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On Hilbert I-convergent sequence spaces

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Abstract

In this paper, we define some new sequence spaces $c_0^I(H)$, $c^I(H)$, $\ell_\infty^I(H)$, and $\ell_\infty(H)$ as a domain of triangle Hilbert matrix and study some topological and algebraic properties of these spaces. Further, we study some inclusion relations concerning these spaces.

Keywords: Hilbert matrix, Hilbert I-convergence, Hilbert I-Cauchy, Hilbert I-bounded.

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$, denote the sets of all natural and real numbers, respectively. By ω, we denotes the vector space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. A family of sets $I \subset P(X)$ (where $P(X)$ is the power set of a non–empty set X) of subsets of X is said to be ideal in X if and only if (i) $\emptyset \in I$, (ii) for each $A, B \in I$ we have $A \cup B \in I$, (iii) for each $A \in I$ and $B \subset A$ we have $B \in I$ and I is called an admissible in X if and only if $I \neq X$ and it contains all singletons. A filter on X is a non-empty family of sets $\mathcal{F} \subset P(X)$ satisfying (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) for each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$. For each ideal I there is a filter $\mathcal{F}(I)$ corresponding to I, that is, $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$, where $K^c = X \backslash K$. Depending on the structure of ideals of subsets of $\mathbb{N}$, Kostyrko et al. [18] defined the notion of I-convergence as a generalization of statistical convergence introduced by Fast [6] and Steinhaus [25]. Later, the notion of I-convergence was further investigated from the sequence space point of view and linked with the summability theory by Šalát et al. [23, 24], Khan and Nazneen [15], Khan et al. [17], Filipów and Tryba [7] and many other authors. For further details on ideal convergence, we refer to [10–13, 16, 26–28].

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Throughout the paper, by ℓ_∞, c, and c_0 we denote the spaces of all bounded, convergent, and null sequences, respectively. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$ and let $x = (x_k) \in \omega$. Consider the new sequence

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k, \quad \text{for each } n \in \mathbb{N}, \tag{1.1}$$

that is, the sequence obtained on transforming the sequence x by the matrix A. Assume that the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$. Then the transformation A applies to the sequence x, and the sequence $A_n(x)$ is called the A-transform of x. The necessary and sufficient conditions for those matrices A that maps convergent sequence into another convergent sequence were given by well known Kojima-Schur Theorem 4.1,I (see, [4], p. 63) as follows:

(i) $\sum_{k=1}^{\infty} |a_{nk}| \leqslant M$ for every $n > n_0$;
(ii) $\lim_{n \to \infty} a_{nk} = \alpha_k$ for every fixed k;
(iii) $\sum_{k=1}^{\infty} a_{nk} = A_n \to \alpha$ as $n \to \infty$.

An infinite matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. Also a triangle matrix $A = (a_{nk})$ uniquely has an inverse A^{-1} which is a triangle matrix. Let λ be an arbitrary sequence space. Then for every sequence $x = (x_k) \in \lambda$, the matrix domain of an infinite matrix A in a sequence space λ is a sequence space defined by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}. \tag{1.2}$$

Motivated by various properties of matrix domains of triangles such as, if A is a triangle and λ is a BK-space, then λ_A is also a BK-space with the norm given by $\|x\|_{\lambda_A} = \|Ax\|_\lambda$ for all $x \in \lambda_A$ (see [2, Theorem 8.1.4]), the study of such matrices attracted the attention of many researchers to dig deeper in this area, for instance [1, 3, 19–21], and the references therein.

Recall in [9] the Hilbert matrix is an infinite matrix $H = (a_{nk})$ whose entries are $a_{nk} = (n + k - 1)^{-1}$ for $n, k \in \mathbb{N}$. The Hilbert matrix H was used in the theory of sequence spaces and considered as a bounded linear operator on the spaces of all p-summable sequences ℓ_p with norm $\|H\|_p = \frac{\pi}{\sin(\pi/p)}$ for $1 < p < \infty$ (see, [8]). Also, the Hilbert matrix can be viewed as an operator on spaces of analytic functions by its action on Taylor coefficients in [5]. Recently, by using the square Hilbert matrix H of order n, Polat [22] has introduced some new sequence spaces h_∞, h_c and h_0 as the sets of all sequences whose H-transforms of the sequence $x = (x_k) \in \omega$ are in the spaces ℓ_∞, c and c_0, respectively, that is

$$\lambda_H = \left\{ x = (x_k) \in \omega : \left(\sum_{k=1}^{m} \frac{x_k}{n + k - 1} \right) \in \lambda \right\} \quad \text{for } \lambda \in \{c_0, c, \ell_\infty\}.$$

Throughout the paper c_0^I, c^I and ℓ_∞^I, denote the spaces of all I-null, I-convergent, and I-bounded sequences, respectively. In this paper, by using the triangle Hilbert matrix $H = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} \frac{1}{n+k-1}, & \text{if } 1 \leqslant k \leqslant n, \\ 0, & \text{if } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, one can easily check that the conditions (i), (ii), and (iii) hold for the triangle Hilbert matrix H, and the notion of ideal convergence we define new sequence spaces $c_0^I(H)$, $c^I(H)$, $\ell_\infty^I(H)$ and $\ell_\infty(H)$ as the sets of all sequences whose H-transforms are in the spaces c_0^I, c^I, ℓ_∞^I and ℓ_∞, respectively. We define the sequence $H_n(x)$ that will be frequently used, as H-transform of the sequence $x = (x_k) \in \omega$, as follows:

$$H_n(x) = \sum_{k=1}^{n} \frac{x_k}{n + k - 1} \text{ for } n, k \in \mathbb{N}. \tag{1.3}$$

Further, we study some topological and algebraic properties and present some inclusion relations on these results.

Now, we recall some definitions and lemmas which will be used throughout the paper.

Definition 1.1 ([25]). If $K = \{k \in K : k \leqslant n\}$ is a subset of $\mathbb{N}$, then the natural density of the set K is given by

$$d(K) = \lim_{n \to \infty} \frac{1}{n} |K|$$

whenever the limit exists, here $|B|$ denotes the cardinality of the set B.

Definition 1.2 ([6]). A sequence $x = (x_k) \in \omega$ is said to be statistically convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, $d(\{k \in \mathbb{N} : |x_k - \ell| \geqslant \epsilon\}) = 0$. And we write $st-\lim x_k = \ell$. In case, $\ell = 0$ then $x = (x_k) \in \omega$ is said to be st-null.

Definition 1.3 ([23]). A sequence $x = (x_k) \in \omega$ is said to be I-Cauchy if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_N| \geqslant \epsilon\} \in I$.

Definition 1.4 ([18]). A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, $\{k \in \mathbb{N} : |x_k - \ell| \geqslant \epsilon\} \in I$. And we write $I-\lim x_k = \ell$. In case, $\ell = 0$ then $(x_k) \in \omega$ is said to be I-null.

Definition 1.5 ([14]). A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists $K > 0$, such that $\{k \in \mathbb{N} : |x_k| > K\} \in I$.

Definition 1.6 ([23]). Let $x = (x_k)$ and $z = (z_k)$ be two sequences. We say that $x_k = z_k$ for almost all k relative to I (in short a.a.k.r.I) if $\{k \in \mathbb{N} : x_k \neq z_k\} \in I$.

Definition 1.7 ([23]). A sequence space E is said to be solid or normal, if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence of scalars $(\alpha_k) \in \omega$ with $|\alpha_k| < 1$, for every $k \in \mathbb{N}$.

Definition 1.8 ([23]). Let $K = \{k_i \in \mathbb{N} : k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda_K^E = \{(x_{k_i}) \in \omega : (x_k) \in E\}.$$

A canonical pre-image of a sequence $(x_{k_i}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E, i.e., y is in canonical pre-image of λ_K^E iff y is canonical pre-image of some element $x \in \lambda_K^E$.

Definition 1.9 ([23]). A sequence space E is said to be monotone, if it contains the canonical pre-images of its step space.

Lemma 1.10 ([23]). *Every solid space is monotone.*

Lemma 1.11 ([24]). *Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

2. Main results

In this section, we define new sequence spaces $c_0^I(H)$, $c^I(H)$, $\ell_\infty^I(H)$, and $\ell_\infty(H)$ as the sets of all sequences whose H-transforms are in the spaces c_0^I, c^I, ℓ_∞^I, and ℓ_∞, respectively. Further, we study some topological and algebraic properties and present some inclusion relations on these resulting. Throughout the paper, we suppose that the sequence $x = (x_k) \in \omega$ and $H_n(x)$ are connected with the relation (1.3) and I is an admissible ideal of subset of $\mathbb{N}$.

$$c_0^I(H) := \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |H_n(x)| \geqslant \epsilon\} \in I\},$$

$$c^I(H) := \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon, \text{ for some } \ell \in \mathbb{R}\} \in I\},$$
$$\ell_\infty^I(H) := \{x = (x_k) \in \omega : \exists K > 0 \text{ s.t } \{n \in \mathbb{N} : |H_n(x)| \geqslant K\} \in I\},$$
$$\ell_\infty(H) := \{x = (x_k) \in \omega : \sup_n |H_n(x)| < \infty\}.$$

We write

$$m_0^I(H) := c_0^I(H) \cap \ell_\infty(H), \quad \text{and} \quad m^I(H) := c^I(H) \cap \ell_\infty(H).$$

With the notation of (1.2), the sequence spaces $c_0^I(H)$, $c^I(H)$, $\ell_\infty^I(H)$, $m^I(H)$, and $m_0^I(H)$ can be redefined as follows:

$$c_0^I(H) = (c_0^I)_H, \quad c^I(H) = (c^I)_H, \quad \ell_\infty^I(H) = (\ell_\infty^I)_H, \quad m^I(H) = (m^I)_H, \quad \text{and} \quad m_0^I(H) = (m_0^I)_H.$$

Definition 2.1. Let I is an admissible ideal of subset of $\mathbb{N}$. If for each $\epsilon > 0$ there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |H_n(x) - H_N(x)| \geqslant \epsilon\} \in I$ then a sequence $x = (x_k) \in \omega$ is called Hilbert I-Cauchy.

Example 2.1. Define $I_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. I_f is an admissible ideal in $\mathbb{N}$ and $c^{I_f}(H) = h_c$.

Example 2.2. Define a non-trivial ideal as $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the natural density of the set A. In this case $c^{I_d}(H) = S(H)$. Where we define $S(H)$ as the space of all Hilbert statistically convergent sequences as follows:

$$S(H) := \left\{x = (x_k) \in \omega : d(\{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon\}) = 0, \text{ for some } \ell \in \mathbb{R}\right\}.$$

Remark 2.2. Hilbert convergent sequence is obviously Hilbert statistically convergent since all finite subsets of the natural numbers have density zero. However, the converse is not true. For example, define the sequence $x = (x_k) \in \omega$ such that

$$H_n(x) = \begin{cases} 1, & \text{if } n \text{ is a square}, \\ 0, & \text{otherwise}, \end{cases}$$

that is, $H_n(x) = \{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, \dots\}$ and let $\ell = 0$. Then,

$$\{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon\} \subset \{1, 4, 9, 16, \dots, i^2, \dots\}.$$

Since the set of squares of natural numbers has natural density zero, we have

$$d(\{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon\}) = 0.$$

This implies that the sequence $(x_k) \in S(H)$, but $(x_k) \notin h_c$.

Theorem 2.3. *The sequence spaces* $c^I(H)$, $c_0^I(H)$, $\ell_\infty^I(H)$, $m_0^I(H)$, *and* $m^I(H)$ *are linear over* $\mathbb{R}$.

Proof. Let $x = (x_k)$, $y = (y_k)$ be two arbitrary elements of the space $c^I(H)$ and α, β are scalars. Now, since $x, y \in c^I(H)$, then for given $\epsilon > 0$, there exist $\ell_1, \ell_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : |H_n(x) - \ell_1| \geqslant \frac{\epsilon}{2}\right\} \in I \quad \text{and} \quad \left\{n \in \mathbb{N} : |H_n(y) - \ell_2| \geqslant \frac{\epsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : |H_n(x) - \ell_1| < \frac{\epsilon}{2|\alpha|}\right\} \in \mathcal{F}(I), \quad A_2 = \left\{n \in \mathbb{N} : |H_n(y) - \ell_2| < \frac{\epsilon}{2|\beta|}\right\} \in \mathcal{F}(I),$$

be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \{n \in \mathbb{N} : |H_n(\alpha x + \beta y) - (\alpha \ell_1 + \beta \ell_2)| < \epsilon\}$$
$$\supseteq \left\{ \left\{ n \in \mathbb{N} : |H_n(x) - \ell_1| < \frac{\epsilon}{2|\alpha|} \right\} \cap \left\{ n \in \mathbb{N} : |H_n(y) - \ell_2| < \frac{\epsilon}{2|\beta|} \right\} \right\}. \tag{2.1}$$

Thus, the sets on right hand-side of equation (2.1) belong to $\mathcal{F}(I)$. By definition of filter associated with ideal, the complement of the set on left-hand side of (2.1) belongs to I. This implies that $(\alpha x + \beta y) \in c^I(H)$. Hence, $c^I(H)$ is linear space. The proof of the remaining results is similar. $\square$

Theorem 2.4. *The spaces* $X(H)$ *are normed spaces with the norm*

$$\|x\|_{X(H)} = \sup_n |H_n(x)|, \quad \text{where } X \in \{c^I, c_0^I, \ell_\infty^I, \ell_\infty\}. \tag{2.2}$$

Proof. The proof of the result is easy in view of existing techniques and hence omitted. $\square$

Theorem 2.5. *A sequence* $x = (x_k) \in \omega$ *is Hilbert I-convergent if and only if for every* $\epsilon > 0$, *there exists* $N = N(\epsilon) \in \mathbb{N}$, *such that*

$$\{n \in \mathbb{N} : |H_n(x) - H_N(x)| < \epsilon\} \in \mathcal{F}(I). \tag{2.3}$$

Proof. Suppose that the sequence $x = (x_k) \in \omega$ is Hilbert I-convergent to some number $\ell \in \mathbb{R}$, then for a given $\epsilon > 0$, we have

$$A_\epsilon = \left\{ n \in \mathbb{N} : |H_n(x) - \ell| < \frac{\epsilon}{2} \right\} \in \mathcal{F}(I).$$

Fix an integer $N = N(\epsilon) \in A_\epsilon$. Then, we have

$$|H_n(x) - H_N(x)| \leqslant |H_n(x) - \ell| + |\ell - H_N(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \in A_\epsilon$. Hence, (2.3) holds.

Conversely, suppose that (2.3) holds for all $\epsilon > 0$. Then,

$$B_\epsilon = \{n \in \mathbb{N} : H_n(x) \in [H_n(x) - \epsilon, H_n(x) + \epsilon]\} \in \mathcal{F}(I), \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [H_n(x) - \epsilon, H_n(x) + \epsilon]$. Fixing $\epsilon > 0$, we have $B_\epsilon \in \mathcal{F}(I)$ and $B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence, $B_\epsilon \cap B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset,$$

that is,

$$\{n \in \mathbb{N} : H_n(x) \in J\} \in \mathcal{F}(I)$$

and thus

$$\text{diam } (J) \leqslant \frac{1}{2} \text{ diam } (J_\epsilon),$$

where the diam of J denotes the length of an interval J. Proceeding in this way, by induction we get a sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ such that

$$\text{diam } (I_n) \leqslant \frac{1}{2} \text{ diam } (I_{n-1}), \text{ for } n = (2, 3, \dots)$$

and

$$\{n \in \mathbb{N} : H_n(x) \in I_n\} \in \mathcal{F}(I).$$

Then, there exists a number $\ell \in \bigcap_{n \in \mathbb{N}} I_n$ and it is a routine work to verify that $\ell = I - \lim H_n(x)$ showing that $x = (x_k) \in \omega$ is Hilbert I-convergent. Hence the result. $\square$

Theorem 2.6. *The inclusions $c_0^I(H) \subset c^I(H) \subset \ell_\infty^I(H)$ are strict.*

Proof. The inclusion $c_0^I(H) \subset c^I(H)$ is obvious. Now, to show its strictness, consider the sequence $x = (x_k) \in \omega$ such that $H_n(x) = 2$. It easy to see that the sequence $H_n(x) \in c^I$ but $H_n(x) \notin c_0^I$, that is, $x \in c^I(H) \backslash c_0^I(H)$. Next, let $x = (x_k) \in c^I(H)$. Then there exists $\ell \in \mathbb{R}$ such that $I - \lim H_n(x) = \ell$, that is,

$$\{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon\} \in I.$$

We have

$$|H_n(x)| = |H_n(x) - \ell + \ell| \leqslant |H_n(x) - \ell| + |\ell|.$$

From this it easily follows that the sequence (x_k) must belongs to $\ell_\infty^I(H)$. Further, we show the strictness of the inclusion $c^I(H) \subset \ell_\infty^I(H)$ by constructing the following example.

Example 2.3. Consider the sequence $x = (x_k) \in \omega$ such that

$$H_n(x) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square,} \\ 1, & \text{if } n \text{ is odd non-square,} \\ 0, & \text{if } n \text{ is even non-square.} \end{cases}$$

Then, the sequence $H_n(x) \in \ell_\infty^I$, but $H_n(x) \notin c^I$ which implies that the sequence $x \in \ell_\infty^I(H) \backslash c^I(H)$.

Thus, the inclusions $c_0^I(H) \subset c^I(H) \subset \ell_\infty^I(H)$ is strict. □

Remark 2.7. Hilbert bounded sequence is obviously Hilbert I-bounded as the empty set belongs to the ideal I. However, the converse is not true. For example, consider the sequence $x = (x_k) \in \omega$ such that

$$H_n(x) = \begin{cases} \frac{n^2}{n+1}, & \text{if } n \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $H_n(x)$ is not a bounded sequence. However, $\{n \in \mathbb{N} : |H_n(x)| > 1\} \in I$. Hence, (x_k) is Hilbert I-bounded.

Theorem 2.8. *The sequence spaces:*

   (i) $c^I(H)$ *and* $\ell_\infty(H)$*, overlap but neither one contains the other;*
   (ii) $c_0^I(H)$ *and* $\ell_\infty(H)$*, overlap but neither one contains the other.*

Proof.

(i) We prove that $c^I(H)$ and $\ell_\infty(H)$ are not disjoint. Consider the sequence $x = (x_k) \in \omega$ such that $H_n(x) = \frac{1}{n}$ for $n \in \mathbb{N}$. Then, $x \in c^I(H)$ but $x \in \ell_\infty(H)$. Next, define the sequence $x = (x_k) \in \omega$ such that

$$H_n(x) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $x \in c^I(H)$ but $x \notin \ell_\infty(H)$. Next, choose the sequence $x = (x_k) \in \omega$ such that

$$H_n(x) = \begin{cases} n, & \text{if is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x) \in \ell_\infty(H)$ but $x \notin c^I(H)$.

(ii) The proof is similar to the proof of part one. □

Theorem 2.9. *The spaces* $m^I(H)$ *and* $m_0^I(H)$ *are closed subspace of* $\ell_\infty(H)$*.*

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $m^I(H) \subset \ell_\infty(H)$. Then, $(x_k^{(i)})$ converges in $\ell_\infty(H)$ and we have $\lim_{i\to\infty} H_n^{(i)}(x) = H_n(x)$. Let $I - \lim H_n^{(i)}(x) = \ell_i$ for each $i \in \mathbb{N}$. Then, we have to show that

   (i) (ℓ_i) is convergent say to ℓ;
   (ii) $I - \lim H_n(x) = \ell$.

(i) Since $(x_k^{(i)})$ is a Cauchy sequence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left| H_n^{(i)}(x) - H_n^{(j)}(x) \right| < \frac{\epsilon}{3}, \text{ for all } i,j \geqslant n_0. \tag{2.4}$$

Now let A_i and A_j be the following sets in I:

$$A_i = \left\{ n \in \mathbb{N} : |H_n^{(i)}(x) - \ell_i| \geqslant \frac{\epsilon}{3} \right\} \tag{2.5}$$

and

$$A_j = \left\{ n \in \mathbb{N} : |H_n^{(j)}(x) - \ell_j| \geqslant \frac{\epsilon}{3} \right\}. \tag{2.6}$$

Consider $i,j \geqslant n_0$ and $n \notin A_i \cap A_j$. Then we have

$$|\ell_i - \ell_j| \leqslant |H_n^{(i)}(x) - \ell_i| + |H_n^{(j)}(x) - \ell_j| + |H_n^{(i)}(x) - H_n^{(j)}(x)| < \epsilon$$

by (2.4), (2.5), and (2.6). Thus, (ℓ_i) is a Cauchy sequence in $\mathbb{R}$ and thus convergent say to ℓ, that is, $\lim_{i\to\infty} \ell_i = \ell$.

(ii) Let $\delta > 0$ be given, then we can find m_0 such that

$$|\ell_i - \ell| < \frac{\delta}{3}, \text{ for each } i > m_0. \tag{2.7}$$

We have $(x_k^{(i)}) \to x_k$ as $i \to \infty$. Thus

$$|H_n^{(i)}(x) - H_n(x)| < \frac{\delta}{3}, \text{ for each } i > m_0. \tag{2.8}$$

Since $(H_n^{(j)}(x))$ is I-convergent to ℓ_j, there exists $D \in I$ such that for each $n \notin D$, we have

$$|H_n^{(j)}(x) - \ell_j| < \frac{\delta}{3}. \tag{2.9}$$

Without loss of generality, let $j > m_0$ then for all $n \notin D$, we have by (2.7), (2.8), and (2.9) that

$$|H_n(x) - \ell| \leqslant |H_n(x) - H_n^{(j)}(x)| + |H_n^{(j)}(x) - \ell_j| + |\ell_j - \ell| < \delta.$$

Hence (x_k) is Hilbert I-convergent to ℓ. Thus $m^I(H)$ is a closed subspace of $\ell_\infty(H)$. Similarly the other cases can be established. $\qquad\square$

Theorem 2.10. *The sequence spaces $c^I(H), c_0^I(H)$, and $\ell_\infty^I(H)$ are BK-spaces according to their norms defined by* (2.2).

Proof. We know that the sequence spaces c^I, c_0^I, and ℓ_∞^I are BK-spaces with their sup-norm. Furthermore, (1.2) holds and the Hilbert matrix is a triangle matrix. By taking into account these three facts and Theorem of Wilansky [29], we conclude that the sequence spaces are BK-spaces. This completes the proof of the theorem. $\qquad\square$

Since the spaces $m^I(H) \subset \ell_\infty(H)$ and $m_0^I(H) \subset \ell_\infty(H)$ are strict, in view of Theorem 2.9, we have the following result.

Theorem 2.11. *The spaces* $m^I(H)$ *and* $m_0^I(H)$ *are nowhere dense subset of* $\ell_\infty(H)$.

Theorem 2.12. *The spaces* $c_0^I(H)$ *and* $m_0^I(H)$ *are solid and monotone.*

Proof. We will prove the result for $c_0^I(H)$ and for $m_0^I(H)$ the result can be established similarly. Let $x = (x_k) \in c_0^I(H)$. For $\epsilon > 0$, the set

$$\{n \in \mathbb{N} : |H_n(x)| \geqslant \epsilon\} \tag{2.10}$$

belongs to I. Let $\alpha = (\alpha_k)$ be a sequence of scalars with $|\alpha| \leqslant 1$ for all $k \in \mathbb{N}$. Then,

$$|H_n(\alpha x)| = |\alpha H_n(x)| \leqslant |\alpha| |H_n(x)| \leqslant |H_n(x)|, \text{ for all } n \in \mathbb{N}.$$

From this inequality and from (2.10) we have

$$\{n \in \mathbb{N} : |H_n(\alpha x)| \geqslant \epsilon\} \subseteq \{n \in \mathbb{N} : |H_n(x)| \geqslant \epsilon\} \in I$$

implies that

$$\{n \in \mathbb{N} : |H_n(\alpha x)| \geqslant \epsilon\} \in I.$$

Therefore, $(\alpha x_k) \in c_0^I(H)$. Hence, the space $c_0^I(H)$ is solid, and hence by Lemma 1.10 the space $c_0^I(H)$ is monotone. $\qquad \square$

Theorem 2.13. *If* I *is neither maximal nor* $I = I_f$, *then the spaces* $c^I(H)$ *and* $m^I(H)$ *are neither monotone nor solid.*

Proof. We prove this result with the help of the following example. $\qquad \square$

Example 2.4. Let $I = I_f$. Let $K = \{n \in \mathbb{N} : n \text{ is odd }\}$. Consider the K-step space E_K of E defined as:

$$E_K = \{(x_k) \in \omega : (x_k) \in E\}.$$

Define the sequence $(y_k) \in E_K$ such that

$$H_n(y) = \begin{cases} H_n(x), & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) such that $H_n(x) = 3$, for all $n \in \mathbb{N}$. Then, $(x_k) \in E(H)$, but its K-step space pre–image does not belongs to $E(H)$, where $E = c^I$ and m^I. Thus, $E(H)$ are not monotone. Hence, by Lemma 1.10 the spaces $E(H)$ are not solid.

Theorem 2.14. *Let* $x = (x_k) \in \omega$ *and let* I *be a non-trivial admissible ideal in* $\mathbb{N}$. *If there is a sequence* $y = (y_k) \in c^I(H)$ *such that* $H_n(x) = H_n(y)$ *for almost all* n *relative to* I, *then* $x \in c^I(H)$.

Proof. Suppose that $H_n(x) = H_n(y)$ for almost all n relative to I, that is

$$\{n \in \mathbb{N} : H_n(x) \neq H_n(y)\} \in I.$$

And let (y_k) be a sequence which is Hilbert I-convergent to ℓ. Then, for every $\epsilon > 0$, we have

$$\{n \in \mathbb{N} : |H_n(y) - \ell| \geqslant \epsilon\} \in I.$$

Since I is an admissible ideal, then the result follows from the following inclusion

$$\{n \in \mathbb{N} : |H_n(x) - \ell| \geqslant \epsilon\} \subseteq \{n \in \mathbb{N} : H_n(x) \neq H_n(y)\} \cup \{n \in \mathbb{N} : |H_n(y) - \ell| \geqslant \epsilon\}. \qquad \square$$

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