\left(\frac{x}{3}\right) \\
\leqslant & L^2\frac{(\sqrt{2753}-45)^2}{729}K\Psi(x) + 90\frac{\sqrt{2753}-45}{729}LK\Psi(x) \\
\leqslant & LK\Psi(x),
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leqslant Ld(g, h),$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Moreover, as we did in the proof of Theorem 2.2, it follows from (1.1), (2.2), (2.7), and (2.8) that

$$\begin{aligned}\|f(x)-Jf(x)\| &= \Big\|Df_e\Big(0,\frac{x}{3}\Big)+6Df_e\Big(0,\frac{2x}{9}\Big)+75Df_e\Big(0,\frac{x}{9}\Big)+36Df_e\Big(\frac{x}{9},\frac{x}{9}\Big)\\ &\qquad +Df_o\Big(0,\frac{x}{3}\Big)+4Df_o\Big(0,\frac{2x}{9}\Big)+45Df_o\Big(0,\frac{x}{9}\Big)+16Df_o\Big(\frac{x}{9},\frac{x}{9}\Big)\Big\|\\ &\leqslant 2\varphi_e\Big(0,\frac{x}{3}\Big)+10\varphi_e\Big(0,\frac{2x}{9}\Big)+120\varphi_e\Big(0,\frac{x}{9}\Big)+52\varphi_e\Big(\frac{x}{9},\frac{x}{9}\Big)\\ &=\Psi(x),\end{aligned}$$

for all $x \in V$. It means that $d(f, Jf) \leqslant 1 < \infty$ from the definition of d.

Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.13) for all $x \in V$. Notice that

$$d(f,F) \leqslant \frac{1}{1-L} d(f,Jf) \leqslant \frac{1}{1-L},$$

and this inequality implies (2.12). By the definition of F, together with (2.2) and (2.11), we get

$$\begin{aligned}\|DF(x,y)\| &= \lim_{n\to\infty}\|DJ^n f(x,y)\|\\ &= \lim_{n\to\infty}\Big\|\sum_{i=0}^{n} {}_nC_i 30^i(-81)^{n-i} f_o\Big(\frac{x}{3^{2n-i}},\frac{y}{3^{2n-i}}\Big)\\ &\qquad +\sum_{i=0}^{n} {}_nC_i 90^i(-729)^{n-i} f_e\Big(\frac{x}{3^{2n-i}},\frac{y}{3^{2n-i}}\Big)\Big\|\\ &\leqslant \lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i(30^i 81^{n-i}+90^{n-i}729^i)\varphi_e\Big(\frac{x}{3^{2n-i}},\frac{y}{3^{2n-i}}\Big)\\ &\leqslant 2\lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i 90^i 729^{n-i}\varphi_e\Big(\frac{x}{3^{2n-i}},\frac{y}{3^{2n-i}}\Big)\\ &\leqslant 2\lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i 90^i(\sqrt{2753}-45)^{n-i}L^{n-i}\varphi_e\Big(\frac{x}{3^n},\frac{y}{3^n}\Big)\\ &\leqslant 2\lim_{n\to\infty}(90+(\sqrt{2753}-45)L)^n\varphi_e\Big(\frac{x}{3^n},\frac{y}{3^n}\Big)\\ &\leqslant 2\lim_{n\to\infty}\frac{(\sqrt{2753}+45)^n(\sqrt{2753}-45)^n}{729^n}L^n\varphi_e(x,y)\\ &= 2\lim_{n\to\infty}L^n\varphi_e(x,y)\\ &=0,\end{aligned}$$

for all $x, y \in V$, i.e., F is a solution of functional equation (1.1).

Finally, we notice that if F is a solution of functional equation (1.1), then it follows from the equality

$$\begin{aligned}F(x)-JF(x) =& DF_e\Big(0,\frac{x}{3}\Big)+6DF_e\Big(0,\frac{2x}{9}\Big)+75DF_e\Big(0,\frac{x}{9}\Big)+36DF_e\Big(\frac{x}{9},\frac{x}{9}\Big)\\ &+DF_o\Big(0,\frac{x}{3}\Big)+4DF_o\Big(0,\frac{2x}{9}\Big)+45DF_o\Big(0,\frac{x}{9}\Big)+16DF_o\Big(\frac{x}{9},\frac{x}{9}\Big),\end{aligned}$$

that F is a fixed point of J. □

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