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Fisher type fixed point results in controlled metric spaces

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Abstract

In the present paper, we define a rational contractive condition of Fisher type in the context of controlled metric space and obtain some generalized fixed point results in this space. These results will unify and amend many well-known results of literature. Some consequences and an example has been presented at the end to show the authenticity of the established results.

Keywords: Fixed point, rational contraction, controlled metric spaces.

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1. Introduction

In 1906, the self-evident progress of a metric space was basically given by Frechet. Banach's fixed point theorem (BFPT) [9] is one of the crucial problems of fixed point theory and its utility, which states that, if $\mathcal{V}$: $(\Omega, \varphi) \to (\Omega, \varphi)$ (complete metric space) and $\exists \alpha \in [0, 1)$ such that

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha\varphi(\rho, \kappa), \quad \forall \rho, \kappa \in \Omega,$$

then $\exists \rho^* \in \Omega$ such that $\mathcal{V}\rho^* = \rho^*$. It is evident from the above contractive condition that $\mathcal{V}$ is a continuous function. Due to importance and simplicity of BFPT, many authors (see [1–23] and references therein) have obtained lots of fascinating upgrade and adjunct of it.

In 1980 Fisher [11] established a result for mapping satisfying

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha\varphi(\rho, \kappa) + \beta\frac{\varphi(\rho, \mathcal{V}\rho)\varphi(\kappa, \mathcal{V}\kappa)}{1 + \varphi(\rho, \kappa)}, \quad \forall \rho, \kappa \in \Omega,$$

where $\alpha, \beta \in [0, 1)$. Several researchers followed Fisher [11] paper using different types of contractive conditions in metric spaces.

Inspired from this innovative idea, several mathematicians generalized and extended this conception in the recent years as: semi metric spaces, rectangular metric spaces, quasi-semimetric spaces, quasi metric spaces, probabilistic metric spaces, pseudo metric spaces, extended b-metric space (EbMs), partial metric spaces, b-metric space (bMS) and controlled metric space (C-MS), etc.

Czerwik [10] defined the notion of (bMS) as follows:

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Definition 1.1 (see [10]). Let $\Omega \neq \emptyset$ and $s \geqslant 1$ and $\varphi : \Omega \times \Omega \to [0,\infty)$. If

(b1) $\varphi(\rho, \kappa) = 0 \Leftrightarrow \rho = \kappa$;

(b2) $\varphi(\rho, \kappa) = \varphi(\kappa, \rho)$, $\forall \rho, \kappa \in \Omega$;

(b3) $\varphi(\rho, \omega) \leqslant s[\varphi(\rho, \kappa) + \varphi(\kappa, \omega)]$, $\forall \rho, \kappa, \omega \in \Omega$,

then (Ω, φ) is called a b-MS.

In 2017, Kamran et al. [16] initiated the concept of (EbMS).

Definition 1.2. Let $\Omega \neq \emptyset$ and $\sigma : \Omega \times \Omega \to [1, \infty)$ and $\varphi : \Omega \times \Omega \to [0, \infty)$. If

(i) $\varphi(\rho, \kappa) = 0 \Leftrightarrow \rho = \kappa$;

(ii) $\varphi(\rho, \kappa) = \varphi(\kappa, \rho)$;

(iii) $\varphi(\rho, \kappa) \leqslant \sigma(\rho, \kappa)[\varphi(\rho, \omega) + \varphi(\kappa, \omega)]$,

then (Ω, φ) is called an EbMS.

Recently, a new kind of a generalized bMS introduced by Mlaiki et al. [21].

Definition 1.3 ([21]). Let $\Omega \neq \emptyset$ and $\sigma : \Omega \times \Omega \to [1, \infty)$ and $\varphi : \Omega \times \Omega \to [0, \infty)$. If

(i) $\varphi(\rho, \kappa) = 0 \Leftrightarrow \rho = \kappa$;

(ii) $\varphi(\rho, \kappa) = \varphi(\kappa, \rho)$;

(iii) $\varphi(\rho, \kappa) \leqslant \sigma(\rho, \omega)\varphi(\rho, \omega) + \sigma(\kappa, \omega)\varphi(\kappa, \omega)$,

then $(\Omega, \varphi, \sigma)$ is called a (C-MS).

Definition 1.4 ([21]). Let $\{\rho_r\}_{r\geqslant 0}$ be a sequence in $(\Omega, \varphi, \sigma)$.

(1) $\{\rho_r\}_{r\geqslant 0} \to \rho$ in Ω, is convergent if $\forall\ \epsilon > 0$, $\exists\ N = N(\epsilon) \in \mathbb{N}$ such that $\varphi(\rho_r, \rho) < \epsilon$, $\forall r \geqslant N$.

(2) $\{\rho_r\}_{r\geqslant 0}$ is Cauchy, if $\forall\ \epsilon > 0$, $\exists\ N = N(\epsilon) \in \mathbb{N}$ such that $\varphi(\rho_m, \rho_r) < \epsilon$, $\forall\ m, r \geqslant N$.

(3) If every Cauchy sequence in $(\Omega, \varphi, \sigma)$ is convergent then $(\Omega, \varphi, \sigma)$ is complete.

Theorem 1.5 ([21]). *Let* $(\Omega, \varphi, \sigma)$ *be a complete* C-MS. *Let* $\mathcal{V} : \Omega \to \Omega$ *be such that*

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha(\varphi(\rho, \kappa)), \quad \forall \rho, \kappa \in \Omega,$$

where $\alpha \in [0, 1)$. *For* $\rho_0 \in \Omega$, *take* $\rho_r = \mathcal{V}^r \rho_0$. *Suppose that*

$$\sup_{m\geqslant 1} \lim_{i\to\infty} \frac{\sigma(\rho_{i+1}, \rho_{i+2})\sigma(\rho_{i+1}, \rho_m)}{\sigma(\rho_i, \rho_{i+1})} < \frac{1}{\lambda}.$$

Assume that $\forall \rho \in \Omega$, $\lim_{r\to\infty} \sigma(\rho_r, \rho)$ *and* $\lim_{r\to\infty} \sigma(\rho, \rho_r)$ *exist and are finite. Then,* $\mathcal{V}$ *has a unique fixed point.*

In this paper, we define rational contraction of Fisher [11] type in the context of C-MS and prove some new fixed point result. Also, we present a non trivial example to illustrate importance of proved results.

2. Main results

Theorem 2.1. *Let $(\mathcal{Q}, \varphi, \sigma)$ be a complete C-MS. Let $\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$ be such that*

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha\varphi(\rho, \kappa) + \beta\frac{\varphi(\rho, \mathcal{V}\rho)\varphi(\kappa, \mathcal{V}\kappa)}{1 + \varphi(\rho, \kappa)}, \tag{2.1}$$

$\forall \rho, \kappa \in \mathcal{Q}$, where $\alpha, \beta \in [0, 1)$ such that $\lambda = \alpha + \beta < 1$. For $\rho_0 \in \mathcal{Q}$, take $\rho_r = \mathcal{V}^r\rho_0$. Suppose that

$$\sup_{m\geqslant 1} \lim_{i\to\infty} \frac{\sigma(\rho_{i+1}, \rho_{i+2})\sigma(\rho_{i+1}, \rho_m)}{\sigma(\rho_i, \rho_{i+1})} < \frac{1}{\lambda}.$$

Next, assume that $\forall \rho \in \mathcal{Q}$, we have $\lim_{r\to\infty} \sigma(\rho_r, \rho)$ and $\lim_{r\to\infty} \sigma(\rho, \rho_r)$ exist and are finite. Then, $\mathcal{V}$ has a unique fixed point.

Proof. Let $\rho_0 \in \mathcal{Q}$. We construct $\{\rho_r\}$ in $\mathcal{Q}$ by $\rho_{r+1} = \mathcal{V}\rho_r$, $\forall r \in \mathbb{N}$. If $\exists r_0 \in \mathbb{N}$ for which $\rho_{r_0+1} = \rho_{r_0}$, then $\mathcal{V}\rho_{r_0} = \rho_{r_0}$. Hence the proof is finished. Now, we assume that $\rho_{r+1} \neq \rho_r$ $\forall r \in \mathbb{N}$. Thus by (2.1), we get

$$\begin{aligned}
\varphi(\rho_r, \rho_{r+1}) &= \varphi(\mathcal{V}\rho_{r-1}, \mathcal{V}\rho_r) \\
&\leqslant \alpha\varphi(\rho_{r-1}, \rho_r) + \beta\frac{\varphi(\rho_{r-1}, \mathcal{V}\rho_{r-1})\varphi(\rho_r, \mathcal{V}\rho_r)}{1 + \varphi(\rho_{r-1}, \rho_r)} \\
&= \alpha\varphi(\rho_{r-1}, \rho_r) + \beta\frac{\varphi(\rho_{r-1}, \rho_r)\varphi(\rho_r, \rho_{r+1})}{1 + \varphi(\rho_{r-1}, \rho_r)} \\
&\leqslant \alpha\varphi(\rho_{r-1}, \rho_r) + \beta\varphi(\rho_r, \rho_{r+1}),
\end{aligned}$$

which implies

$$\varphi(\rho_r, \rho_{r+1}) \leqslant (\frac{\alpha}{1-\beta})\varphi(\rho_{r-1}, \rho_r) = \lambda\varphi(\rho_{r-1}, \rho_r).$$

Similarly,

$$\begin{aligned}
\varphi(\rho_{r-1}, \rho_r) &= \varphi(\mathcal{V}\rho_{r-2}, \mathcal{V}\rho_{r-1}) \\
&\leqslant \alpha\varphi(\rho_{r-2}, \rho_{r-1}) + \beta\frac{\varphi(\rho_{r-2}, \mathcal{V}\rho_{r-2})\varphi(\rho_{r-1}, \mathcal{V}\rho_{r-1})}{1 + \varphi(\rho_{r-2}, \rho_{r-1})} \\
&= \alpha\varphi(\rho_{r-2}, \rho_{r-1}) + \beta\frac{\varphi(\rho_{r-2}, \rho_{r-1})\varphi(\rho_{r-1}, \rho_r)}{1 + \varphi(\rho_{r-2}, \rho_{r-1})} \\
&= \alpha\varphi(\rho_{r-2}, \rho_{r-1}) + \beta\varphi(\rho_{r-1}, \rho_r),
\end{aligned}$$

which implies that

$$\varphi(\rho_{r-1}, \rho_r) \leqslant (\frac{\alpha}{1-\beta})\varphi(\rho_{r-2}, \rho_{r-1}) = \lambda\varphi(\rho_{r-2}, \rho_{r-1}).$$

Pursuing in this direction, we get

$$\begin{aligned}
\varphi(\rho_r, \rho_{r+1}) &\leqslant \lambda\varphi(\rho_{r-1}, \rho_r) \\
&\leqslant \lambda^2\varphi(\rho_{r-2}, \rho_{r-1}) \\
&\leqslant \\
&\vdots \\
&\leqslant \lambda^r\varphi(\rho_0, \rho_1).
\end{aligned}$$

Thus,

$$\varphi(\rho_r, \rho_{r+1}) \leqslant \lambda^r\varphi(\rho_0, \rho_1).$$

For all $r, m \in \mathbb{N}(r < m)$, we have

$$\begin{aligned}
\varphi(\rho_r, \rho_m) &\leqslant \sigma(\rho_r, \rho_{r+1})\varphi(\rho_r, \rho_{r+1}) + \sigma(\rho_{r+1}, \rho_m)\varphi(\rho_{r+1}, \rho_m) \\
&\leqslant \sigma(\rho_r, \rho_{r+1})\varphi(\rho_r, \rho_{r+1}) + \sigma(\rho_{r+1}, \rho_m)\sigma(\rho_{r+1}, \rho_{r+2})\varphi(\rho_{r+1}, \rho_{r+2}) \\
&\quad + \sigma(\rho_{r+1}, \rho_m)\sigma(\rho_{r+2}, \rho_m)\varphi(\rho_{r+2}, \rho_m) \\
&\leqslant \sigma(\rho_r, \rho_{r+1})\varphi(\rho_r, \rho_{r+1}) + \sigma(\rho_{r+1}, \rho_m)\sigma(\rho_{r+1}, \rho_{r+2})\varphi(\rho_{r+1}, \rho_{r+2}) \\
&\quad + \sigma(\rho_{r+1}, \rho_m)\sigma(\rho_{r+2}, \rho_m)\sigma(\rho_{r+2}, \rho_{r+3})\varphi(\rho_{r+2}, \rho_{r+3}) \\
&\quad + \sigma(\rho_{r+1}, \rho_m)\sigma(\rho_{r+2}, \rho_m)\sigma(\rho_{r+3}, \rho_m)\varphi(\rho_{r+3}, \rho_m) \\
&\leqslant \\
&\vdots \\
&\leqslant \sigma(\rho_r, \rho_{r+1})\varphi(\rho_r, \rho_{r+1}) + \sum_{i=r+1}^{m-2}\left(\prod_{j=r+1}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\varphi(\rho_i, \rho_{i+1}) \\
&\quad + \prod_{i=r+1}^{m-1}\sigma(\rho_i, \rho_m)\varphi(\rho_{m-1}, \rho_m),
\end{aligned}$$

which further implies that

$$\begin{aligned}
\varphi(\rho_r, \rho_m) &\leqslant \sigma(\rho_r, \rho_{r+1})\varphi(\rho_r, \rho_{r+1}) + \sum_{i=r+1}^{m-2}\left(\prod_{j=r+1}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\varphi(\rho_i, \rho_{i+1}) \\
&\quad + \left(\prod_{i=r+1}^{m-1}\sigma(\rho_i, \rho_m)\right)\sigma(\rho_{m-1}, \rho_m)\varphi(\rho_{m-1}, \rho_m) \\
&\leqslant \sigma(\rho_r, \rho_{r+1})\lambda^r\varphi(\rho_0, \rho_1) + \sum_{i=r+1}^{m-2}\left(\prod_{j=r+1}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\lambda^i\varphi(\rho_0, \rho_1) \\
&\quad + \left(\prod_{i=r+1}^{m-1}\sigma(\rho_i, \rho_m)\right)\sigma(\rho_{m-1}, \rho_m)\lambda^{m-1}\varphi(\rho_0, \rho_1) \\
&= \sigma(\rho_r, \rho_{r+1})\lambda^r\varphi(\rho_0, \rho_1) + \sum_{i=r+1}^{m-1}\left(\prod_{j=r+1}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\lambda^i\varphi(\rho_0, \rho_1).
\end{aligned}$$

Thus

$$\varphi(\rho_r, \rho_m) \leqslant \sigma(\rho_r, \rho_{r+1})\lambda^r\varphi(\rho_0, \rho_1) + \sum_{i=r+1}^{m-1}\left(\prod_{j=r+1}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\lambda^i\varphi(\rho_0, \rho_1). \tag{2.2}$$

Let

$$S_l = \sum_{i=0}^{l}\left(\prod_{j=0}^{i}\sigma(\rho_j, \rho_m)\right)\sigma(\rho_i, \rho_{i+1})\lambda^i\varphi(\rho_0, \rho_1).$$

From (2.2), we get

$$\varphi(\rho_r, \rho_m) \leqslant \varphi(\rho_0, \rho_1)[\lambda^r\sigma(\rho_r, \rho_{r+1}) + (S_{m-1} - S_r)]. \tag{2.3}$$

As above, using $\sigma(\rho, \kappa) \geqslant 1$, and ratio test, $\lim_{r\to\infty} S_r$ exists. Thus $\{S_r\}$ is Cauchy. Finally, letting $r, m \to \infty$ in (2.3), we conclude that

$$\lim_{r,m\to\infty}\varphi(\rho_r, \rho_m) = 0. \tag{2.4}$$

Thus, $\{\rho_r\}$ is a Cauchy $(\mathcal{Q}, \varphi, \sigma)$. So $\exists \rho^* \in \mathcal{Q}$ such that

$$\lim_{r\to\infty} \varphi(\rho_r, \rho^*) = 0, \tag{2.5}$$

that is, $\rho_r \to \rho^*$ as $r \to \infty$. Now, by (2.1) and condition (iii), we get

$$\begin{aligned}
\varphi(\rho^*, \mathcal{V}\rho^*) &\leqslant \sigma(\rho^*, \rho_{r+1})\varphi(\rho^*, \rho_{r+1}) + \sigma(\rho_{r+1}, \mathcal{V}\rho^*)\varphi(\rho_{r+1}, \mathcal{V}\rho^*)\\
&= \sigma(\rho^*, \rho_{r+1})\varphi(\rho^*, \rho_{r+1}) + \sigma(\rho_{r+1}, \mathcal{V}\rho^*)\varphi(\mathcal{V}\rho_r, \mathcal{V}\rho^*)\\
&\leqslant \sigma(\rho^*, \rho_{r+1})\varphi(\rho^*, \rho_{r+1}) + \sigma(\rho_{r+1}, \mathcal{V}\rho^*)[\alpha\varphi(\rho_r, \rho^*) + \beta\frac{\varphi(\rho_r, \mathcal{V}\rho_r)\varphi(\rho^*, \mathcal{V}\rho^*)}{1+\varphi(\rho_r, \rho^*)}]\\
&= \sigma(\rho^*, \rho_{r+1})\varphi(\rho^*, \rho_{r+1}) + \sigma(\rho_{r+1}, \mathcal{V}\rho^*)[\alpha\varphi(\rho_r, \rho^*) + \beta\frac{\varphi(\rho_r, \rho_{r+1})\varphi(\rho^*, \mathcal{V}\rho^*)}{1+\varphi(\rho_r, \rho^*)}].
\end{aligned}$$

Taking the limit as $r \to \infty$ and using (2.5), we get a contradiction to $\varphi(\rho^*, \mathcal{V}\rho^*) > 0$. Thus $\varphi(\rho^*, \mathcal{V}\rho^*) = 0$. This yields that $\rho^* = \mathcal{V}\rho^*$. Hence it is proved. □

Example 2.2. Let $\mathcal{Q} = \{0, 1, 2\}$. Define $\sigma : \mathcal{Q}\times\mathcal{Q} \to [1, \infty)$ and $\varphi : \mathcal{Q}\times\mathcal{Q} \to [1, \infty)$ as $\sigma(\rho, \kappa) = 1 + \rho\kappa$ and

$$\begin{aligned}
&\varphi(2, 2) = \varphi(0, 0) = \varphi(1, 1) = 0,\\
&\varphi(2, 0) = \varphi(0, 2) = 5, \quad \varphi(0, 1) = \varphi(0, 1) = 10,\\
&\varphi(1, 2) = \varphi(2, 1) = 30.
\end{aligned}$$

Now, define

$$\mathcal{V} : \mathcal{Q} \to \mathcal{Q},$$

by

$$\mathcal{V}\rho = \begin{cases} 0, & \text{if } \rho\in\{0, 2\},\\ 2, & \text{if } \rho = 1,\end{cases}$$

and choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.
Case 01: If $\rho = 0, \kappa = 1$, we have

$$\varphi(\mathcal{V}0, \mathcal{V}1) = \varphi(0, 2) = 5 = \frac{1}{2}(10) = \frac{1}{2}\varphi(0, 1) + \frac{1}{3}\frac{\varphi(0, 0)\varphi(1, 2)}{1+\varphi(0, 1)}.$$

Case 02: If $\rho = 0, \kappa = 2$, we have

$$\varphi(\mathcal{V}0, \mathcal{V}2) = \varphi(0, 0) = 0 < \frac{1}{2}(5) = \frac{1}{2}\varphi(0, 2) + \frac{1}{3}\frac{\varphi(0, 0)\varphi(2, 0)}{1+\varphi(0, 2)}.$$

Case 03: If $\rho = 1, \kappa = 2$, we have

$$\varphi(\mathcal{V}1, \mathcal{V}2) = \varphi(2, 0) = 5 < 15 + \frac{50}{31} = \frac{1}{2}\varphi(1, 2) + \frac{1}{3}\frac{\varphi(1, 2)\varphi(2, 0)}{1+\varphi(1, 2)}).$$

Case 04: If $\rho = \kappa = 0, \rho = \kappa = 1, \rho = \kappa = 2$, we have

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) = 0.$$

Consequently,

$$\varphi(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha(\varphi(\rho, \mathcal{V}\rho) + \varphi(\kappa, \mathcal{V}\kappa)),$$

$\forall \rho, \kappa \in \mathcal{Q}$. Thus the assertions of above result are fulfilled and $\mathcal{V}$ has a unique fixed point, that is, $\rho = 0$.

Special Cases:

Corollary 2.3. *Let $(\mathcal{Q}, \varphi_e)$ be a complete EbMS. Let $\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$ be such that*

$$\varphi_e(\mathcal{V}\rho, \mathcal{V}\kappa) \leqslant \alpha\varphi_e(\rho, \kappa) + \beta\frac{\varphi_e(\rho, \mathcal{V}\rho)\varphi_e(\kappa, \mathcal{V}\kappa)}{1+\varphi_e(\rho, \kappa)},$$

$\forall \rho, \kappa \in \mathcal{Q}$, where $\alpha, \beta \in [0, 1)$ such that $\lambda = \alpha + \beta < 1$. For $\rho_0 \in \mathcal{Q}$, take $\rho_r = \mathcal{V}^r\rho_0$. Suppose that

$$\sup_{m\geqslant 1}\lim_{i\to\infty}\frac{\sigma(\rho_{i+1},\rho_{i+2})\sigma(\rho_{i+1},\rho_m)}{\sigma(\rho_i,\rho_{i+1})}<\frac{1}{\lambda}.$$

Also, assume that $\forall\rho\in\Omega$, *we have* $\lim_{r\to\infty}\sigma(\rho_r,\rho)$ *and* $\lim_{r\to\infty}\sigma(\rho,\rho_r)$ *exist and are finite. Then,* $\exists\rho^*\in\Omega$ *such that* $\mathcal{V}\rho^*=\rho^*$.

Proof. Take $\sigma(\rho,\omega)=\sigma(\omega,\kappa)$ in Theorem 2.1. □

Corollary 2.4. *Let* (Ω,φ_b) *be a complete bMS and let* $\mathcal{V}:\Omega\to\Omega$. *If* $\exists\alpha,\beta\in[0,1)$ *such that* $\lambda=\alpha+\beta<1$ *and*

$$\varphi_b(\mathcal{V}\rho,\mathcal{V}\kappa)\leqslant\alpha\varphi_b(\rho,\kappa)+\beta\frac{\varphi_b(\rho,\mathcal{V}\rho)\varphi_b(\kappa,\mathcal{V}\kappa)}{1+\varphi_b(\rho,\kappa)},\quad\forall\rho,\kappa\in\Omega,$$

then $\exists\rho^*\in\Omega$ *such that* $\mathcal{V}\rho^*=\rho^*$.

Proof. Take $\sigma(\rho,\omega)=\sigma(\omega,\kappa)=s\geqslant 1$ in Theorem 2.1. □

Corollary 2.5. *Let* (Ω,φ) *be a complete MS and let* $\mathcal{V}:\Omega\to\Omega$. *If* $\exists\alpha,\beta\in[0,1)$ *such that* $\lambda=\alpha+\beta<1$ *and*

$$\varphi(\mathcal{V}\rho,\mathcal{V}\kappa)\leqslant\alpha\varphi(\rho,\kappa)+\beta\frac{\varphi(\rho,\mathcal{V}\rho)\varphi(\kappa,\mathcal{V}\kappa)}{1+\varphi(\rho,\kappa)},\quad\forall\rho,\kappa\in\Omega,$$

then $\exists\rho^*\in\Omega$ *such that* $\mathcal{V}\rho^*=\rho^*$.

Proof. Take $\sigma(\rho,\omega)=\sigma(\omega,\kappa)=1$ in Theorem 2.1. □

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