



## Numerical solution of Volterra integro-differential equation with delay



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### Abstract

We consider an initial value problem for a linear first-order Volterra delay integro-differential equation. We develop a novel difference scheme for the approximate solution of this problem via a finite difference method. The method is based on the fitted difference scheme on a uniform mesh which is achieved by using the method of integral identities which includes the exponential basis functions and applying to interpolate quadrature formulas that contain the remainder term in integral form. Also, the method is proved to be first-order convergent in the discrete maximum norm. Furthermore, a numerical experiment is performed to verify the theoretical results. Finally, the proposed scheme is compared with the implicit Euler scheme.

**Keywords:** Volterra delay integro-differential equation, finite difference method, error estimate.

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### 1. Introduction

Volterra delay integro-differential equations (VDIDEs) arise widely in mathematical models of biology, medicine, physics phenomena. In particular, the propagation of nervous impulse, population dynamics, polymeric liquids can be modelled by these equations [5, 9, 12, 13, 15, 17].

Motivated by the above works, we consider the following VDIDE in the interval  $\bar{I} = [0, T]$  :

$$Lu := u'(t) + a(t)u(t) + b(t)u(t-r) = f(t) + \int_{t-r}^t K(t,s)u(s)ds, \quad t \in I, \quad (1.1)$$

$$u(t) = \varphi(t), \quad t \in I_0, \quad (1.2)$$

where  $I = (0, T] = \cup_{p=1}^m I_p$ ,  $I_p = \{t : r_{p-1} < t \leq r_p\}$ ,  $1 \leq p \leq m$  and  $r_s = sr$ ,  $0 \leq s \leq m$  and,  $I_0 = [-r, 0]$  (for simplicity we suppose that  $T/r$  is integer; i.e.,  $T = mr$ ).  $a(t) \geq \alpha > 0$ ,  $b(t)$ ,  $f(t)$ ,  $\varphi(t)$  and  $K(t, s)$  are given sufficiently smooth functions,  $r$  is a positive constant large delay. The existence and uniqueness of solution to VDIDEs is discussed in [3, 6–8] (see also references cited in them).

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In recent years, various stability analysis and numerical approaches for VDIDEs have been increasingly studied via many methods. For instance, Koto [14] and Rihan et al. [18] studied stability of Runge-Kutta method for VDIDEs with a constant delay, Zang and Vandewalle [21] proposed an alternative approach method by using the general linear methods with compound quadrature rules for VDIDEs, the dissipativity of  $\theta$ -methods proposed by Gan [10] the analytical and numerical solution of nonlinear VDIDEs, the stability of linear multistep methods for VDIDEs discussed by Huang [11]. A collocation method with standard software for solving VDIDEs proposed by Shakourifar and Enright in [19]. Zhao et al. [23] proposed some stability results of numerical solution of linear VDIDEs with real coefficients. A Taylor polynomial method for solving VDIDEs presented by Belloura and Bousselsal [4]. Zhao et al. [22] constructed a methodology based on the Sinc collocation technique to approximate pantograph VDIDEs. Abdi et. al. [1] proposed barycentric rational method for solving VDIDEs. Kudu et al. [16] and Yapman et. al. [20] studied finite difference method to approximate solution of singularly perturbed VDIDEs.

In this paper, we develop a novel numerical approach to solve (1.1)-(1.2). Our approach is based on the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. And consequently, the local truncation errors occurs that includes up to first order derivative of the exact solution and thus facilitates examination of the convergence. The remainder of this paper is as follows. In Section 2, we give some important properties of the exact solution of (1.1)-(1.2). We describe the finite difference discretization on uniform mesh in Section 3. We present the error analysis of the proposed scheme and the convergence is proved in the discrete maximum norm in Section 4. In Section 5, we carry out the numerical experiment which validate the theoretical analysis computationally. Also, we compare our results with the results by obtained classical Euler method. Finally, in the last section, we give some conclusions.

**Notation.** Throughout the paper,  $C$  will denote a generic positive constant independent of the mesh parameter. Also, some specific fixed constants of this kind are indicated by sub-scripting  $C$  and  $D$ .  $\|g\|_\infty$  denotes the continuous maximum norm on the corresponding interval for any continuous function  $g(t)$ .

## 2. Some properties of the problem

In this section, we give a priori estimates for the exact solution of (1.1)-(1.2), which are used in later sections in the analysis of the approximate solution.

**Lemma 2.1.** *Let  $a, b, f \in C(\bar{I})$ ,  $\varphi \in C(I_0)$  and  $K \in C(\bar{I} \times \bar{I})$ . Then for  $u$  which is the exact solution of the problem (1.1)-(1.2) the following estimates hold:*

$$\|u\|_{\infty,p} \leq C_p, \quad 1 \leq p \leq m, \quad (2.1)$$

$$\|u'\|_{\infty,p} \leq D_p, \quad 1 \leq p \leq m, \quad (2.2)$$

where

$$C_p = \|\varphi\|_{\infty,0} \delta^p + \alpha^{-1} \frac{1 - \delta^p}{1 - \delta} (\|f\|_{\infty,p} + \bar{K} \|\varphi\|_{1,0}) e^{-\alpha^{-1} \bar{K} T}, \quad 1 \leq p \leq m,$$

$$D_1 = (\|a\|_{\infty,1} + \bar{K} T) C_1 + \|b\|_{\infty,1} \|\varphi\|_{\infty,0} + \|f\|_{\infty,1} + \bar{K} \|\varphi\|_{1,0},$$

$$D_p = (\|a\|_{\infty,p} + \bar{K} T) C_p + \|b\|_{\infty,p} C_{p-1} + \|f\|_{\infty,p} + \bar{K} \|\varphi\|_{1,0}, \quad 2 \leq p \leq m,$$

and

$$\delta = (1 + \alpha^{-1} \|b\|_{\infty,p}) e^{\alpha^{-1} \bar{K} T}, \quad \bar{K} = \max_{(t,s) \in \bar{I} \times \bar{I}} |K(t,s)|,$$

$$\|\varphi\|_{1,0} = \int_{-r}^0 |\varphi(s)| ds.$$

*Proof.* Since

$$\left| \int_{t-r}^t K(t,s)u(s)ds \right| \leq \bar{K} \int_{t-r}^t |u(s)| ds = \bar{K} \begin{cases} \int_{t-r}^0 |\varphi(s)| ds + \int_0^t |u(s)| ds, & \text{for } 0 < t \leq r, \\ \int_{t-r}^t |u(s)| ds, & \text{for } t > r, \end{cases}$$

then we can write

$$\left| \int_{t-r}^t K(t,s)u(s)ds \right| \leq \bar{K}(\|\varphi\|_{1,0} + \int_0^t |u(s)| ds). \tag{2.3}$$

Now, for  $t \in I_p$ , from (1.1) we have

$$u(t) = u(r_{p-1}) e^{-\int_{r_{p-1}}^t a(\tau)d\tau} + \int_{r_{p-1}}^t F(\xi) e^{-\int_{\xi}^t a(\tau)d\tau} d\xi$$

with

$$F(t) = f(t) - b(t)u(t-r) + \int_{t-r}^t K(t,s)u(s)ds.$$

So,

$$\begin{aligned} |u(t)| &\leq |u(r_{p-1})| e^{-\int_{r_{p-1}}^t a(\tau)d\tau} + \int_{r_{p-1}}^t [|f(\xi)| + |b(\xi)||u(\xi-r)| + \int_{\xi-r}^{\xi} |K(\xi,s)||u(s)| ds] e^{-\int_{\xi}^t a(\tau)d\tau} d\xi \\ &\leq |u(r_{p-1})| e^{-\alpha(t-r_{p-1})} + \int_{r_{p-1}}^t [|f(\xi)| + |b(\xi)||u(\xi-r)| + \int_{\xi-r}^{\xi} |K(\xi,s)||u(s)| ds] e^{-\alpha(t-\xi)} d\xi. \end{aligned} \tag{2.4}$$

Then, taking into account the (2.3) inequality in (2.4), we get

$$\begin{aligned} \|u\|_{\infty,p} &\leq \|u\|_{\infty,p-1} + \alpha^{-1}[\|f\|_{\infty,p} + \|b\|_{\infty,p} \|u\|_{\infty,p-1} + \bar{K}(\|\varphi\|_{1,0} + \int_0^t |u(s)| ds)] \\ &\leq (1 + \alpha^{-1} \|b\|_{\infty,p}) \|u\|_{\infty,p-1} + \alpha^{-1}[\|f\|_{\infty,p} + \bar{K}(\|\varphi\|_{1,0} + \int_0^T |u(s)| ds)]. \end{aligned}$$

Next, by using the Gronwall’s inequality it follows that

$$\|u\|_{\infty,p} \leq [(1 + \alpha^{-1} \|b\|_{\infty,p}) \|u\|_{\infty,p-1} + \alpha^{-1}(\|f\|_{\infty,p} + \bar{K} \|\varphi\|_{1,0})] e^{\alpha^{-1} \bar{K} T}.$$

From here, the following first order difference inequality follows

$$v_p \leq qv_{p-1} + \lambda$$

with

$$v_p = \|u\|_{\infty,p}, \quad q = (1 + \alpha^{-1} \|b\|_{\infty,p}) e^{\alpha^{-1} \bar{K} T}, \quad \lambda = \alpha^{-1}(\|f\|_{\infty,p} + \bar{K} \|\varphi\|_{1,0}) e^{\alpha^{-1} \bar{K} T},$$

which yields the estimate

$$v_p \leq v_0 q^p + \lambda \sum_{s=1}^p q^{p-s}$$

and so we arrive at (2.1). The proof of (2.2) is by induction in  $p$ . From (1.1) and (2.3), we have

$$|u'(t)| \leq |a(t)||u(t)| + |b(t)||u(t-r)| + |f(t)| + \bar{K}(\|\varphi\|_{1,0} + \int_0^t |u(s)| ds). \tag{2.5}$$

Now, for  $p = 1$  ( $t \in I_1$ ), we have

$$|u'(t)| \leq \|a\|_{\infty,1} \|u\|_{\infty,1} + \|b\|_{\infty,1} \|\varphi\|_{\infty,0} + \|f\|_{\infty,1} + \bar{K}(\|\varphi\|_{1,0} + \|u\|_{\infty,1} T) \equiv D_1.$$

Let the inequality (2.2) be true for  $p = k$ . That is

$$D_k = (\|a\|_{\infty,k} + \bar{K}T)C_k + \|b\|_{\infty,k} C_{k-1} + \|f\|_{\infty,k} + \bar{K}\|\varphi\|_{1,0}.$$

For  $t \in I_{k+1}$  because of (2.5) we get

$$|u'(t)| \leq \|a\|_{\infty,k+1} \|u\|_{\infty,k+1} + \|b\|_{\infty,k+1} \|u\|_{\infty,k} + \|f\|_{\infty,k+1} + \bar{K}(\|\varphi\|_{1,0} + \|u\|_{\infty,k+1} T)$$

and hence the inequality (2.2) hold for  $p = k + 1$ .  $\square$

### 3. Construction of the difference scheme

Let  $\omega_{N_0}$  be a uniform mesh on  $\bar{I}$ :

$$\omega_{N_0} = \{t_i = ih, i = 1, 2, \dots, N_0, h = T/N_0 = r/N\},$$

which contains by  $N$  mesh point at each subinterval  $I_p$  ( $1 \leq p \leq m$ ):

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \leq i \leq pN, 1 \leq p \leq m\}$$

and consequently

$$\omega_{N_0} = \cup_{p=1}^m \omega_{N_p}.$$

For any mesh function  $g(t)$ , we use  $g_i = g(t_i)$  and moreover  $y_i$  denotes an approximation of  $u(t)$  at  $t_i$  and

$$g_{\bar{i},i} = (g_i - g_{i-1})/h, \|g\|_{\infty,p} = \|g\|_{\infty,\omega_{N,p}} := \max_{1 \leq i \leq N} |g_i|.$$

For the difference approximation the problem (1.1), we are using the following identity

$$h^{-1} \int_{t_{i-1}}^{t_i} Lu(t)\psi_i(t)dt = h^{-1} \int_{t_{i-1}}^{t_i} [f(t) + \int_{t-r}^t K(t,s)u(s)ds]\psi_i(t)dt, 1 \leq i \leq N_0, \quad (3.1)$$

with basis function

$$\psi_i(t) = e^{-\int_t^{t_i} a(\tau)d\tau}, t_{i-1} \leq t \leq t_i,$$

which is the solution of the following problem

$$-\psi_i'(t) + a(t)\psi_i(t) = 0, t_{i-1} < t \leq t_i, \psi_i(t_i) = 1. \quad (3.2)$$

The relation (3.1) is rewritten as

$$\begin{aligned} & h^{-1} \int_{t_{i-1}}^{t_i} u'(t)\psi_i(t)dt + h^{-1} \int_{t_{i-1}}^{t_i} a(t)u(t)\psi_i(t)dt + h^{-1} \int_{t_{i-1}}^{t_i} b(t)u(t-r)\psi_i(t)dt \\ & = h^{-1} \int_{t_{i-1}}^{t_i} f(t)\psi_i(t)dt + h^{-1} \int_{t_{i-1}}^{t_i} [\int_{t-r}^t K(t,s)u(s)ds]\psi_i(t)dt. \end{aligned} \quad (3.3)$$

Firstly, using the formulas (2.1) and (2.2) from [2] on each interval  $(t_{i-1}, t_i)$  taking into account the left hand side Eq. (3.3) we have

$$h^{-1} \int_{t_{i-1}}^{t_i} u'(t)\psi_i(t)dt + h^{-1} \int_{t_{i-1}}^{t_i} a(t)u(t)\psi_i(t)dt + h^{-1} \int_{t_{i-1}}^{t_i} b(t)u(t-r)\psi_i(t)dt$$

$$= A_i u_{\bar{t}_i} + B_i u_{\bar{t}_{i-N}} + C_i u_i + D_i u_{i-N} + R_i^{(1)},$$

where

$$A_i = h^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t) dt + h^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) a(t) \psi_i(t) dt, \tag{3.4}$$

$$B_i = h^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) b(t) \psi_i(t) dt, \tag{3.5}$$

$$C_i = h^{-1} \int_{t_{i-1}}^{t_i} a(t) \psi_i(t) dt, \tag{3.6}$$

$$D_i = h^{-1} \int_{t_{i-1}}^{t_i} b(t) \psi_i(t) dt, \tag{3.7}$$

$$R_i^{(1)} = h^{-1} \int_{t_{i-1}}^{t_i} dt b(t) \psi_i(t) \int_{t_{i-1}}^{t_i} u'(s-r) [T_0(t-s) - h^{-1}(t-t_{i-1})] ds. \tag{3.8}$$

Secondly, for the integral term from the right hand side Eq. (3.3), after applying the appropriate quadrature rules, we have

$$\begin{aligned} h^{-1} \int_{t_{i-1}}^{t_i} \left[ \int_{t-r}^t K(t,s) u(s) ds \right] \psi_i(t) dt &= h^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t) dt \int_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K(t_{i-\frac{1}{2}}, s) u(s) ds + R_i^{(2)} \\ &= h^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t) dt h \sum_{j=i-N}^{i-1} K(t_{i-\frac{1}{2}}, s_j) u_j + R_i^{(2)} + R_i^{(3)}, \end{aligned}$$

where

$$R_i^{(2)} = h^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t) dt \int_{t_{i-1}}^{t_i} \frac{d}{dt} \left( \int_{\xi-r}^{\xi} K(\xi, s) u(s) ds \right) [T_0(t-\xi) - T_0(t_{i-\frac{1}{2}}-\xi)] d\xi, \tag{3.9}$$

$$R_i^{(3)} = \sum_{j=i-N}^{i-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} (t_{j+\frac{1}{2}} - \xi - h T_0(t_j - \xi)) \frac{d}{ds} (K(t_{i-\frac{1}{2}}, s) u(s)) ds, \tag{3.10}$$

and  $T_0(t) = 1, t > 0; T_0(t) = 0, t \leq 0$ .

Hereby, we write the exact relation for  $u(t_i)$ :

$$\ell u_i \equiv A_i u_{\bar{t}_i} + B_i u_{\bar{t}_{i-N}} + C_i u_i + D_i u_{i-N} = F_i + E_i h \sum_{j=i-N}^{i-1} K_{i-\frac{1}{2},j} u_j + R_i, \quad 1 \leq i \leq N_0, \tag{3.11}$$

with

$$E_i = h^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t) dt, \quad F_i = h^{-1} \int_{t_{i-1}}^{t_i} f(t) \psi_i(t) dt, \quad R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}, \tag{3.12}$$

where  $A_i, B_i, C_i, D_i$  and  $R_i^{(k)}$  ( $k = 1, 2, 3$ ) are determined by (3.4)-(3.7) and (3.8)-(3.10), respectively. By virtue of (3.11) we suggest the following difference scheme for approximating (1.1)-(1.2):

$$\ell y_i \equiv A_i y_{\bar{t}_i} + B_i y_{\bar{t}_{i-N}} + C_i y_i + D_i y_{i-N} = F_i + E_i h \sum_{j=i-N}^{i-1} K_{i-\frac{1}{2},j} y_j, \quad 1 \leq i \leq N_0, \tag{3.13}$$

$$y_i = \varphi_i, \quad -N \leq i \leq 0. \tag{3.14}$$

We also propose another difference scheme that can be easily obtained using the implicit Euler method and appropriate quadrature rules:

$$y_{\bar{t},i} + a_i y_i + b_i y_{i-N} = f_i + h \sum_{j=i-N}^{i-1} K_{i-\frac{1}{2},j} y_j, \quad 1 \leq i \leq N_0, \tag{3.15}$$

$$y_i = \varphi_i, \quad -N \leq i \leq 0. \tag{3.16}$$

#### 4. Error analysis

In order to investigate the convergence of this method, note that the error function  $z_i = y_i - u_i$ ,  $0 \leq i \leq N_0$  is the solution of the following discrete problem

$$\ell z_i = R_i, \quad 1 \leq i \leq N_0, \tag{4.1}$$

$$z_i = 0, \quad -N_0 \leq i \leq 0, \tag{4.2}$$

where the truncation error  $R_i$  is given by (3.12).

**Lemma 4.1.** *If  $\alpha, b, f \in C(\bar{I})$ ,  $\varphi \in C(I_0)$  and  $K \in C^1(\bar{I} \times \bar{I})$ , then for the truncation error  $R_i$  we have*

$$\|R\|_{\infty,p} \leq Ch, \quad 1 \leq p \leq m.$$

*Proof.* From (3.8), we write

$$|R_i^{(1)}| \leq Ch^{-1} \int_{t_{i-1}}^{t_i} dt |b(t)| \psi_i(t) \int_{t_{i-1}}^{t_i} |u'(\xi - r)| d\xi.$$

By virtue of Lemma 2.1 and  $0 < \psi_i(t) \leq 1$

$$|R_i^{(1)}| \leq Ch.$$

For the estimate  $R_i^{(2)}$ , from (3.9) we have

$$|R_i^{(2)}| \leq Ch^{-1} \int_{t_{i-1}}^{t_i} dt \psi_i(t) \int_{t_{i-1}}^{t_i} \left[ \left| \frac{\partial}{\partial \xi} K(\xi, s) \right| |u(s)| + |K(\xi, \xi)| |u(\xi)| + |K(\xi, \xi - r)| |u(\xi)| \right] d\xi$$

and in view of Lemma 2.1,  $|\frac{\partial}{\partial t} K(t, s)| \leq C$ , and  $0 < \psi_i(t) \leq 1$ ,

$$|R_i^{(2)}| \leq Ch.$$

Now, for the  $R_i^{(3)}$ , from (3.10) we have

$$|R_i^{(3)}| \leq Ch \sum_{j=i-N}^{i-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left[ \left| \frac{\partial}{\partial s} K(t_{i-\frac{1}{2}}, s) \right| |u(s)| + |K(t_{i-\frac{1}{2}}, s)| |u'(s)| \right] ds$$

and in view of Lemma 2.1 and  $|\frac{\partial}{\partial s} K(t, s)| \leq C$

$$|R_i^{(3)}| \leq Ch \sum_{j=i-N}^{i-1} (t_{j+\frac{1}{2}} - t_{j-\frac{1}{2}}) = Ch^2 N. \quad \square$$

**Lemma 4.2.** *Let  $z_i$  be the solution of (4.1)-(4.2) holds true. Then*

$$\|z\|_{\infty,p} \leq C \sum_{k=1}^p \|R\|_{\infty,k}, \quad 1 \leq p \leq m.$$

*Proof.* Since

$$\left| h \sum_{j=i-N}^{i-1} K_{i-\frac{1}{2},j} z_j \right| \leq h\bar{K} \sum_{j=i-N}^{i-1} |z_j| \leq \begin{cases} h\bar{K} \sum_{j=i-N}^0 |\varphi_j|, & i = 1, \\ h\bar{K} \sum_{j=i-N}^0 |\varphi_j| + h\bar{K} \sum_{j=1}^{i-1} |z_j|, & 1 < i \leq N, \\ h\bar{K} \sum_{j=i-N}^{i-1} |z_j|, & i > N, \end{cases}$$

then it is not hard to see that

$$\left| f_i + h \sum_{j=i-N}^{i-1} K_{i-\frac{1}{2},j} z_j \right| \leq \|f\|_{\infty,p} + \bar{K} \|\varphi\|_{1,0} + h\bar{K} \sum_{j=1}^i |z_{j-1}|.$$

From the solution (4.1)-(4.2)

$$\|z\|_{\infty,p} \leq \alpha^{-1} \sum_{k=1}^p \|R\|_{\infty,k} Q_{p-k}, \quad 1 \leq p \leq m,$$

where

$$Q_{p-k} = \begin{cases} 1, & \text{for } k = p, \\ \prod_{s=k+1}^p (1 + \alpha^{-1} (\|a\|_{\infty,I_s} + \|c\|_{\infty,I_s})), & \text{for } 0 \leq k \leq p - 1. \end{cases}$$

Hence, the proof is followed easily by induction in  $p$ . □

Combining Lemmas 4.1 and 4.2 gives us the following convergence result.

**Theorem 4.3.** *If  $u$  be the solution of (1.1)-(1.2) and  $y$  the solution of (3.1)-(3.2), then the following estimate holds true:*

$$\|y - u\|_{\infty, \omega_{N_0}} \leq Ch.$$

### 5. Numerical results

In this section, we present numerical experiment for the problem (1.1)-(1.2) to verify the theoretical results. We obtain numerical results by applying the new difference scheme (3.13)-(3.14) to the particular problem. Also, we present numerical results obtained by using implicit Euler method in (3.15)-(3.16). In addition, we compare both methods with respect to maximum pointwise errors.

**Example 1.** *We consider the test problem:*

$$u'(t) + 2u(t) - u(t-1) = e^{t-1} - 2t + \int_{t-1}^t (t-s)u(s)ds, \quad 0 < t \leq 2,$$

subject to the interval condition

$$u(t) = e^t, \quad -1 \leq t \leq 0,$$

the exact solution of the problem is given by

$$u(t) = \begin{cases} e^{-t}, & t \in (0, 1], \\ \frac{4}{\sqrt{5}} [\delta^- e^{\delta^-(t-1)} - \delta^+ e^{\delta^+(t-1)}] + \frac{1}{2} e^{t-1} + (\frac{9}{2} + e^{-1} - t) e^{1-t}, & t \in (1, 2], \end{cases}$$

where

$$\delta^\pm = (-1 \pm \sqrt{5})/2.$$

We define the exact error  $e_i^N$  and the computed maximum pointwise error  $e^N$  for any  $N$  as follows:

$$e_i^N = |y_i - u_i|, \quad e^N = \max_{0 \leq i \leq N} e_i^N,$$

where  $y_i$  is the numerical approximation to exact value  $u_i$  for the nodes  $t_i$ . The computational results of the test problem obtained by using both present method (PM) and implicit Euler method (EM) are given in the Tables 1-3.

Table 1: The numerical results for the test problem on  $(0, 2]$  (PM).

$t_i$	$u_i$	$y_i$ (N = 64)	$e_i^{64}$	$y_i$ (N = 128)	$e_i^{128}$
0.125	0.8824969	0.8824977	$8.158E-7$	0.8824971	$2.040E-7$
0.250	0.7788008	0.7788030	$2.225E-6$	0.7788013	$5.563E-7$
0.375	0.6872893	0.6872935	$4.251E-6$	0.6872903	$1.063E-6$
0.500	0.6065307	0.6065376	$6.921E-6$	0.6065324	$1.731E-6$
0.625	0.5352614	0.5352717	$1.027E-5$	0.5352640	$2.568E-6$
0.750	0.4723666	0.4723809	$1.434E-5$	0.4723701	$3.584E-6$
0.875	0.4168620	0.4168812	$1.918E-5$	0.4168668	$4.794E-6$
1.000	0.3678794	0.3679043	$2.485E-5$	0.3678857	$6.214E-6$
1.125	0.3108644	0.3108857	$2.128E-5$	0.3108697	$5.319E-6$
1.250	0.2378586	0.2378772	$1.859E-5$	0.2378632	$4.648E-6$
1.375	0.1563964	0.1564131	$1.672E-5$	0.1564006	$4.179E-6$
1.500	0.0723224	0.0723380	$1.559E-5$	0.0723263	$3.897E-6$
1.625	-0.0097872	-0.0097720	$1.514E-5$	-0.0097834	$3.785E-6$
1.750	-0.0862851	-0.0862697	$1.531E-5$	-0.0862812	$3.828E-6$
1.875	-0.1541774	-0.1541614	$1.604E-5$	-0.1541734	$4.009E-6$
2.000	-0.2108996	-0.2108823	$1.726E-5$	-0.2108953	$4.314E-6$

Table 2: The numerical results for the test problem on  $(0, 2]$  (EM).

$t_i$	$u_i$	$y_i$ (N = 64)	$e_i^{64}$	$y_i$ (N = 128)	$e_i^{128}$
0.125	0.8824969	0.8830591	$5.622E-3$	0.8827802	$2.833E-4$
0.250	0.7788008	0.7797059	$9.051E-4$	0.7792562	$4.554E-4$
0.375	0.6872893	0.6884010	$1.112E-3$	0.6878477	$5.584E-4$
0.500	0.6065307	0.6077771	$1.246E-3$	0.6071556	$6.249E-4$
0.625	0.5352614	0.5366208	$1.359E-3$	0.5359418	$6.804E-4$
0.750	0.4723666	0.4738566	$1.490E-3$	0.4731112	$7.446E-4$
0.875	0.4168620	0.4185319	$1.670E-3$	0.4176955	$8.336E-4$
1.000	0.3678794	0.3698046	$1.925E-3$	0.3688397	$9.603E-4$
1.125	0.3108644	0.3116622	$7.978E-4$	0.3112550	$3.906E-4$
1.250	0.2378586	0.2382602	$4.016E-4$	0.2380500	$1.914E-4$
1.375	0.1563964	0.1568536	$4.572E-4$	0.1566167	$2.203E-4$
1.500	0.0723224	0.0731035	$7.811E-4$	0.0727067	$3.843E-4$
1.625	-0.0097872	-0.0085294	$1.258E-3$	-0.0091625	$6.247E-4$
1.750	-0.0862851	-0.0844663	$1.819E-3$	-0.0853781	$9.070E-4$
1.875	-0.1541774	-0.1517501	$2.427E-3$	-0.1529648	$1.213E-3$
2.000	-0.2108996	-0.2078312	$3.068E-3$	-0.2093656	$1.534E-3$

Table 3: Comparison of  $e^N$  both methods on  $(0, 2]$ .

N	$e^N$ (EM)	$e^N$ (PM)	N	$e^N$ (EM)	$e^N$ (PM)
32	$6.138E-3$	$9.938E-5$	256	$7.670E-4$	$1.553E-6$
64	$3.068E-3$	$2.485E-5$	512	$3.835E-4$	$3.884E-7$
128	$1.534E-3$	$6.214E-6$	1024	$1.917E-4$	$9.709E-8$

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