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On BF-semigroups and Fuzzy BF-semigroups

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Abstract

We introduce and establish the notion of BF-semigroups. We construct quotient BF-semigroups via BF-ideals and we investigate homomorphisms of BF-semigroups and establish the isomorphism theorems for BF-semigroups. Moreover, we apply the concept of fuzzy sets to BF-semigroups.

Keywords: BF/BF_1/BF_2-semigroup, sub BF-semigroup, BF-ideal, quotient BF-semigroup, fuzzy sub BF-semigroup.

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1. Introduction

The concept of BH-algebras was introduced by Jun et al. [4]. They defined a BH-algebra as an algebra of $(A; *, 0)$ of type (2, 0) (that is, a nonempty set A with a binary operation $*$ and a constant 0) satisfying the following axioms:

(B1) $x * x = 0$;

(B2) $x * 0 = x$;

(BH) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In [5], Kim and Kim introduced BG-algebras. An algebra $(A; *, 0)$ of type (2, 0) is a BG-algebra if it obeys (B1), (B2), and

(BG) $x = (x * y) * (0 * y)$.

In [7], Walendziak introduced BF-algebras, together with BF_1/BF_2-algebras. An algebra $(A; *, 0)$ of type (2, 0) is a BF-algebra if it obeys (B1), (B2), and

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(BF) $0 * (x * y) = y * x$.

A BF_1-algebra is a BF-algebra satisfying (BG) and a BF_2-algebra is a BF-algebra satisfying (BH). Let $(A; *, 0)$ be a BF-algebra. A nonempty subset N of A is called a subalgebra of A if $x * y \in N$ for any $x, y \in N$. A subset I of A is called an ideal of A if it satisfies:

(I1) $0 \in I$;

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

We say that an ideal I is normal if it satisfies:

(I3) $x * y \in I$ implies $(z * x) * (z * y) \in I$, for any $z \in A$.

The concept of fuzzy set, which was introduced by Zadeh [8] provides an extension of the classical notion of set. Since then a number of researches, both in theory and application, involving fuzzy sets have been established. In particular, Borumand Saeid and Rezvani [1] introduced fuzzy BF-algebras and they established some properties of this concept. Moreover, Hadipour [3] generalized the concept of fuzzy BF-algebras. We introduce and establish the notion of BF-semigroups. We construct quotient BF-semigroups via BF-ideals and we investigate homomorphisms of BF-semigroups and establish the isomorphism theorems for BF-semigroups. Moreover, we apply the concept of fuzzy sets to BF-semigroups.

2. BF-semigroups

Definition 2.1. A BF-semigroup is a nonempty set X together with two binary operations $*$ and $\cdot$ and a constant 0 satisfying:

(i) $(X; *, 0)$ is a BF-algebra;

(ii) $(X, \cdot)$ is a semigroup;

(iii) $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

Definition 2.2. A BF-semigroup is called a BF_1-semigroup (resp. a BF_2-semigroup) if it obeys (BG) (resp. (BH)).

Example 2.3. Let X = {0, a, b, c} be a set with the following table of operations:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Then $(X; *, \cdot, 0)$ is a BF-semigroup.

Example 2.4. Let X = {0, a, b} be a set with the following table of operations:

$*$	0	a	b
0	0	a	b
a	a	0	a
b	b	a	0

$\cdot$	0	a	b
0	0	0	0
a	0	b	0
b	0	0	0

Then $(X; *, \cdot, 0)$ is a BF-semigroup.

Example 2.5. Let X = {0, a, b} be a set with the following table of operations:

$*$	0	a	b
0	0	a	b
a	a	0	0
b	b	0	0

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	b	a

Then $(X; *, \cdot, 0)$ is a BF-semigroup.

Example 2.6. If $(X; *, 0)$ is a BF-algebra, then $(X; *, \cdot, 0)$ is a BF-semigroup, where $x \cdot y = 0$ for any $x, y \in X$.

Every BF_1-semigroup is a BF_2-semigroup (see [7, Proposition 2.12]). The BF-semigroup in Example 2.5 is not a BF_2-semigroup since $a * b = 0$ and $b * a = 0$ but $a \neq b$. Example 2.4 is a BF_2-semigroup which is not a BF_1-semigroup since $(b * a) * (0 * a) = a * a = 0 \neq b$.

Proposition 2.7 ([7, Proposition 2.5])**.** *If $(A; *, 0)$ is a BF-algebra, then for any $x, y \in A$,*

(a) $0 * (0 * x) = x$;

(b) $0 * x = 0 * y$ *implies* $x = y$;

(c) $x * y = 0$ *implies* $y * x = 0$.

From now on, let X stands for BF-semigroup $(X; *, \cdot, 0)$.

Proposition 2.8. *For all $a, b, c \in X$,*

(a) $a \cdot 0 = 0 \cdot a = 0$;

(b) $a \cdot (0 * b) = (0 * a) \cdot b = 0 * (a \cdot b)$;

(c) $(0 * a) \cdot (0 * b) = a \cdot b$;

(d) $a \cdot (b * (0 * c)) = (a \cdot b) * (0 * (a \cdot c))$, $(b * (0 * c)) \cdot a = (b \cdot a) * (0 * (c \cdot a))$.

Proof.

(a): By (B2), distributive property, and (B1), $a \cdot 0 = a \cdot (0 * 0) = (a \cdot 0) * (a \cdot 0) = 0$. Similarly, $0 \cdot a = 0$.

(b): By distributive property and (a), $a \cdot (0 * b) = (a \cdot 0) * (a \cdot b) = 0 * (a \cdot b) = (0 \cdot b) * (a \cdot b) = (0 * a) \cdot b$.

(c): By (b), distributive property, (a), and Proposition 2.7(a), $(0 * a) \cdot (0 * b) = 0 * (a \cdot (0 * b)) = 0 * ((a \cdot 0) * (a \cdot b)) = 0 * (0 * (a \cdot b)) = a \cdot b$.

(d): By distributive property and (b), $a \cdot (b * (0 * c)) = (a \cdot b) * (a \cdot (0 * c)) = (a \cdot b) * (0 * (a \cdot c))$. Similarly, $(b * (0 * c)) \cdot a = (b \cdot a) * (0 * (c \cdot a))$. □

Definition 2.9. A nonempty subset S of X is called a sub BF-semigroup of X if $x * y, x \cdot y \in S$ for all $x, y \in S$.

Clearly, $\{0\}$ and X are sub BF-semigroups of X.

Example 2.10. Let X = {0, a, b, c} be a set with the following table of operations:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	a
c	0	c	a	b

Then $(X; *, \cdot, 0)$ is a BF-semigroup. The set $S_1 = \{0, a\}$ is a sub BF-semigroup of X, while the set $S_2 = \{0, a, b\}$ is not since $a * b = c \notin S_2$.

3. Ideals in BF-semigroups

Definition 3.1. A nonempty subset I of X is called a BF-ideal of X if it obeys (I1), (I2), (I3), and

(I4) $a \cdot x, x \cdot a \in I$ for any $a \in I$, $x \in X$.

This means I is a BF-ideal of X if and only if I is a normal ideal of X satisfying (I4). Also, if I is a BF-ideal of X, then I is a sub BF-semigroup of X. The set $I_1 = \{0, b\}$ in Example 2.4 is a BF-ideal of X while $I_2 = \{0, a\}$ is not since $a \cdot a = b \notin I_2$. Clearly, X is a BF-ideal of X. However, $\{0\}$ need not be a BF-ideal of X (see [7, Example 3.2]). If X is a BF_2-semigroup, then $\{0\}$ is a BF-ideal of X. Indeed, if $x, y, z \in X$, then clearly (I1) and (I2) hold for $\{0\}$. Suppose that $x * y = 0$. Then from [7, Proposition 2.9], $x = y$ and so $(z * x) * (z * y) = 0$. Now, (I4) follows from Proposition 2.8(a). Therefore, $\{0\}$ is a BF-ideal of X.

Theorem 3.2.

(a) *The intersection of sub BF-semigroups of X is a sub BF-semigroup of X; and*

(b) *the intersection of BF-ideals of X is a BF-ideal of X.*

In [2], if I and J are ideals of BF-algebra $(X; *, 0)$, then we define the subset IJ of X to be the set $\{x \in X : x = i * (0 * j) \text{ for some } i \in I, j \in J\}$. From [2, Lemma 3.3(iii)], if I and J are normal ideals of a BF-algebra $(X; *, 0)$, then for any $i_1, i_2 \in I$, $j_1, j_2 \in J$, (SI) implies IJ is a subalgebra of X, where (SI) is given below

$$\text{(SI)}\ (i_1 * j_1) * (i_2 * j_2) = (i_1 * i_2) * (j_1 * j_2).$$

Proposition 3.3. *Let I and J be BF-ideals of a BF-semigroup X such that X satisfies (SI). Then IJ is a sub BF-semigroup of X.*

Proof. Let $x, y \in IJ$. Then there exist $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $x = i_1 * (0 * j_1)$ and $y = i_2 * (0 * j_2)$. Since I is a BF-ideal of X and $i_2 \in I$, $(i_1 * (0 * j_1)) \cdot i_2 \in I$. Since J is a BF-ideal of X and $j_2 \in J$, $(i_1 * (0 * j_1)) \cdot j_2 \in J$. Thus, by distributive property and Proposition 2.8 (b), we have

$$\begin{aligned} x \cdot y &= (i_1 * (0 * j_1)) \cdot (i_2 * (0 * j_2)) \\ &= [(i_1 * (0 * j_1)) \cdot i_2] * [(i_1 * (0 * j_1)) \cdot (0 * j_2)] \\ &= [(i_1 * (0 * j_1)) \cdot i_2] * [0 * ((i_1 * (0 * j_1)) \cdot j_2)] \in IJ. \end{aligned}$$

Therefore, IJ is a sub BF-semigroup of X. □

Definition 3.4. A map $\varphi : X \to Y$ is called a BF-semigroup homomorphism (or simply BFs-homomorphism) if $\varphi(x * y) = \varphi(x) * \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for any $x, y \in X$. We denote by $\ker \varphi$ the subset $\{x \in X : \varphi(x) = 0\}$ of X (it is the kernel of the BFs-homomorphism φ).

Lemma 3.5. *Let $\varphi : X \to Y$ be a BFs-homomorphism.*

(a) *If I is a sub BF-semigroup of X, then $\varphi(I)$ is a sub BF-semigroup of Y.*

(b) *If J is a sub BF-semigroup of Y, then $\varphi^{-1}(J)$ is a sub BF-semigroup of X.*

(c) *If I is a BF-ideal of X and φ is one-to-one, then $\varphi(I)$ is a BF-ideal of $\varphi(X)$.*

(d) *If J is a BF-ideal of Y, then $\varphi^{-1}(J)$ is a BF-ideal of X.*

Proof. The proof is analogous to the proof of [2, Lemma 2.6]. □

If φ is a BFs-homomorphism, then $\ker \varphi$ need not be a BF-ideal of X (see [7, Example 3.2] with identity mapping). Moreover, there is a BFs-homomorphism with kernel equal to $\{0\}$ but is not one-to-one (see [7, Example 3.13]).

Proposition 3.6. *Let* X *and* Y *be* BF_2*-semigroups and let* $\varphi : X \to Y$ *be a* BFs*-homomorphism. Then*

(a) φ *is one-to-one if and only if ker* $\varphi = \{0\}$*;*

(b) *ker* φ *is a* BF*-ideal of* X*.*

Proof. This follows from [7, Proposition 3.14]. □

Next, we construct quotient BF-semigroups via BF-ideals. Let I be a BF-ideal of X. Then I is a normal ideal of the BF-algebra $(X; *, 0)$. From [7], $(X/I; *', 0/I)$ is a BF-algebra (the quotient BF-algebra of X modulo I), where $*'$ is defined by $x/I *' y/I = (x * y)/I$. Note that for any $x \in X$, $x/I = \{y \in X : x \sim_I y\}$ is the congruence class containing x, where the relation $\sim_I$ is defined by $x \sim_I y$ if and only if $x * y \in I$. Now, we define $\cdot'$ on X/I by $x/I \cdot' y/I = (x \cdot y)/I$. The operation $\cdot'$ is well-defined. To see this, let $x/I, x'/I, y/I, y'/I \in X/I$. Suppose that $x/I = x'/I$ and $y/I = y'/I$. Then $x * x' \in I$ and $y * y' \in I$. Since I is a BF-ideal of X and $y * y' \in I$, it follows that $x \cdot (y * y') \in I$. By distributive property, $(x \cdot y) * (x \cdot y') = x \cdot (y * y') \in I$ and so $x \cdot y \sim_I x \cdot y'$. Similarly, since $x * x' \in I$, it follows that $x \cdot y' \sim_I x' \cdot y'$. By transitive property, $x \cdot y \sim_I x' \cdot y'$. Hence, $x/I \cdot' y/I = (x \cdot y)/I = (x' \cdot y')/I = x'/I \cdot' y'/I$. It is easy to see that $(X/I; *', \cdot', 0/I)$ is a BF-semigroup. This BF-semigroup is called the quotient BF-semigroup of X modulo I.

Theorem 3.7. *Let* X *and* Y *be* BF_2*-semigroups and let* $\varphi : X \to Y$ *be a* BFs*-homomorphism from* X *onto* Y*. Then* X*/ker* φ *is isomorphic to* Y*.*

Proof. The proof is analogous to the proof of [7, Theorem 3.16]. □

If I and J are BF-ideals of X, then $I \cap J$ is a BF-ideal of X. Since I is sub BF-semigroup of X, $I \cap J$ is a BF-ideal of I. Thus, $I/(I \cap J)$ is well-defined. If X satisfies (SI), then by Proposition 3.3 and [2, Lemma 3.3 (i)], $(IJ)/J$ is well-defined.

Theorem 3.8. *If* I *and* J *are* BF*-ideals of* X *such that* X *satisfies* (SI)*, then* $I/(I \cap J)$ *is isomorphic to* $(IJ)/J$*.*

Proof. The proof is analogous to the proof of [2, Theorem 3.4]. □

Lemma 3.9. *If* I *and* J *are* BF*-ideals of a* BF*-semigroup* X *such that* $I \subseteq J$*, then* J/I *is a* BF*-ideal of* X/I*.*

Proof. From [2, Lemma 3.5], J/I is a normal ideal of X/I. Let $j/I \in J/I$ and $x/I \in X/I$. Then $j \in J$ and $x \in X$. Since J is a BF-ideal, $j \cdot x, x \cdot j \in J$. Hence, $j/I \cdot x/I = (j \cdot x)/I \in J/I$ and $x/I \cdot j/I = (x \cdot j)/I \in J/I$. Therefore, J/I is a BF-ideal of X/I. □

Let I and J be BF-ideals of a BF_2-semigroup X such that $I \subseteq J$. Then by Lemma 3.9, $(X/I)/(J/I)$ is well-defined.

Theorem 3.10. *If* I *and* J *are* BF*-ideals of a* BF_2*-semigroup* X *such that* $I \subseteq J$*, then* $(X/I)/(J/I)$ *is isomorphic to* X/J*.*

Proof. The proof is analogous to the proof of [2, Theorem 3.6]. □

4. Fuzzy BF-semigroups

Definition 4.1. A fuzzy set μ of a BF-semigroup X is called a fuzzy sub BF-semigroup of X if it satisfies the following axioms for all $x, y \in X$:

(i) $\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$;

(ii) $\mu(x \cdot y) \geqslant \min\{\mu(x), \mu(y)\}$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with the following table of operations:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

Then $(X;*,\cdot,0)$ is a BF-semigroup. Now, μ is a fuzzy sub BF-semigroup of X, where $\mu(0) = 0.7 = \mu(a)$ and $\mu(b) = 0.4 = \mu(c)$.

Lemma 4.3. *Let μ be a fuzzy sub BF-semigroup of X. Then $\mu(0) \geqslant \mu(x)$ for all $x \in X$. Moreover, if μ is onto, then $\mu(0) = 1$.*

Proof. The first statement follows from ([1, Lemma 3.3]). Now, if μ is onto, then there exists an $x \in X$ such that $\mu(x) = 1$. Therefore, $1 = \mu(x) \leqslant \mu(0) \leqslant 1$, that is, $\mu(0) = 1$. □

Theorem 4.4. *Let μ be a fuzzy sub BF-semigroup of X. Then there exists a sequence $\langle x_n \rangle$ in X such that $\lim_{n\to\infty} \mu(x_n) = 1$ if and only if $\mu(0) = 1$.*

Proof. The first part of the theorem follows from ([1, Theorem 3.5]). Suppose that $\mu(0) = 1$. Consider the sequence $\langle x_n \rangle = \langle 0, 0, \cdots \rangle$ in X. Then clearly $\lim_{n\to\infty} \mu(x_n) = 1$. □

Proposition 4.5. *Let μ be a fuzzy sub BF-semigroup of X. Then the set $X_\mu = \{x \in X : \mu(x) = \mu(0)\}$ is a sub BF-semigroup of X.*

Proof. Let $x, y \in X_\mu$. Then $\mu(x) = \mu(0) = \mu(y)$. From ([1, Theorem 3.14]), $x * y \in X_\mu$. Now,

$$\mu(x \cdot y) \geqslant \min\{\mu(x), \mu(y)\} = \mu(0).$$

Applying Lemma 4.3, $\mu(x \cdot y) = \mu(0)$. Hence, $x \cdot y \in X_\mu$. Therefore, X_μ is a sub BF-semigroup of X. □

Lemma 4.6. *If μ is a fuzzy sub BF-semigroup of X, then*

(a) $\mu(0 * x) = \mu(x)$;

(b) $\mu(x * (0 * y)) \geqslant \min\{\mu(x), \mu(y)\}$;

(c) $\mu(x * y) = \mu(y * x)$.

Proof.

(a): By Lemma 4.3, $\mu(0 * x) \geqslant \min\{\mu(0), \mu(x)\} = \mu(x)$. By Proposition 2.7 (a) and Lemma 4.3, $\mu(x) = \mu(0 * (0 * x)) \geqslant \min\{\mu(0), \mu(0 * x)\} = \mu(0 * x)$. Therefore, $\mu(0 * x) = \mu(x)$.

(b): By (a), $\mu(x * (0 * y)) \geqslant \min\{\mu(x), \mu(0 * y)\} = \min\{\mu(x), \mu(y)\}$.

(c): By (BF) and (a), $\mu(x * y) = \mu(0 * (y * x)) = \mu(y * x)$. □

Theorem 4.7. *Suppose that μ is a fuzzy sub BF-semigroup of a BF_1-semigroup X. If $\mu(x * y) = \mu(0)$, then $\mu(x) = \mu(y)$.*

Proof. Let $x, y \in X$ and $\mu(x * y) = \mu(0)$. By (BG), $x = (x * y) * (0 * y)$. Thus, by Lemma 4.6 (a) and Lemma 4.3, we have $\mu(x) = \mu((x * y) * (0 * y)) \geqslant \min\{\mu(x * y), \mu(0 * y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. By Lemma 4.6 (c), $\mu(0) = \mu(x * y) = \mu(y * x)$. Applying similar argument as above, we obtain $\mu(y) \geqslant \mu(x)$. Therefore, $\mu(x) = \mu(y)$. □

Theorem 4.7 does not hold for BF-semigroup as shown in the following example.

Example 4.8. Consider the BF-semigroup $(X;*,\cdot,0)$ in Example 2.5. Now, μ is a fuzzy sub BF-semigroup of X, where $\mu(0) = 0.5 = \mu(a)$ and $\mu(b) = 0.2$. Moreover, $\mu(a * b) = \mu(0)$ but $\mu(a) = 0.5 \neq 0.2 = \mu(b)$.

The upper level set $U(\mu;t)$ is the set $U(\mu;t) = \{x \in X : \mu(x) \geqslant t\}$, where $0 \leqslant t \leqslant 1$.

Theorem 4.9. *Suppose that $U(\mu;t)$ is nonempty. Then μ is a fuzzy sub BF-semigroup of X if and only if $U(\mu;t)$ is a sub BF-semigroup of X.*

Proof. The "only if" statement is clear. Suppose that $U(\mu;t)$ is a sub BF-semigroup of X. If there are $x_0, y_0 \in X$ such that $\mu(x_0 * y_0) < \min\{\mu(x_0), \mu(y_0)\}$, then taking $t_0 = \frac{1}{2}(\mu(x_0 * y_0) + \min\{\mu(x_0), \mu(y_0)\})$, $\mu(x_0 * y_0) < t_0 < \min\{\mu(x_0), \mu(y_0)\}$. Hence, $x_0, y_0 \in U(\mu;t_0)$, but $x_0 * y_0 \notin U(\mu;t_0)$, a contradiction. This means that $\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Similarly, $\mu(x \cdot y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Therefore, μ is a fuzzy sub BF-semigroup of X. □

Theorem 4.10. *Let N be a nonempty subset of a BF-semigroup X. Suppose that μ_N is a fuzzy set in X defined by*

$$\mu_N(x) = \begin{cases} \alpha, & x \in N, \\ \beta, & \textit{otherwise}, \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha > \beta$. Then μ_N is a fuzzy sub BF-semigroup of X if and only if N is a sub BF-semigroup of X. Moreover, in this case, $X_{\mu_N} = N$.

Proof. Straightforward. □

Corollary 4.11. *A subset N of X is a sub BF-semigroup X if and only if χ_N is a fuzzy sub BF-semigroup of X.*

Proof. Take $\alpha = 1$ and $\beta = 0$ in Theorem 4.10. □

Corollary 4.12. *If N is a sub BF-semigroup of X, then there exists a fuzzy sub BF-semigroup μ of X such that $U(\mu;t) = N$ for any $0 < t < 1$.*

Let X and Y be two nonempty sets and $f : X \to Y$ a mapping. Let μ a fuzzy set of Y. The preimage of μ under f, denoted by μ^f, is the fuzzy set of X defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$, that is, $\mu^f = \mu \circ f$. Let μ be a fuzzy set of X. The mapping $f(\mu) : Y \to [0,1]$ defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)}\{\mu(x)\}, & f^{-1}(y) \neq \varnothing, \\ 0, & f^{-1}(y) = \varnothing, \end{cases}$$

is called the image of μ under f, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Let $f : X \to Y$ be a BFs-homomorphism. If μ is a fuzzy sub BF-semigroup, then $f(\mu)(0) = \mu(0)$ and $\mu^f(0) = \mu(0)$. Let N be a nonempty subset of a BF-semigroup X. A fuzzy set μ is said to have the supremum property if there exists $n_0 \in N$ such that $\mu(n_0) = \sup\limits_{n \in N}\{\mu(n)\}$. If X is a finite set, then $\mu : X \to [0,1]$ has the supremum property. The following theorem shows that the homomorphic image $f(\mu)$ of a fuzzy sub BF-semigroup μ is a fuzzy sub BF-semigroup of the image of f.

Theorem 4.13. *Let $f : X \to Y$ be a BFs-epimorphism. If μ is a fuzzy sub BF-semigroup of X with the supremum property, then $f(\mu)$ is a fuzzy sub BF-semigroup of Y.*

Proof. Let μ be a fuzzy sub BF-semigroup of X with the supremum property. Suppose that $y_1, y_2 \in Y$. Since f is onto, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are nonempty subsets of X. Since μ has the supremum property, there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that $\mu(x_1) = \sup\limits_{x \in f^{-1}(y_1)} \mu(x)$ and $\mu(x_2) = \sup\limits_{x \in f^{-1}(y_2)} \mu(x)$. Since $f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2, x_1 * x_2 \in f^{-1}(y_1 * y_2)$. Hence,

$$\begin{aligned} f(\mu)(y_1 * y_2) &= \sup_{x \in f^{-1}(y_1 * y_2)} \mu(x) \\ &\geqslant \mu(x_1 * x_2) \end{aligned}$$

$$\begin{aligned}
&\geqslant \min\{\mu(x_1), \mu(x_2)\} \\
&= \min\left\{ \sup_{x \in f^{-1}(y_1)} \mu(x), \sup_{x \in f^{-1}(y_2)} \mu(x) \right\} \\
&= \min\{f(\mu)(y_1), f(\mu)(y_2)\}.
\end{aligned}$$

Similarly, $f(\mu)(y_1 \cdot y_2) \geqslant \min\{f(\mu)(y_1), f(\mu)(y_2)\}$. Therefore, $f(\mu)$ is a fuzzy sub BF-semigroup of Y. □

If X is finite, then μ has the supremum property.

Corollary 4.14. *If* X *is finite,* $f : X \to Y$ *is a* BFs*-epimorphism and* μ *is a fuzzy sub* BF*-semigroup of* X*, then* $f(\mu)$ *is a fuzzy sub* BF*-semigroup of* Y*.*

Let $f : X \to Y$ be a BFs-homomorphism. The following theorem shows that the homomorphic preimage μ^f of a fuzzy sub BF-semigroup μ of Y is a fuzzy sub BF-semigroup of X.

Theorem 4.15. *Let* $f : X \to Y$ *be a* BFs*-homomorphism. If* μ *is a fuzzy sub* BF*-semigroup of* Y*, then* μ^f *is a fuzzy sub* BF*-semigroup of* X*. If* f *is onto, then the converse holds.*

Proof. Let μ is a fuzzy sub BF-semigroup of Y. Suppose that $x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Then

$$\begin{aligned}
\mu^f(x * y) &= \mu(f(x * y)) \\
&= \mu(f(x) * f(y)) \\
&= \mu(a * b) \\
&\geqslant \min\{\mu(a), \mu(b)\} \\
&= \min\{\mu(f(x)), \mu(f(y))\} \\
&= \min\{\mu^f(x), \mu^f(y)\}.
\end{aligned}$$

Similarly, $\mu^f(x \cdot y) \geqslant \min\{\mu^f(x), \mu^f(y)\}$. Hence, μ^f is a fuzzy sub BF-semigroup of X. For the converse, suppose that f is onto and μ^f is a fuzzy sub BF-semigroup of X. Let $a, b \in Y$. Since f is onto, $a = f(x)$ and $b = f(y)$ for some $x, y \in X$. Hence,

$$\begin{aligned}
\mu(a * b) &= \mu(f(x) * f(y)) \\
&= \mu(f(x * y)) \\
&= \mu^f(x * y) \\
&\geqslant \min\{\mu^f(x), \mu^f(y)\} \\
&= \min\{\mu(f(x)), \mu(f(y))\} \\
&= \min\{\mu(a), \mu(b)\}.
\end{aligned}$$

Similarly, $\mu(a \cdot b) \geqslant \min\{\mu(a), \mu(b)\}$. Therefore, μ is a fuzzy sub BF-semigroup of Y. □

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