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Analytical properties of extended Hermite-Bernoulli polynomials

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Abstract

This article aims to present a new family of extended Hermite-Bernoulli polynomials by making use of the Mittag-Leffler function. We also derive some analytical properties of our proposed extended Hermite-Bernoulli polynomials systematically. Furthermore, some concluding remarks of our present investigation are also pointed out in the last section.

Keywords: Hermite polynomials, Bernoulli polynomials, Hermite-Bernoulli polynomials, Mittag-Leffler function.

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1. Introduction

The two variable Kampé de Fériet generalization of the Hermite polynomials is defined by (see [15])

$$\mathbb{H}_n(u,v) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{v^r u^{n-2r}}{r!(n-2r)!},$$

where the polynomials $\mathbb{H}_n(u,v)$ have the following generating function:

$$e^{ut+vt^2} = \sum_{n=0}^{\infty} \mathbb{H}_n(u,v)\frac{t^n}{n!}.$$

If we set $v=-1$ and replace u by $2u$ then the polynomials $\mathbb{H}_n(u,v)$ reduce to the ordinary Hermite polynomials $\mathbb{H}_n(u)$ (see [15]).

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The Bernoulli numbers $\mathbb{B}_n$, Bernoulli polynomials $\mathbb{B}_n(x)$, and their generalization $\mathbb{B}_n^{(\alpha)}(x)$ of order α (real or complex) are defined, respectively, by means of the following generating functions (see [1–3, 15–17]):

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\mathbb{B}_n\frac{t^n}{n!},\quad (|t|<2\pi),$$

$$\left(\frac{t}{e^t-1}\right)e^{ut}=\sum_{n=0}^{\infty}\mathbb{B}_n(u)\frac{t^n}{n!},\quad (|t|<2\pi), \tag{1.1}$$

and

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{ut}=\sum_{n=0}^{\infty}\mathbb{B}_n^{(\alpha)}(u)\frac{t^n}{n!},\quad (|t|<2\pi,\ 1^{\alpha}:=1). \tag{1.2}$$

Moreover, Natalini and Bernardini [10] and Kurt [9] considered two new generalizations of the Bernoulli polynomials, which are given, respectively, by the following generating functions:

$$\left(\frac{t^p}{e^t-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)e^{ut}=\sum_{n=0}^{\infty}\mathbb{B}_n^{[p-1]}(u)\frac{t^n}{n!},\quad (|t|<2\pi,\ l\in\mathbb{N}), \tag{1.3}$$

and

$$\left(\frac{t^p}{e^t-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha}e^{ut}=\sum_{n=0}^{\infty}\mathbb{B}_n^{[\alpha,p-1]}(u)\frac{t^n}{n!},\quad (|t|<2\pi,\ 1^{\alpha}:=1,\ l\in\mathbb{N}). \tag{1.4}$$

Clearly, for $p=1$, equations (1.3) and (1.4) reduce to (1.1) and (1.2), respectively.

More recently, Ghayasuddin et al. [5] introduced a new family of Bernoulli polynomials $\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u)$ by means of the following generating function:

$$\left(\frac{t^p}{E_{\lambda}(t)-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha}e^{ut}=\sum_{n=0}^{\infty}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u)\frac{t^n}{n!}, \tag{1.5}$$

where $E_{\lambda}(t)$ is the well known Mittag-Leffler function given by (see [11])

$$E_{\lambda}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(1+\lambda k)}\qquad (t\in\mathbb{C}\text{ and }\Re(\lambda)>0). \tag{1.6}$$

On setting $\lambda=1$ in (1.5) and by using the fact $E_1(t)=e^t$, we easily get the generalized Bernoulli polynomials given in (1.4).

In 1999, Dattoli et al. [4] defined the Hermite-Bernoulli polynomials ${}_H\mathbb{B}_n(u,v)$ by the under mentioned generating function:

$$\left(\frac{t}{e^t-1}\right)e^{ut+vt^2}=\sum_{n=0}^{\infty}{}_H\mathbb{B}_n(u,v)\frac{t^n}{n!}.$$

Afterwards, Pathan [12] proposed a new family of Hermite-Bernoulli polynomials as follows:

$$\left(\frac{t^p}{e^t-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)e^{ut+vt^2}=\sum_{n=0}^{\infty}{}_H\mathbb{B}_n^{[p-1]}(u,v)\frac{t^n}{n!}. \tag{1.7}$$

Pathan and Khan [14] defined a further generalization of (1.7) by means of the following generating function:

$$\left(\frac{t^p}{e^t - \sum\limits_{l=0}^{p-1} \frac{t^l}{l!}}\right)^\alpha e^{ut+vt^2} = \sum_{n=0}^{\infty} {}_H\mathbb{B}_n^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}. \tag{1.8}$$

On taking $\alpha = 1$, (1.8) easily reduces to (1.7). For more details about the Bernoulli numbers, Bernoulli polynomials and Hermite-Bernoulli polynomials, we refer to see, for example, [6–8] and the references cited therein.

The aim of this article is to propose a new family of Hermite-Bernoulli polynomials in a unified and generalized form, which is given in the next section. We develop various fundamental properties and establish the implicit summation formulae, symmetry identities and bilateral series expansion for the newly introduced Hermite-Bernoulli polynomials by using different analysis on the generating function.

2. A new family of Hermite-Bernoulli polynomials

In this section, we propose a new extension of Hermite-Bernoulli polynomials ${}_H\mathbb{B}_n^{[\alpha,p-1]}(u,v)$ defined by Pathan and Khan [14] by making use of the definition of extended Bernoulli polynomials given in (1.5).

Definition 2.1. For real and complex parameter α and $\Re(\lambda) > 0$, the extended Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ are defined by

$$\left(\frac{t^p}{E_\lambda(t) - \sum\limits_{l=0}^{p-1} \frac{t^l}{l!}}\right)^\alpha e^{ut+vt^2} = \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}, \tag{2.1}$$

where $E_\lambda(t)$ is the well known Mittag-Leffler function given in (1.6).

It is noticed that the case $\lambda = 1$ in (2.1), gives the known generalization of Hermite-Bernoulli polynomials given by (1.8), i.e.,

$${}_H\mathbb{B}_{n,1}^{[\alpha,p-1]}(u,v) = {}_H\mathbb{B}_n^{[\alpha,p-1]}(u,v).$$

Also, for $u = v = 0$ in (2.1), we get certain new generalized Hermite-Bernoulli numbers ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}$, i.e.,

$${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(0,0) = {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}.$$

Theorem 2.2. *The following summation formula for extended Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(u-w)\, \mathbb{H}_m(w,v). \tag{2.2}$$

Proof. Taking

$$\left(\frac{t^p}{E_\lambda(t) - \sum\limits_{l=0}^{p-1} \frac{t^l}{l!}}\right)^\alpha e^{ut+vt^2} = \left(\frac{t^p}{E_\lambda(t) - \sum\limits_{l=0}^{p-1} \frac{t^l}{l!}}\right)^\alpha e^{(u-w)t}\, e^{wt+vt^2}$$

$$= \sum_{n=0}^{\infty} \mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u-w)\frac{t^n}{n!} \sum_{m=0}^{\infty} \mathbb{H}_m(w,v)\frac{t^m}{m!}, \tag{2.3}$$

$$\sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(u-w)\mathbb{H}_m(w,v)\frac{t^n}{m!(n-m)!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (2.3), we arrive at our claimed result (2.2). □

Corollary 2.3. *On putting $w = u$ in (2.2), we have*

$$ {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}\, \mathbb{H}_m(u,v). \tag{2.4}$$

Remark 2.4. For $\lambda = 1$, equation (2.2) and (2.4) are easily reduces to the known results of Pathan and Khan [14, Eq.(2.16) and Eq.(2.18)].

Theorem 2.5. *The following summation formula for extended Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$$ {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+w,v) = \sum_{m=0}^{n} \binom{n}{m} w^m\, {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(u,v). \tag{2.5}$$

Proof. Taking

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+w,v)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!} \\
&= \left(\frac{t^p}{E_\lambda(t) - \sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha} \left(e^{wt}-1\right) e^{ut+vt^2} \\
&= \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}\left(\sum_{m=0}^{\infty}\frac{(wt)^m}{m!} - 1\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} w^m\, {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{m!\,(n-m)!} - \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}.
\end{aligned}$$

Finally, by comparing the likes powers of t, we attain our required result (2.5).

□

Corollary 2.6. *On setting $w = 1$ in (2.5), we have*

$$ {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+1,v) = \sum_{m=0}^{n} \binom{n}{m} {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(u,v).$$

3. Implicit summation formulae involving extended Hermite-Bernoulli polynomials

This section deals with some implicit summation formulae for our newly introduced polynomials in the following theorems.

Theorem 3.1. *The following implicit summation formula for the extended Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$$ {}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v) = \sum_{r=0}^{j}\sum_{s=0}^{k} \binom{j}{r}\binom{k}{s} (y-u)^{r+s}\, {}_H\mathbb{B}_{j+k-r-s,\lambda}^{[\alpha,p-1]}(u,v). \tag{3.1}$$

Proof. On replacing t by $t+x$ in (2.1), we find

$$\left(\frac{(t+x)^p}{E_\lambda(t+x)-\sum_{l=0}^{p-1}\frac{(t+x)^l}{l!}}\right)^\alpha e^{u(t+x)+v(t+x)^2}=\sum_{n=0}^{\infty}{}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{(t+x)^n}{n!}. \tag{3.2}$$

Now by applying the result (see [17, p.52(2)])

$$\sum_{R=0}^{\infty}f(R)\frac{(p+q)^R}{R!}=\sum_{r,s=0}^{\infty}f(r+s)\frac{p^r}{r!}\frac{q^s}{s!} \tag{3.3}$$

in (3.2), we get

$$\left(\frac{(t+x)^p}{E_\lambda(t+x)-\sum_{l=0}^{p-1}\frac{(t+x)^l}{l!}}\right)^\alpha e^{v(t+x)^2}=e^{-u(t+x)}\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}. \tag{3.4}$$

Since the left-hand side of (3.4) is independent of u so we can replace u by y, to get

$$\left(\frac{(t+x)^p}{E_\lambda(t+x)-\sum_{l=0}^{p-1}\frac{(t+x)^l}{l!}}\right)^\alpha e^{v(t+x)^2}=e^{-y(t+x)}\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}. \tag{3.5}$$

On equating the right-hand sides of (3.4) and (3.5), we have

$$\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}=e^{(y-u)(t+x)}\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}. \tag{3.6}$$

Further, using the result given in (3.3) on the right-hand side of (3.6), we get

$$\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}=\sum_{r,s=0}^{\infty}(y-u)^{r+s}\frac{t^r}{r!}\frac{x^s}{s!}\sum_{j,k=0}^{\infty}{}_H\mathbb{B}_{j+k,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^j}{j!}\frac{x^k}{k!}. \tag{3.7}$$

Finally, replacing j by $j-r$ and k by $k-s$ on the right-hand side of (3.7), and after equating the likes powers of t and x, we easily attain our claimed result (3.1). □

Theorem 3.2. *The following summation formula for the extended Hermite-Bernoulli polynomials* ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ *holds true:*

$${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+x,v+y)=\sum_{k=0}^{n}\binom{n}{k}{}_H\mathbb{B}_{n-k,\lambda}^{[\alpha,p-1]}(u,v)\,\mathbb{H}_k(x,y). \tag{3.8}$$

Proof. Replacing $u\to u+x$ and $v\to v+y$ in (2.1), we get

$$\left(\frac{t^p}{E_\lambda(t)-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^\alpha e^{(u+x)t+(v+y)t^2}=\left(\frac{t^p}{E_\lambda(t)-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^\alpha e^{ut+vt^2}e^{xt+yt^2},$$

$$\sum_{n=0}^{\infty} {}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+x,v+y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}\sum_{k=0}^{\infty} \mathbb{H}_k(x,y)\frac{t^k}{k!},$$
$$\sum_{n=0}^{\infty} {}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u+x,v+y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_{H}\mathbb{B}_{n-k,\lambda}^{[\alpha,p-1]}(u,v)\mathbb{H}_k(x,y)\frac{t^n}{(n-k)!k!}. \tag{3.9}$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.9), we get our claimed result (3.8). □

Corollary 3.3. *On putting $x = u$ and $y = v$ in (3.8), we have*

$${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(2u,2v) = \sum_{k=0}^{n}\binom{n}{k} {}_{H}\mathbb{B}_{n-k,\lambda}^{[\alpha,p-1]}(u,v)\ \mathbb{H}_k(u,v).$$

Theorem 3.4. *The following summation formula for the extended Hermite-Bernoulli polynomials ${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(v,u) = \sum_{m=0}^{[\frac{n}{2}]}\frac{n!u^m}{m!(n-2m)!}\mathbb{B}_{n-2m,\lambda}^{[\alpha,p-1]}(v). \tag{3.10}$$

Proof. Replacing $u \to v$ and $v \to u$ in (2.1), we find

$$\left(\frac{t^p}{E_\lambda(t) - \sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha} e^{vt+ut^2} = \sum_{n=0}^{\infty} {}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(v,u)\frac{t^n}{n!},$$
$$\sum_{n=0}^{\infty}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(v)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{u^m t^{2m}}{m!} = \sum_{n=0}^{\infty} {}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(v,u)\frac{t^n}{n!}. \tag{3.11}$$

On applying Lemma 11 of [17, p.57(7)] in (3.11) and then by comparing the likes powers of t, we obtain our needed result (3.10). □

4. Symmetry Identities of ${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$

In this section, we derive the following symmetry identities for the extended Hermite-Bernoulli polynomials ${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$.

Theorem 4.1. *The following symmetry identity for extended Hermite-Bernoulli polynomials ${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m\ {}_{H}\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(bu,b^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}\\ &\quad=\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m\ {}_{H}\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(au,a^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}.\end{aligned} \tag{4.1}$$

Proof. Let us consider

$$f(t) = \left[\left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)\right]^{\alpha} e^{abut+a^2b^2vt^2}$$

$$= \left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)^\alpha e^{buat+b^2v(at)^2}\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)^\alpha$$
$$= \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(bu, b^2v)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}\frac{(bt)^m}{m!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m \, {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(bu, b^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}\frac{t^n}{n!}.$$

Since $f(t)$ is symmetric in a and b, therefore above expression can also be expressed as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m \, {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(au, a^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}\frac{t^n}{n!}.$$

By comparing the like powers of t on the right-hand side of the last two equations, we get our needed result (4.1). □

Theorem 4.2. *The general symmetry identity for extended Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m\sum_{r=0}^{a-1}\sum_{s=0}^{b-1} {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(bu + \frac{b}{a}r + s, b^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}(ax)$$
$$= \sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m\sum_{r=0}^{b-1}\sum_{s=0}^{a-1} {}_H\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(au + \frac{a}{b}r + s, a^2v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}(bx). \tag{4.2}$$

Proof. Let us consider

$$h(t) = \left[\left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)\right]^\alpha \frac{(e^{abt}-1)^2 e^{ab(u+x)t+a^2b^2vt^2}}{(e^{at}-1)(e^{bt}-1)},$$
$$h(t) = \left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)^\alpha e^{abut+a^2b^2vt^2}\frac{(e^{abt}-1)}{(e^{bt}-1)}\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)^\alpha e^{abxt}\frac{(e^{abt}-1)}{(e^{at}-1)}$$
$$= \left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)^\alpha e^{abut+a^2b^2vt^2}\sum_{r=0}^{a-1}e^{btr}\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)^\alpha e^{abxt}\sum_{s=0}^{b-1}e^{ats}$$
$$= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\left(\frac{(at)^p}{E_\lambda(at) - \sum_{l=0}^{p-1}\frac{(at)^l}{l!}}\right)^\alpha e^{\left(bu+\frac{b}{a}r+s\right)at+b^2v(at)^2}\left(\frac{(bt)^p}{E_\lambda(bt) - \sum_{l=0}^{p-1}\frac{(bt)^l}{l!}}\right)^\alpha e^{axbt}$$
$$= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(bu + \frac{b}{a}r + s, b^2v)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}(ax)\frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}{}_{H}\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(bu+\frac{b}{a}r+s,b^{2}v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}(ax)\frac{t^{n}}{n!}.$$

Since $h(t)$ is symmetric in a and b, therefore above expression can also be expressed as

$$h(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}\sum_{r=0}^{b-1}\sum_{s=0}^{a-1}{}_{H}\mathbb{B}_{n-m,\lambda}^{[\alpha,p-1]}(au+\frac{a}{b}r+s,a^{2}v)\mathbb{B}_{m,\lambda}^{[\alpha,p-1]}(bx)\frac{t^{n}}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right-hand side of the last two equations, we arrive at our claimed result (4.2). □

5. Bilateral expansion

In this section, we derive the following bilateral series expression for our extended Hermite-Bernoulli polynomials ${}_{H}\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$.

Theorem 5.1. *The following expansion holds true:*

$$G = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^{m}t^{n}\sum_{K=0}^{n}\frac{(-w)^{K}\ {}_{H}\mathbb{B}_{n-K,\lambda}^{[\alpha,p-1]}(u,v)}{K!(m+K)!(n-K)!}, \tag{5.1}$$

where $G = \left(\frac{t^p}{E_\lambda(t)-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha} e^{s-\frac{wt}{s}+ut+vt^2}$ and $m^* = \max[0,-m]$ $(m \in \mathbb{Z} := \{0,\pm1,\pm2,\ldots\})$.

Proof. On applying the definition of extended Hermite-Bernoulli polynomials and expanding the exponential function, we get

$$\begin{aligned} G &= \sum_{M=0}^{\infty}\frac{s^M}{M!}\sum_{K=0}^{\infty}\left(\frac{-wt}{s}\right)^{K}\frac{1}{K!}\sum_{N=0}^{\infty}{}_{H}\mathbb{B}_{N,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^N}{N!} \\ &= \sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\sum_{K=0}^{\infty}\frac{(-w)^{K}\ {}_{H}\mathbb{B}_{N,\lambda}^{[\alpha,p-1]}(u,v)s^{M-K}t^{N+K}}{M!N!K!}. \end{aligned}$$

On setting $N+K=n$ and $M-K=m$, and after rearrangement justified by the absolute convergence of the above series, we easily arrive at our claimed result (5.1). □

Corollary 5.2. *On setting $v=0$ in (5.1), we have*

$$\left(\frac{t^p}{E_\lambda(t)-\sum_{l=0}^{p-1}\frac{t^l}{l!}}\right)^{\alpha} e^{s-\frac{wt}{s}+ut} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^{m}t^{n}\sum_{K=0}^{n}\frac{(-w)^{K}\ \mathbb{B}_{n-K,\lambda}^{[\alpha,p-1]}(u)}{K!(m+K)!(n-K)!}.$$

Corollary 5.3. *If we assign $s=t=\frac{w}{2}$, $\alpha=0$, and $u=1$ in (5.1), then we have*

$$e^{\frac{vw^2}{4}} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\left(\frac{w}{2}\right)^{m+n}\sum_{K=0}^{n}\frac{(-w)^{K}\ {}_{H}\mathbb{B}_{n-K,\lambda}^{[0,p-1]}(1,v)}{K!(m+K)!(n-K)!}.$$

6. Concluding remarks

In this article, we have presented a new class of Hermite-Bernoulli polynomials ${}_H\mathbb{B}_{n,\lambda}^{[\alpha,p-1]}(u,v)$. Moreover, we have developed various fundamental and useful properties of our newly introduced polynomials. In this section, we briefly discuss about the new extension of Hermite-Euler polynomials given by Pathan and Khan [13].

In [5], authors have also introduced a further extension of Euler polynomials by means of the following generating function:

$$\left(\frac{2^p}{E_\lambda(t)+\sum\limits_{l=0}^{p-1}\frac{t^l}{l!}}\right)^\alpha e^{ut}=\sum_{n=0}^{\infty}\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}(u)\frac{t^n}{n!}, \tag{6.1}$$

where $\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}(u)$ denotes the extended Euler polynomials.

For $\lambda=1$, these extended Euler polynomials are easily reduces to the generalized Euler polynomials $\mathbb{E}_{n}^{[\alpha,p-1]}(u)$ given in [7]. From (6.1), we conclude that the Hermite-Euler polynomials ${}_H\mathbb{E}_{n}^{[\alpha,p-1]}(u,v)$ defined by Pathan and Khan [13] can be generalized as follows:

$$\left(\frac{2^p}{E_\lambda(t)+\sum\limits_{l=0}^{p-1}\frac{t^l}{l!}}\right)^\alpha e^{ut+vt^2}=\sum_{n=0}^{\infty}{}_H\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}(u,v)\frac{t^n}{n!}. \tag{6.2}$$

Here ${}_H\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}(u,v)$ denotes our new extended Hermite-Euler polynomials.

On setting $\lambda=1$ in (6.2), we have

$${}_H\mathbb{E}_{n,1}^{[\alpha,p-1]}(u,v)={}_H\mathbb{E}_{n}^{[\alpha,p-1]}(u,v),$$

where ${}_H\mathbb{E}_{n}^{[\alpha,p-1]}(u,v)$ are the Hermite-Euler polynomials given in [13].

Also, it is possible to define the extended Hermite-Euler numbers ${}_H\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}$ by considering that

$${}_H\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}(0,0)={}_H\mathbb{E}_{n,\lambda}^{[\alpha,p-1]}.$$

Furthermore, we can derive some interesting properties for our newly proposed Hermite-Euler polynomials in a same manner given in Sections 2, 3, 4, 5. Derivations of such properties are still an open problem for further research.

References

[1] L. C. Andrews, *Special functions for engineers and applied mathematicians*, Macmillan Co., New York, (1985). 1

[2] T. M. Apostol, *On the Lerch Zeta function*, Pacific. J. Math., **1** (1951), 161–167.

[3] E. T. Bell, *Exponential polynomials*, Ann. of Math. (2), **35** (1934), 258–277. 1

[4] G. Dattoli, S. Lorenzutta, C. Cesarano, *Finite sums and generalized forms of Bernoulli polynomials*, Rend. Mat. Appl., **19** (1999), 385–391. 1

[5] M. Ghayasuddin, N. U. Khan, J. Choi, *A new family of generalized Bernoulli polynomials and numbers*, 2019. (Communicated) 1, 6

[6] W. A. Khan, *Some properties of the generalized Apostol type Hermite-Based Polynomials*, KyungPook Math. J., **55** (2015), 597–614. 1

[7] W. A. Khan, S. Araci, M. Acikgoz, *A new class of Laguerre-based Apostol type polynomials*, Cogent Math., **3** (2016), 17 pages. 6

[8] W. A. Khan, H. Haroon, *Some symmetric identities for the generalized Bernoulli, Euler and Genocchi polynomials associated with Hermite polynomials*, SpringerPlus, **5** (2016), 1–21. 1

[9] B. Kurt, *A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials* $B_n^2(x,y)$, Appl. Math. Sci. (Ruse), **4** (2010), 2315–2322. 1
[10] P. Natalini, A. Bernardini, *A generalization of the Bernoulli polynomials*, J. Appl. Math., **3** (2003), 155–163. 1
[11] G. M. Mittag-Leffler, *Sur la représentation analytique d'une branche uniforme d'une fonction monogéne*, Acta Math., **29** (1905), 101–182. 1
[12] M. A. Pathan, *A new class of generalized Hermite-Bernoulli polynomials*, Georgian Math. J., **19** (2012), 559–573. 1
[13] M. A. Pathan, W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Euler polynomials*, East-West J. Math., **16** (2014), 92–109. 6, 6, 6
[14] M. A. Pathan, W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials*, Mediterr. J. Math., **12** (2015), 679–695. 1, 2, 2.4
[15] E. D. Rainville, *Special functions*, Macmillan Co., New York, (1960). 1, 1
[16] H. M. Srivastava, J. S. Choi, *Zeta and q-Zeta functions and associated series and integrals*, Elsevier, Amsterdam, (2012).
[17] H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, John Wiley & Sons, New York, (1984). 1, 3, 3