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Taylor's expansion for fractional matrix functions: theory and applications

Ahmad El-Ajou

Department of Mathematics, Faculty of Science, Al Balqa Applied University, Salt 19117, Jordan.

Department of Mathematics, Faculty of Science, Taibah University, Madina, KSA.

Abstract

In this paper, several aims and tasks have been accomplished that can be summarized in the following points. Firstly, we recover some nice results related to the convergence and radii of convergence for the matrix fractional power series formula. Secondly, the Frobenius norm approximations for the matrix fractional derivatives in Caputo sense and fractional integrals in Riemann-Liouville sense are presented. Thirdly, we present the general exact and numerical solutions of four important and interesting matrix fractional differential equations and a new computational technique is also applied for getting the general solutions of the non-linear case in Caputo sense. Finally, some illustrated examples and special cases are also given and considered to show our new approach.

Keywords: Caputo fractional derivative, Riemann-Liouville integral, matrix Mittag-Leffler functions, matrix fractional differential equations.

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1. Introduction

The use of matrix differential equations (MDEs) are appeared in many applications and real life problems such as in matrix theory, control theory, physics phenomena, engineering problems, decay-growth problems, mortgage problems, modeling of signal processing, modeling of best predictions, simulation-reduction problems, matrix time-varying descriptor systems, computing system and large scale benchmark problems, state-space problem and constrained least-squares problems [1–4, 10, 19, 23, 24, 27, 30, 39]. The general form of the first order non-homogeneous linear MDE with appropriate orders of matrices is formulated by [1, 19, 23, 24, 30]:

$$X'(t) = \sum_{i=1}^{s} A_i X(t) B_i + G(t), \; X(t_0) = E. \tag{1.1}$$

In addition, many interesting and important cases can be formulated from Eq. (1.1) and one of the

Email address: ajou44@bau.edu.jo, ajou42@yahoo.com (Ahmad El-Ajou)

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simplest and well-known cases from Eq. (1.1) is the following MDE [1, 3, 19, 24, 30]:

$$X'(t) = AX(t) + G(t),\ X(0) = B, \tag{1.2}$$

and the general exact solution of Eq. (1.2) is presented by:

$$X(t) = e^{At}B + \int_0^t e^{A(t-s)}G(s)\,ds,$$

where $e^{At} \in M_m$ is the matrix exponential function (M_m and $M_{m,n}$ stand for the set of all matrices of order $m \times m$ and $m \times n$, respectively). Due to the introduction of fractional calculus, scientists nowadays are paying attentions to the topic of fractional operators and its applications, since they found this topic is more fitting for extending and generalizing many classical differential equations (systems) and also many phenomena in physics, engineering and problems in control theory can be modeled mathematically by matrix fractional differential equations (MFDEs) and systems such as fractional L system, fractional Chen systems, fractional Lorenz system, descriptor and dynamic systems, nonlinear oscillation of earthquake, population of fractional oscillators and electromagnetic wave [4, 7–9, 11–13, 20, 22, 25, 28, 29, 31, 32, 34–38, 40].

There are many definitions of fractional integral and fractional derivative [9, 16, 17, 21, 31, 34, 35, 37], the most important of which are the definition of Riemann-Liouville fractional integral and Caputo fractional derivative, which are defined as follows.

- The Riemann-Liouville fractional integral of order $\alpha \geqslant 0$ of $\varphi(t)$ is defined by:

$$J_s^\alpha \varphi(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^t (t-x)^{\alpha-1}\varphi(x)\,dt, & t > x > s \geqslant 0,\ \alpha > 0, \\ \varphi(t), & \alpha = 0. \end{cases}$$

- The Caputo fractional derivative of order $\alpha > 0$ of φ is defined by:

$$D_s^\alpha \varphi(t) = \begin{cases} J_s^{n-\alpha}\varphi^{(n)}(t), & t > s \geqslant 0,\ n-1 < \alpha < n, \\ \frac{d^n \varphi(t)}{dt^n}, & \alpha = n. \end{cases}$$

Many of the properties of the previous definitions exist in the references [6, 9, 14, 15, 18, 31, 33–35, 37]. The most important of these properties that we will need during this work can be summarized in the following lemma.

Lemma 1.1. *For $\varphi(t)$, $t \geqslant s$, $\alpha, \beta \geqslant 0$, $C \in \mathbb{R}$, and $\gamma \geqslant -1$, we have:*

1. $J_s^\alpha J_s^\beta \varphi(t) = J_s^{\alpha+\beta}\varphi(t) = J_s^\beta J_s^\alpha \varphi(t)$,
2. $J_s^\alpha C = \frac{C}{\Gamma(\alpha+1)}(t-s)^\alpha$,
3. $J_s^\alpha (t-s)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-s)^{\alpha+\gamma}$,
4. $D_s^\alpha C = 0$,
5. $D_s^\alpha (t-s)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(t-s)^{\gamma-\alpha}$,
6. $D_s^\alpha J_s^\alpha \varphi(t) = \varphi(t)$,
7. $J_s^\alpha D_s^\alpha \varphi(t) = \varphi(t) - \sum_{j=0}^{n-1} \varphi(s^+)\frac{(t-s)^j}{j!}, n-1 < \alpha \leqslant n$.

Recently, Al-Zhour [2] used the Kronecer product method for solving the following non-homogeneous MFDEs in Caputo sense:

$$D^\alpha X(t) = A\,X(t) + G(t),\ X(t_0) = B,\ 0 < \alpha \leqslant 1,$$

and it has the following solution:

$$X(t) = E_\alpha(At^\alpha)B + \int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)G(s)\,ds,$$

where $E_\alpha(At^\alpha)$, $A \in M_m$ is the one-parameter Matrix Mittag-Leffler function (MM-LF) and given by the following formula:

$$E_{\alpha}\left(At^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{A^{k}t^{\alpha k}}{\Gamma\left(k\alpha+1\right)}, \tag{1.3}$$

whereas the two-parameter MM-LF is given by the following formula [2, 6–8, 32]:

$$E_{\alpha,\beta}\left(At^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{A^{k}t^{\alpha k}}{\Gamma\left(k\alpha+\beta\right)}.$$

In a case, if we set $\alpha=1$ in Eq. (1.3), then we have:

$$E_{1}\left(At\right)=\sum_{k=0}^{\infty}\frac{A^{k}t^{k}}{\Gamma\left(k+1\right)}=e^{At}.$$

Very recently, Al-Zhour [4], Al-Zuhiri et.al. [5], and Kilicman and Ahmood [26] used, respectively, the Kronecker product method, Hadamard product method and fractional Laplace transform method for solving very restricted matrix fractional differential equations.

In this paper, we find the exact and numerical solutions of some interesting and attractive linear and non-linear MFDEs in Caputo sense by using matrix fractional power series (MFPS) method and establish some new results related to the convergence of this method. Moreover, the Frobenius norm (which is defined below) approximation for the matrix fractional derivatives in Caputo sense and fractional integrals in Riemann-Liouville sense of a given function are presented with some illustrated numerical examples to show our new approach. Note that the results shown in this paper are established in Caputo fractional derivative D_s^{α}, since it has suitable for modeling MFDEs and FDEs.

The outcome of this paper is organized as follows. In the next section, we extend the generalized fractional Taylor's series to the matrix form and establish some nice results related to the convergence of the extension formula. In Section 3, we give two applications for computing the Frobenius norm approximation for the matrix fractional derivatives and fractional integrals of a given matrix function. Section 4 deals with four applications related to the linear and nonlinear MFDEs with some illustrated numerical computations in order to show the capability and simplicity of our new approach on the field of matrix theory. The conclusion of the present work is given as in the last section.

2. Matrix fractional power series (MFPS)

This section extends the fractional power series that discussed in the references [14, 16–18] to the matrix case, as general, and establish some new nice results related to the convergent and radii of convergence for MFPS in Caputo sense. In addition, we study some important definitions and theorems which are very useful to investigate our results in the approximation of the matrix fractional derivative in Caputo sense and fractional integral in Riemann-Liouville sense and also in the solutions of some MFDEs as in Sections 3 and 4.

Definition 2.1. Let $A_k \in M_{m,n}$ $(k=0,1,2,\ldots)$. Then the sequence $\{A_k\}_{k=0}^{\infty}$ converges to a matrix $A \in M_{m,n}$ with respect to a matrix norm $\|\cdot\|$ on $M_{m,n}$ if and only if $\lim_{k\to\infty}\|A_k-A\|=0$. If $\{A_k\}$ converges to A with respect to $\|\cdot\|$, we write $\lim_{k\to\infty}A_k=A$.

Definition 2.2. Given a matrix series $\sum_{j=0}^{\infty}A_j$ such that $A_j\in M_{m,n}$ and S_k denotes its kth partial sum, $S_k=\sum_{j=0}^{k}A_j$. If $\{S_k\}$ is convergent and $\lim_{k\to\infty}S_k=S$ is existing, then $\sum_{j=0}^{\infty}A_j$ is convergent to the matrix S. Otherwise, the series is divergent.

Theorem 2.3. *Let $A_k\in M_{m,n}$ $(k=0,1,2,\ldots)$ and the matrix series $\sum_{k=0}^{\infty}A_k$ be convergent, then $\lim_{k\to\infty}A_k=0$.*

Proof. Since $\sum_{k=0}^{\infty} A_k$ is convergent, then $\{S_k\}$ is convergent and also since $\lim_{k\to\infty} S_k = S$, then $\lim_{k\to\infty} S_{k-1} = S$. Therefore, $\lim_{k\to\infty} A_k = \lim_{k\to\infty}(S_k - S_{k-1}) = 0$. □

Definition 2.4. Let $A_k \in M_{m,n}$ $(k = 0, 1, 2, \dots)$ be a sequence of constant matrices, $0 \leqslant r-1 < \alpha \leqslant r$ and the variable $t \geqslant t_0$, then the following series

$$\sum_{k=0}^{\infty} A_k (t-t_0)^{k\alpha}, \tag{2.1}$$

is called the MFPS about t_0 and A_k $(k = 0, 1, 2, \dots)$ are called the coefficients of the MFPS.

Remark 2.5. The MFPS as in Eq. (2.1) always converges when $t = t_0$.

Remark 2.6. We shall treat the MFPS as in Eq. (2.1) about $t_0 = 0$ since the translation $t' = t - t_0$ reduces the MFPS about t_0 to the case about 0.

Theorem 2.7. *Let $A_k \in M_{m,n}$ $(k = 0, 1, 2, \dots)$. Then for $t \geqslant 0$, we have:*

(i) *if the MFPS $\sum_{k=0}^{\infty} A_k t^{k\alpha}$ converges when $t = \lambda > 0$ with respect to a matrix norm $\|\cdot\|$, then it also converges when $0 \leqslant t < \lambda$;*
(ii) *if the MFPS $\sum_{k=0}^{\infty} A_k t^{k\alpha}$ diverges when $t = \rho > 0$, then it also diverges when $t > \rho$.*

Proof.
(i). Assume that $\sum A_k \lambda^{k\alpha}$ converges, then $\lim_{k\to\infty} A_k \lambda^{k\alpha} = 0$ (by Theorem 2.3). That is $\exists$ a positive number N and a matrix norm $\|\cdot\|$ on $M_{m,n}$ such that $\|A_k \lambda^{k\alpha}\| < \varepsilon = 1$ when $k \geqslant N$. Thus, for $k \geqslant N$, we have:

$$\|A_k \lambda^{k\alpha}\| = \|\frac{A_k \lambda^{k\alpha} t^{k\alpha}}{\lambda^{k\alpha}}\| = \|A_k \lambda^{k\alpha}\| |\frac{t}{\lambda}|^{k\alpha} < |\frac{t}{\lambda}|^{k\alpha}.$$

Now, if $0 \leqslant t < \lambda$, then $|\frac{t}{\lambda}|^{\alpha} < 1$ and $\sum |\frac{t}{\lambda}|^{k\alpha}$ is convergent and so $\sum_{k=0}^{\infty} \|A_k t^{k\alpha}\|$ is a convergent series (by comparison test) which implies that $\sum A_k t^{k\alpha}$ is convergent.

(ii). Part (ii) follows by applying Part (i) of Theorem 2.7. □

Theorem 2.8. *The MFPS $\sum_{k=0}^{\infty} A_k t^{k\alpha}$ has the following three cases:*

(i) *converges at $t = 0$;*
(ii) *converges for each $t \geqslant 0$;*
(iii) *converges when $0 \leqslant t < R$ and diverges when $t > R$, where R is a positive integer number and called the "radius of convergence" of the MFPS.*

Proof. Follows by the same techniques of the proof as in [14, Theorem 3.2]. □

Remark 2.9. It is clear that $R = 0$ in case (i) and $R = \infty$ in case (ii) of Theorem 2.8.

Theorem 2.10. *The matrix classical power series (MCPS) $\sum_{k=0}^{\infty} A_k t^k$, $-\infty < t < \infty$ has a radius of convergence R if and only if the MFPS $\sum_{k=0}^{\infty} A_k t^{k\alpha}, t \geqslant 0$ has radius of convergence $R^{1/\alpha}$.*

Proof. Straightforward by changing the variable $t = x^{\alpha}$, $x \geqslant 0$ and conversely by changing the variable $t = x^{1/\alpha}$, $x \geqslant 0$. □

Theorem 2.11. *If the MFPS $\sum_{k=0}^{\infty} A_k t^{k\alpha}, t \geqslant 0$ has radius of convergence $R > 0$ and $X(t) = \sum_{k=0}^{\infty} A_k t^{k\alpha} \in M_{m,n}$, $0 \leqslant t < R$, $0 \leqslant r-1 < \alpha \leqslant r$, then we have*

$$D_0^{\alpha} X(t) = \sum_{k=1}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha}, \tag{2.2}$$

$$J_0^{\alpha} X(t) = \sum_{k=0}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)} t^{(k+1)\alpha}.$$

Proof. Define $Y(x)=\sum_{k=0}^{\infty}A_k x^k$ for $0\leqslant x<R^{\alpha}$, then for $0\leqslant \tau<x<R^{\alpha}$, we have:

$$\begin{aligned} D_0^{\alpha}Y(x) &= \frac{1}{\Gamma(r-\alpha)}\int_0^x (x-\tau)^{r-\alpha-1}Y^{(r)}(\tau)\,d\tau \\ &= \frac{1}{\Gamma(r-\alpha)}\int_0^x (x-\tau)^{r-\alpha-1}\left(\frac{d^r}{d\tau^r}\sum_{k=0}^{\infty}A_k\tau^k\right)d\tau \\ &= \sum_{k=0}^{\infty}A_k\left(\frac{1}{\Gamma(r-\alpha)}\int_0^x (x-\tau)^{r-\alpha-1}\left(\frac{d^r}{d\tau^r}\tau^k\right)d\tau\right)=\sum_{k=0}^{\infty}A_k D_0^{\alpha}\left(x^k\right). \end{aligned} \tag{2.3}$$

Now by setting $x=t^{\alpha}, t\geqslant 0$ in Eq. (2.3), we get:

$$\begin{aligned} D_0^{\alpha}X(t)=D_0^{\alpha}Y(t^{\alpha}) &= \sum_{k=0}^{\infty}A_k D_0^{\alpha}\left(t^{k\alpha}\right), 0\leqslant t^{\alpha}<R^{\alpha} \\ &= \sum_{k=1}^{\infty}A_k\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha}, 0\leqslant t<R. \end{aligned}$$

Likewise, we find that:

$$\begin{aligned} J_0^{\alpha}Y(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}Y(\tau)\,d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\left(\sum_{k=0}^{\infty}A_k\tau^k\right)d\tau \\ &= \sum_{k=0}^{\infty}A_k\left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\left(\tau^k\right)d\tau\right)=\sum_{k=0}^{\infty}A_k J_0^{\alpha}\left(x^k\right). \end{aligned} \tag{2.4}$$

Substitute $x=t^{\alpha}, t\geqslant 0$ in Eq. (2.4), then we get

$$\begin{aligned} J_0^{\alpha}X(t)=J_0^{\alpha}Y(t^{\alpha}) &= \sum_{k=0}^{\infty}A_k J_0^{\alpha}\left(t^{k\alpha}\right), 0\leqslant t^{\alpha}<R^{\alpha} \\ &= \sum_{k=0}^{\infty}A_k\frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}t^{(k+1)\alpha}, 0\leqslant t<R, \end{aligned}$$

and so the proof ends. □

Theorem 2.12. *Assume that $X(t)=\left[x_{ij}(t)\right]\in M_{m,n}$ has a MFPS representation at t_0 of the form*

$$X(t)=\sum_{k=0}^{\infty}A_k(t-t_0)^{k\alpha}, 0\leqslant r-1<\alpha\leqslant r, t_0\leqslant t<t_0+R. \tag{2.5}$$

Then

(i) $X(t)$ *is analytic matrix function on* (t_0,t_0+R);
(ii) *if* $X(t)$ *and* $D_{t_0}^{k\alpha}X(t)\in C[t_0,t_0+R)$, $k=0,1,2,\ldots$, *then*

$$A_k=\frac{D_{t_0}^{k\alpha}X(t_0)}{\Gamma(k\alpha+1)}, \tag{2.6}$$

where $D_{t_0}^{k\alpha}=D_{t_0}^{\alpha}\cdot D_{t_0}^{\alpha}\cdots D_{t_0}^{\alpha}$ (k *-times). That is*

$$X(t)=\sum_{k=0}^{\infty}\frac{D_{t_0}^{k\alpha}X(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}, 0\leqslant r-1<\alpha\leqslant r, t_0\leqslant t<t_0+R, \tag{2.7}$$

which is called the "Matrix fractional Taylor's series (MFTS)" about t_0.

As a special case, if $\alpha = 1$, then we obtain the so-called "Matrix classical Taylor's series (MCTS)" about t_0.

Proof.
(i). Since $Y(t) = \sum_{k=0}^{\infty} A_k t^k$ is analytic matrix function on $|t| < R^{\alpha}$ and $g(t) = (t-t_0)^{\alpha}$ is analytic function on $t_0 < t < t_0 + R$, $0 \leqslant r-1 < \alpha \leqslant r$, and so $(Y \circ g)(t) = X(t)$ is analytic matrix function on $(t_0, t_0 + R)$.
(ii). Set $t = t_0$ in Eq. (2.5), then each term after the first one vanishes and thus we get: $A_0 = X(t_0)$. Now by using part (5) 0f Lemma 1.1, then for $t_0 \leqslant t < t_0 + R$, we have

$$D_{t_0}^{\alpha} X(t) = \Gamma(\alpha+1) A_1 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} (t-t_0)^{\alpha} A_2 + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} (t-t_0)^{2\alpha} A_3 + \cdots . \tag{2.8}$$

By substituting $t = t_0$ in Eq. (2.8), we obtain

$$A_1 = \frac{D_{t_0}^{\alpha} X(t_0)}{\Gamma(\alpha+1)}.$$

Again, by applying Eq. (2.2) on Eq. (2.8), then for $t_0 \leqslant t < t_0 + R$, we have

$$D_0^{2\alpha} X(t) = \Gamma(2\alpha+1) A_2 + \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)} (t-t_0)^{\alpha} A_3 + \frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)} (t-t_0)^{2\alpha} A_4 + \cdots . \tag{2.9}$$

Hence, if we put $t = t_0$ in Eq. (2.9), then we have

$$A_2 = \frac{D_{t_0}^{2\alpha} X(t_0)}{\Gamma(2\alpha+1)}.$$

By the same way, if we apply the operator $D_{t_0}^{\alpha}(\cdot)$ k-times on the Eq. (2.5) and substitute $t = t_0$ in the resulting equation for each time, then it's easy to get the result as in Eq. (2.6) and by substituting the formula as in Eq. (2.6) into Eq. (2.5) we get the result as in Eq. (2.7). □

Theorem 2.13. *Assume that $X(t) = [x_{ij}(t)] \in M_{m,n}$ has a MFTS representation at t_0 as in Eq. (2.5), $D_{t_0}^{k\alpha} X(t) \in C(t_0, t_0 + R)$, $k = 0, 1, 2, \ldots$ and $Y(t) = X\left((t-t_0)^{1/\alpha} + t_0\right)$, $t_0 \leqslant t < t_0 + R^{\alpha}$. Then*

$$D_{t_0}^{k\alpha} X(t_0) = \frac{\Gamma(k\alpha+1)}{k!} Y^{(k)}(t_0). \tag{2.10}$$

Proof. Set $t = (x-t_0)^{1/\alpha} + t_0$, $t_0 \leqslant x < t_0 + R^{\alpha}$ in Eq. (2.7), then we get

$$Y(x) = X\left((x-t_0)^{1/\alpha} + t_0\right) = \sum_{k=0}^{\infty} \frac{D_{t_0}^{k\alpha} X(t_0)}{\Gamma(k\alpha+1)} (x-t_0)^k, \ t_0 \leqslant x < t_0 + R^{\alpha}. \tag{2.11}$$

Also, the MCTS of $Y(x)$ about t_0 can be represented by

$$Y(x) = \sum_{n=0}^{\infty} \frac{Y^{(n)}(t_0)}{n!} (x-t_0)^n, \ t_0 \leqslant x < t_0 + R^{\alpha}. \tag{2.12}$$

Now by comparing the corresponding coefficients in Eq. (2.11) and (2.12), then we obtain the result as in Eq. (2.10). □

Remark 2.14. The kth-partial sum of the MFTS of $X(t) \in M_{m,n}$ is

$$T_k(t) = \sum_{j=0}^{k} \frac{D_{t_0}^{j\alpha} X(t_0)}{\Gamma(j\alpha+1)} (t-t_0)^{j\alpha},$$

and $R_k(t) = X(t) - T_k(t)$ is called the "Remainder of the MFTS".

Theorem 2.15. *Let* $X(t) \in M_{m,n}$ *such that* $X(t)$ *and* $D_{t_0}^{j\alpha}X(t) \in C(t_0, t_0+R)$, $j = 0,1,2,\ldots,k+1$ *and* $0 < \alpha \leqslant 1$. *Then the matrix function* $X(t)$ *can be represented by*

$$X(t) = \sum_{j=0}^{k} \frac{\left(D_{t_0}^{j\alpha}X\right)(t_0)}{\Gamma(j\alpha+1)}(t-t_0)^{j\alpha} + J_{t_0}^{(k+1)\alpha}D_{t_0}^{(k+1)\alpha}X(t), \quad t_0 \leqslant t \leqslant t_0+R. \tag{2.13}$$

Proof. By applying Lemma 1.1, we get

$$\begin{aligned} J_{t_0}^{(k+1)\alpha}D_{t_0}^{(k+1)\alpha}X(t) &= J_{t_0}^{k\alpha}\left(\left(J_{t_0}^{\alpha}D_{t_0}^{\alpha}\right)D_{t_0}^{k\alpha}X(t)\right) \\ &= J_{t_0}^{k\alpha}\left(\left(J_{t_0}D_{t_0}\right)D_{t_0}^{k\alpha}X(t)\right) \\ &= J_{t_0}^{k\alpha}\left(D_{t_0}^{k\alpha}X(t) - D_{t_0}^{k\alpha}X(t_0)\right) \\ &= J_{t_0}^{k\alpha}D_{t_0}^{k\alpha}X(t) - J_{t_0}^{k\alpha}\left(D_{t_0}^{k\alpha}X(t_0)\right) \\ &= J_{t_0}^{(k-1)\alpha}\left(\left(J_{t_0}D_{t_0}\right)D_{t_0}^{(k-1)\alpha}X(t)\right) - \left(\frac{\left(D_{t_0}^{k\alpha}X\right)(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\right) \\ &= J_{t_0}^{(k-1)\alpha}\left(D_{t_0}^{(k-1)\alpha}X(t) - D_{t_0}^{(k-1)\alpha}X(t_0)\right) - \left(\frac{\left(D_{t_0}^{k\alpha}X\right)(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\right) \\ &= J_{t_0}^{(k-2)\alpha}\left(\left(J_{t_0}D_{t_0}\right)D_{t_0}^{(k-2)\alpha}X(t)\right) - \left(\frac{\left(D_{t_0}^{(k-1)\alpha}X\right)(t_0)}{\Gamma((k-1)\alpha+1)}(t-t_0)^{(k-1)\alpha}\right) \\ &\quad - \left(\frac{\left(D_{t_0}^{k\alpha}X\right)(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\right). \end{aligned} \tag{2.14}$$

Now, we can get the result as in Eq. (2.13) after repeating previous procedure of calculations in Eq. (2.14) k-times. □

Theorem 2.16. *Let* $X(t) \in M_{m,n}$ *and for* $0 < \alpha \leqslant 1$,

$$\|D_{t_0}^{(k+1)\alpha}X(t)\| \leqslant M, \quad t_0 \leqslant t \leqslant \lambda.$$

Then the reminder $R_k(t) \in M_{m,n}$ *of the MFTS satisfies the inequality:*

$$\|R_k(t)\| \leqslant \frac{M}{\Gamma((k+1)\alpha+1)}(t-t_0)^{(k+1)\alpha}, \quad t_0 \leqslant t \leqslant \lambda.$$

Proof. Suppose that $D_{t_0}^{j\alpha}X(t)$ exist for $j = 0,1,2,\ldots,k+1$ and since

$$R_k(t) = X(t) - \sum_{j=0}^{k} \frac{D_{t_0}^{j\alpha}X(t_0)}{\Gamma(j\alpha+1)}(t-t_0)^{j\alpha},$$

we get by applying Theorem 2.15

$$R_k(t) = J_{t_0}^{(k+1)\alpha}D_{t_0}^{(k+1)\alpha}X(t).$$

Thus for $t_0 \leqslant \tau \leqslant t \leqslant \lambda$, we have

$$\begin{aligned} \|R_k(t)\| &= \|J_{t_0}^{(k+1)\alpha}D_{t_0}^{(k+1)\alpha}X(t)\| \\ &= \left\|\frac{1}{\Gamma((k+1)\alpha)}\int_{t_0}^{t}(t-\tau)^{(k+1)\alpha-1}D_{t_0}^{(k+1)\alpha}X(\tau)\,d\tau\right\| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\Gamma((k+1)\alpha)} \int_{t_0}^{t} |(t-\tau)^{(k+1)\alpha-1}| \| D_{t_0}^{(k+1)\alpha} X(\tau) \| d\tau \\
&\leqslant \frac{1}{\Gamma((k+1)\alpha)} \int_{t_0}^{t} |(t-\tau)^{(k+1)\alpha-1}| M d\tau \\
&= \frac{M}{\Gamma((k+1)\alpha)} \int_{t_0}^{t} (t-\tau)^{(k+1)\alpha-1} d\tau \\
&= \frac{M}{\Gamma((k+1)\alpha+1)} (t-t_0)^{(k+1)\alpha}.
\end{aligned}
$$

□

3. Frobenius norm approximation of matrix fractional derivatives and integrals

In this section, we present the Frobenius norm approximation for the matrix fractional derivatives and integrals of the matrix function $X(t) \in M_{m,n}$ at a given point based on our results as obtained in Section 2 (Theorems 2.11, 2.12, 2.13 and MFTS method). Here we use Mathematica software packages for getting the numerical computations.

Prior anything, we recall the definition of the Frobenius norm contained in the following formula

$$
\|X(t)\|_F = \sqrt{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij}(t)|^2 \right)}, \; X(t) = \left[x_{ij}(t)\right] \in M_{m,n}.
$$

Problem 3.1. Given the following matrix function:

$$
X(t) = \frac{I_4}{1-t^\alpha} \in M_{4\times 4}, \quad \alpha > 0, t \geqslant 0,
$$

where I_4 is an identity matrix of order 4.

The MFTS of $X(t)$ about $t = 0$ is written by

$$
X(t) = \sum_{k=0}^{\infty} \frac{D_0^{k\alpha} X(0)}{\Gamma(k\alpha+1)} t^{k\alpha}, \; \alpha > 0, t \geqslant 0,
$$

which is called the "Matrix fractional Maclaurin's series (MFMS)" of $X(t)$. According to Theorem 2.13, we have

$$
D_0^{k\alpha} X(0) = \frac{\Gamma(k\alpha+1)}{k!} Y^{(k)}(0), \; \alpha > 0, t \geqslant 0,
$$

where

$$
Y(t) = X\left(t^{1/\alpha}\right) = \frac{I_4}{1-t}, \; Y^{(k)}(0) = I_4 k!.
$$

That is, the MFMS of $X(t)$ can be represented as follows

$$
X(t) = \sum_{k=0}^{\infty} I_4 t^{k\alpha}, \; \alpha > 0, t \geqslant 0,
$$

which is a geometric MFPS with ratio t^α and converges for each $0 \leqslant t^\alpha < 1$, and so for each $0 \leqslant t < 1$.

Now according to Theorem 2.11, we can approximate the matrix fractional derivative $\|D_0^\alpha X(t)\|_F$ and the matrix fractional integral $\|J_0^\alpha X(t)\|_F$ of a matrix function $X(t)$ on $0 \leqslant t < 1$, beginningwith $\|D_0^\alpha X(t)\|_F$

which can be bounded by the m^{th}-partial sum of its expansion by

$$\begin{aligned}\|D_0^{\alpha}X(t)\|_F &\cong \|I_4\sum_{k=1}^{m}\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha}\|_F,\ 0\leqslant t<1,\\ &\leqslant \|I_4\|_F\sum_{k=1}^{m}|\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha}|,\\ &=2\sum_{k=1}^{m}\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha}.\end{aligned}$$

Table 1 shows the numerical values of $\|D_0^{\alpha}X(t)\|_F$ for distinct values of t and α on $0\leqslant t<1$ in step of 0.1 when $m=10$.

Table 1: Numerical values of $\|D_0^{\alpha}X(t)\|_F$ when $m=10$.

t	$\alpha=1/2$	$\alpha=3/4$	$\alpha=3/2$	$\alpha=2$
0.0	1.77245	1.83813	2.65868	4.00000
0.1	2.89754	2.50723	2.9625	4.24611
0.2	3.83615	3.23901	3.62818	5.06366
0.3	4.99905	4.22712	4.77134	6.74124
0.4	6.52266	5.65090	6.74233	9.98811
0.5	8.55521	7.79886	10.3588	16.5913
0.6	11.2710	11.1383	17.6872	31.6794
0.7	14.8752	16.4033	34.4080	72.7418
0.8	19.6062	24.7069	76.7787	208.884
0.9	25.7382	37.6799	190.973	731.952

Similarly, the m^{th}-partial sum expansion for approximating $\|J_0^{\alpha}X(t)\|_F$ can be obtained by Theorem 2.11 as follows

$$\begin{aligned}\|J_0^{\alpha}X(t)\|_F &\cong \|I_4\sum_{k=0}^{m}\frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}t^{(k+1)\alpha}\|_F,\ \alpha>0,\ 0\leqslant t<1,\\ &\leqslant \sum_{k=0}^{m}\frac{2\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}t^{(k+1)\alpha}.\end{aligned}$$

Table 2 shows the numerical values of $\|J_0^{\alpha}X(t)\|_F$ for distinct values of t and α on $0\leqslant t<1$ when $m=10$.

Table 2: Numerical values of $\|J_0^{\alpha}X(t)\|_F$ when $m=10$.

t	$\alpha=1/2$	$\alpha=3/4$	$\alpha=3/2$	$\alpha=2$
0.0	0.00000	0.00000	0.0000	0.00000
0.1	0.24349	0.05059	0.00045	0.00002
0.2	0.57892	0.16072	0.00372	0.00027
0.3	1.01824	0.33195	0.01310	0.00140
0.4	1.59074	0.57880	0.03280	0.00457
0.5	2.33971	0.92714	0.06857	0.01162
0.6	3.32452	1.41993	0.12897	0.02549
0.7	4.62349	2.12725	0.22806	0.05087
0.8	6.33708	3.16197	0.39193	0.09609
0.9	8.59139	4.70257	0.67637	0.17828

Problem 3.2. Given the following MM-LF of $A \in M_n$:

$$E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \alpha > 0, t \geqslant 0.$$

Since the general solutions of many fractional differential system are represented in terms of $E_\alpha(At^\alpha)$, then we need here to approximate: $\|D_0^\alpha(E_\alpha(At^\alpha))\|_F$ and $\|J_0^\alpha(E_\alpha(At^\alpha))\|_F$, respectively, for m^{th}-partial sums based on Theorem 2.11 as follow

$$\|D_0^\alpha(E_\alpha(At^\alpha))\|_F = \|AE_\alpha(At^\alpha)\|_F \cong \|\sum_{k=0}^{m} \frac{A^{k+1} t^{k\alpha}}{\Gamma(k\alpha+1)}\|_F \leqslant \sum_{k=0}^{m} \frac{\|A\|_F^{k+1} t^{k\alpha}}{\Gamma(k\alpha+1)}, \alpha > 0, t \geqslant 0,$$

$$\|J_0^\alpha(E_\alpha(At^\alpha))\|_F \cong \|\sum_{k=0}^{m} \frac{A^k t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}\|_F \leqslant \sum_{k=0}^{m} \frac{\|A\|_F^k t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}, \alpha > 0, t \geqslant 0.$$

To show the validity of our MFPS representation for the approximating $E_\alpha(At^\alpha)$, consider

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Hence, the numerical values of $\|D_0^\alpha(E_\alpha(At^\alpha))\|_F$ and $\|J_0^\alpha(E_\alpha(At^\alpha))\|_F$ for distinct values of α and t on $0 \leqslant t \leqslant 4$ when $m = 10$ are given, respectively, as in Tables 3 and 4.

Table 3: Numerical values of $\|D_0^\alpha(E_\alpha(At^\alpha))\|_F$ when $m = 10$.

t	$\alpha = 1/2$	$\alpha = 3/4$	$\alpha = 3/2$	$\alpha = 2$
0.0	0.50000	0.50000	0.50000	0.50000
0.4	0.74338	0.66409	0.54893	0.52013
0.8	0.89951	0.81536	0.64568	0.58216
1.2	1.05385	0.98370	0.78606	0.69106
1.6	1.21501	1.17624	0.97661	0.85562
2.0	1.38709	1.39880	1.22860	1.08909
2.4	1.57296	1.65743	1.55775	1.41027
2.8	1.77504	1.95887	1.98495	1.84502
3.2	1.99563	2.31081	2.53745	2.42837
3.6	2.23704	2.72215	3.25056	3.20728
4.0	2.50164	3.20322	4.16986	4.24448

Table 4: Numerical Values of $\|J_0^\alpha(E_\alpha(At^\alpha))\|_F$ when $m = 10$.

t	$\alpha = 1/2$	$\alpha = 3/4$	$\alpha = 3/2$	$\alpha = 2$
0.0	0.00000	0.00000	0.00000	0.00000
0.4	0.97353	0.65637	0.19572	0.08054
0.8	1.59803	1.26143	0.58273	0.32863
1.2	2.21540	1.93481	1.14425	0.76425
1.6	2.86007	2.70494	1.90645	1.42249
2.0	3.54851	3.59519	2.91440	2.35637
2.4	4.29224	4.62975	4.23100	3.64108
2.8	5.10113	5.83555	5.93980	5.38009
3.2	5.98457	7.24346	8.14981	7.71346
3.6	6.95207	8.88915	11.0023	10.8291
4.0	8.01350	10.8142	14.6794	14.9779

4. MFPS solutions of some linear and non-linear MFDEs

This section focuses on the general exact (numerical) solutions of four important and interesting linear and non-linear MFDEs by using the efficient MFPS technique. Also, a new technique is applied on the non-linear MFDEs as in the last two problems.

Problem 4.1. Given the following linear MFDEs:

$$D_0^{2\alpha}X(t) = -W^2X(t), \quad 0 < \alpha \leqslant 1, t \geqslant 0, \tag{4.1}$$

$$X(0) = P_0, \ D_0^{\alpha}X(0) = P_1, \tag{4.2}$$

where $W \in M_m$, P_0 and $P_1 \in M_{m,n}$ are real or complex constant matrices and $X(t) \in M_{m,n}$.

According to MFMS method, assume that the MFPS solution $X(t)$ of Eq. (4.1) as follows:

$$X(t) = \sum_{k=0}^{\infty} A_k \, t^{k\alpha}. \tag{4.3}$$

Now, by applying the operator D_0^{α} two-times on Eq. (4.3), we have

$$\begin{aligned} D_0^{\alpha}X(t) &= \sum_{k=1}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha}, \\ D_0^{2\alpha}X(t) &= \sum_{k=2}^{\infty} A_k \frac{\Gamma(k+1)}{\Gamma((k-2)\alpha+1)} t^{(k-2)\alpha} = \sum_{k=0}^{\infty} A_{k+2} \frac{\Gamma((k+2)\alpha+1)}{\Gamma(k\alpha+1)} t^{k\alpha}. \end{aligned} \tag{4.4}$$

Substitute Eqs. (4.3) and (4.4) into Eq. (4.1), yields that:

$$\sum_{k=0}^{\infty} A_{k+2} \frac{\Gamma((k+2)\alpha+1)}{\Gamma(k\alpha+1)} t^{k\alpha} + W^2 \sum_{k=0}^{\infty} A_k \, t^{k\alpha} = 0.$$

This formula leads to the following recurrence relation:

$$A_{k+2} = \frac{-\Gamma(k\alpha+1)}{\Gamma((k+2)\alpha+1)} W^2 A_k, k = 0, 1, 2, \ldots.$$

By using the initial conditions as in Eq. (4.2), we have

$$A_0 = P_0, \ A_1 = \frac{1}{\Gamma(\alpha+1)} P_1.$$

Now, the other coefficients of $t^{k\alpha}$ can be partitioned as follows

1. For the terms of even indices are:

$$A_2 = \frac{-1}{\Gamma(2\alpha+1)} W^2 P_0, \ A_4 = \frac{1}{\Gamma(4\alpha+1)} W^4 P_0, \ldots.$$

2. For the terms of odd indices are:

$$A_3 = \frac{-1}{\Gamma(3\alpha+1)} W^2 P_1, \ A_5 = \frac{1}{\Gamma(5\alpha+1)} W^4 P_1, \ldots.$$

Thus, we can obtain the series solution as follows

$$X(t)=\sum_{k=0}^{\infty}W^{2k}P_0\frac{(-1)^k}{\Gamma(2k\alpha+1)}\,t^{2k\alpha}+\sum_{k=0}^{\infty}W^{2k}P_1\frac{(-1)^k}{\Gamma((2k+1)\alpha+1)}\,t^{(2k+1)\alpha}.$$

This solution can be represented in the term of the MM-LF as an exact solution by

$$X(t)=E_{2\alpha}\left(-W^2t^{2\alpha}\right)P_0+t^{\alpha}E_{2\alpha,(\alpha+1)}\left(-W^2t^{2\alpha}\right)P_1.$$

Problem 4.2. Given the following composite linear MFDEs:

$$D_0^2X(t)+D_0^{1/2}X(t)+X(t)=8B,\ t\geqslant 0, \tag{4.5}$$

$$X(0)=X'(0)=0, \tag{4.6}$$

where B and $X(t)\in M_{m,n}$.

According to the MFMS method, suppose that the MFPS solution $X(t)$ of Eq. (4.5) as follows:

$$X(t)=\sum_{k=0}^{\infty}A_kt^{\frac{k}{2}}.$$

The MFMS requires to find the fractional derivatives: $D_0^{1/2}X(t)$, $D_0^1X(t)$, and $D_0^2X(t)$. However, it is easy to find them by using Eq. (2.2) as follow

$$\begin{aligned}
D_0^{1/2}X(t)&=\sum_{k=1}^{\infty}A_k\frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k-1}{2}+1\right)}t^{\frac{k-1}{2}}=\sum_{k=0}^{\infty}A_{k+1}\frac{\Gamma\left(\frac{k+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)}t^{\frac{k}{2}}\ ,\\
D_0^1X(t)&=\frac{dX}{dt}=A_1t^{\frac{-1}{2}}+A_2+\sum_{k=3}^{\infty}A_n\frac{k}{2}t^{\frac{k-2}{2}},\\
D_0^2X(t)&=\frac{d^2X}{dt^2}=\frac{-1}{2}A_1t^{\frac{-3}{2}}+\frac{3}{4}A_3t^{\frac{-3}{2}}+\sum_{k=4}^{\infty}k_n\frac{k}{2}\left(\frac{k}{2}-1\right)t^{\frac{k-4}{2}}.
\end{aligned}$$

Since $t\geqslant 0$, then A_1 and A_3 must be zeros and by using the initial conditions as in Eq. (4.6), we get $A_0=A_2=0$.

Now, the new representation form of the solution is obtained by:

$$\begin{aligned}
X(t)&=\sum_{k=4}^{\infty}A_kt^{\frac{k}{2}}\ ,\\
D_0^{1/2}X(t)&=A_4\frac{2}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}}+\sum_{k=4}^{\infty}A_{k+1}\frac{\Gamma\left(\frac{k+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)}t^{\frac{k}{2}}\ ,\\
D_0^2y(t)&=2A_4+\frac{15}{4}A_5t^{\frac{1}{2}}+6A_6t+\frac{35}{4}A_7t^{\frac{3}{2}}+\sum_{k=4}^{\infty}A_{k+4}\frac{k+4}{2}\left(\frac{k+4}{2}-1\right)t^{\frac{k}{2}}.
\end{aligned}$$

Substitute the expansion formulas above into Eq. (4.5), we obtain

$$\begin{aligned}
&A_4=4B, A_5=A_6=0, A_7=-\frac{128B}{(105\sqrt{\pi})},\\
&A_{k+4}=\frac{-4}{(k+2)(k+4)}\left(A_k+A_{k+1}\times\frac{\left(\frac{k+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)}\right),\ \ k\geqslant 4.
\end{aligned}$$

So that the 15th-truncated series approximation of $X(t)$ is represented by

$$X_{15}(t) = 4Bt^2 - \frac{128B}{105\sqrt{\pi}}t^{\frac{7}{2}} - \frac{B}{3}t^4 + \frac{B}{15}t^5 + \frac{1024B}{10395\sqrt{\pi}}t^{\frac{11}{2}} + \frac{B}{90}t^6 - \frac{1024B}{135135\sqrt{\pi}}t^{\frac{13}{2}} - \frac{B}{210}t^7 - \frac{2048B}{675675\sqrt{\pi}}t^{\frac{15}{2}}. \quad (4.7)$$

In order to examine the approximation solution in Eq. (4.7), we need to calculate the residual error function $(\mathrm{Res}(t))$ with respect to the Frobenius norm for different values t on $0 \leqslant t \leqslant 1$ in step of 0.2, where the residual error function is defined as follows

$$\mathrm{Res}(t) = \|D_0^2 X(t) + D_0^{\frac{1}{2}} X(t) + X(t) - 8B\|_F,$$

and here, we will take a fixed numerical values for the matrix B as follows:

$$B = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Table 5 shows that the 15th-numerical values of $\|X(t)\|_F$, $\|D_0^{1/2}X(t)\|_F$, and $\|D_0^2 X(t)\|_F$ with $\mathrm{Res}(t)$. It indicates that the numerical solution of the problem as in Eq. (4.7) is more accurate at the beginning of the values of the interval. In fact, we can say that the MFPS method is efficient to obtain a good accuracy of the solution.

Table 5: 15th-Numerical values of $\|X(t)\|_F$, $\|D_0^{1/2}X(t)\|_F$, $\|D_0^2X(t)\|_F$ and $\mathrm{Res}(t)$.

t	$\|X(t)\|_F$	$\|D_0^{1/2}X(t)\|_F$	$\|D_0^2X(t)\|_F$	$\mathrm{Res}(t)$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.07852	0.26265	3.65883	3.10574×10^{-7}
0.4	0.30235	0.70661	2.99101	3.08381×10^{-5}
0.6	0.64523	1.21006	2.14425	4.60852×10^{-4}
0.8	1.07361	1.70471	1.21851	3.16383×10^{-3}
1.0	1.55075	2.14109	0.29399	1.41674×10^{-2}

Problem 4.3. Given the following nonlinear MFDE:

$$D_0^{\alpha} X(t) = X^2(t) + I_n, \quad r-1 < \alpha \leqslant r, \ t \geqslant 0, \quad (4.8)$$

$$X^{(i)}(0) = 0, \quad i = 0, 1, \ldots, r-1, \quad (4.9)$$

where I_n is an identity matrix and $X(t) \in M_n$.

Similarly to the previous problems, let the MFPS solution of Eqs. (4.8) and (4.9) as:

$$X(t) = \sum_{k=0}^{\infty} A_k t^{k\alpha}. \quad (4.10)$$

Based on Eq. (4.9), we get $A_0 = 0$ and so Eq. (4.10) becomes:

$$X(t) = \sum_{k=1}^{\infty} A_k t^{k\alpha}. \quad (4.11)$$

In general, it is not easy to find the coefficients A_k from the recurrence relation corresponding to the MFPS representation for nonlinear MFDEs. Therefore, we use a new technique in this problem for finding A_k by defining the so-called αmth-order MFDE as follows:

$$D_0^{\alpha m}\left(D_0^{\alpha} X(t) - X^2(t) - I_n\right) = 0, \ m = 0, 1, 2, \ldots. \quad (4.12)$$

Note that when $m = 0$, then Eq. (4.12) is reduced to Eq. (4.8). So, the MFMS representation in Eq. (4.11) is a solution for the αmth-order MFDE as in Eq. (4.12). That is:

$$D_0^{\alpha(m+1)}\left(\sum_{k=1}^{\infty} A_k t^{k\alpha}\right) - D_0^{\alpha m}\left(\sum_{k=1}^{\infty} A_k t^{k\alpha}\right)^2 - D_0^{\alpha m}\left(I_n\right) = 0,\ m = 0, 1, 2, \ldots. \tag{4.13}$$

Based on Eq. (2.2), then Eq. (4.13) becomes

$$\sum_{k=m+1}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-m-1)\alpha+1)} t^{(k-m-1)\alpha} - \sum_{k=m}^{\infty}\left(\sum_{j=0}^{k} A_j A_{k-j}\right)\frac{\Gamma(k\alpha+1)}{\Gamma((k-m)\alpha+1)} t^{(k-m)\alpha} = \chi_m, \tag{4.14}$$

where $\chi_m = I_n$ if $m = 0$ and $\chi_m = 0$ if $m \geqslant 1$.

Hence by using Theorems (2.7) and (2.11), the αmth derivative of the MFMS representation, Eq. (4.11) converges at least at $t = 0$, for $m = 0, 1, 2, \ldots$.

Now, by setting $t = 0$ in Eq. (4.14), we get the following values of the coefficients A_k of $t^{k\alpha}$:

$$A_0 = 0,\ A_1 = \frac{I}{\Gamma(\alpha+1)},\ A_{m+1} = \frac{\Gamma(m\alpha+1)}{\Gamma((1+m)\alpha+1)}\sum_{j=0}^{m} A_j A_{m-j} \text{ for } m = 1, 2, \ldots.$$

Thus, the general expansion solution of Eqs. (4.8) and (4.9) is obtained as follows:

$$X(t) = \frac{I_n}{\Gamma(\alpha+1)} t^{\alpha} + \frac{\Gamma(2\alpha+1) I_n}{(\Gamma(\alpha+1))^2 \Gamma(3\alpha+1)} t^{3\alpha} + 2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1) I_n}{(\Gamma(\alpha+1))^3 \Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} + \ldots. \tag{4.15}$$

In particular, if $\alpha = 1$, then the Eqs. (4.8) and (4.9) become:

$$X'(t) = X^2(t) + I_n,\ X(0) = 0,\ t \geqslant 0, \tag{4.16}$$

and when applying Eq. (4.15), then the CPS solution of Eq. (4.16) is presented by:

$$X(t) = \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \frac{1382t^{11}}{155925} + \cdots\right) I_n = I_n \tan t.$$

In order to examine the approximate solution as in Eq. (4.15), Table 6 shows that the 15th-approximate of $\|X(t)\|_F$ with $\mathrm{Res}(t)$ for different values of α and t on $0 \leqslant t \leqslant 1$ by considering the identity matrix of order 9 and defining the residual error function of Problem 4.3 with respect to Frobenius norm by:

$$\mathrm{Res}(t) = \left\|D_0^{\alpha} X(t) - X^2(t) - I_9\right\|_F.$$

Table 6: 15th-numerical values of $\|X(t)\|_F$ with $\mathrm{Res}(t)$.

t	$\|X(t;\alpha{=}1.5)\|_F$	$\mathrm{Res}(t;\alpha{=}1.5)$	$\|X(t;\alpha{=}2.5)\|_F$	$\mathrm{Res}(t;\alpha{=}2.5)$
0.0	0.00000	0.00000	0.000000	0.00000
0.2	0.20199	2.034437×10^{-17}	0.016149	3.109055×10^{-16}
0.4	0.57409	4.370361×10^{-17}	0.091350	1.252591×10^{-15}
0.6	1.06871	2.850815×10^{-13}	0.251775	7.275543×10^{-16}
0.8	1.68902	2.897717×10^{-10}	0.517173	1.022700×10^{-15}
1.0	2.46753	6.341391×10^{-8}	0.905028	1.998026×10^{-16}

Table 6 provides that the numerical values for the convergence of the MFPS technique and the results shown here confirm that our new method is very efficient by using only a few approximation terms and higher accuracy can be achieved by computing more components of the solution.

Problem 4.4. Given the following composite nonlinear MFDE:

$$D_0^{2\alpha} X(t) = \left(D_0^{\alpha} X(t)\right)^2 + I_n, \quad \frac{1}{2} < \alpha \leqslant 1, t \geqslant 0, \tag{4.17}$$

$$X(0) = C_0, \quad D_0^{\alpha} X(0) = C_1, \tag{4.18}$$

where I_n is an identity matrix of order n, and C_0, C_1, $X(t) \in M_n$.

Again, the MFPS solution of Eqs. (4.17) and (4.18) can be assumed as follows:

$$X(t) = \sum_{k=0}^{\infty} A_k t^{k\alpha}.$$

Thus, the αmth-order MFDE of Eq. (4.17) is:

$$D_0^{m\alpha} \left(\left(D_0^{2\alpha} \sum_{k=0}^{\infty} A_k t^{k\alpha} \right) - \left(D_0^{\alpha} \sum_{k=0}^{\infty} A_k t^{k\alpha} \right)^2 - I_n \right) = 0, \quad m = 0, 1, 2, \ldots. \tag{4.19}$$

Based on Eq. (4.17) and using Cauchy product for infinite series, then Eq. (4.19) becomes:

$$D_0^{m\alpha} \left(\sum_{k=2}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-2)\alpha+1)} t^{(k-2)\alpha} - \sum_{k=0}^{\infty} B_k t^{k\alpha} - I_n \right) = 0, \tag{4.20}$$

where

$$B_k = \sum_{j=0}^{k} A_{j+1} A_{k-j+1} \frac{\Gamma((j+1)\alpha+1)}{\Gamma(j\alpha+1)} \frac{\Gamma((k-j+1)\alpha+1)}{\Gamma((k-j)\alpha+1)}.$$

In fact, Eq. (4.20) can be easily reduced into the following equivalent form:

$$\sum_{k=m+2}^{\infty} A_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-m-2)\alpha+1)} t^{(k-m-2)\alpha} - \sum_{k=m}^{\infty} B_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-m)\alpha+1)} t^{(k-m)\alpha} = \chi_m, \tag{4.21}$$

where $\chi_m = I_n$ if $m = 0$ and $\chi_m = 0$ if $m \geqslant 1$.

However, setting $t = 0$ in Eq. (4.21) gives the values of A_k, the coefficient of $t^{k\alpha}$, follows as

$$A_0 \text{ and } A_1 \text{ are arbitrary}, \quad A_2 = \frac{I + C_1^2 (\Gamma(\alpha+1))^2}{\Gamma(2\alpha+1)}, \quad A_{m+2} = \frac{\Gamma(m\alpha+1)}{\Gamma((2+m)\alpha+1)} B_m, m = 1, 2, \ldots.$$

Now, by simple computations, we get the exact solution of Eqs. (4.17) and (4.18) which are expanded in the following power series solution

$$X(t) = C_0 + C_1 t^{\alpha} + \frac{I_n + C_1^2 (\Gamma(\alpha+1))^2}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{2C_1 \Gamma(1+\alpha) \left(I_n + C_1^2 (\Gamma(1+\alpha))^2\right)}{\Gamma(1+3\alpha)} t^{3\alpha} + \cdots.$$

Note that so many new special cases can be extracted by giving two constant matrices C_0 and C_1.

5. Conclusions

In this work, we extend the FPS to the MFPS and discuss the convergence and radii of convergence for MFPS in Caputo sense. In addition, we approximate the matrix fractional derivatives and fractional integrals of a given matrix function with respect to Frobenius norm. Finally, new techniques are also applied for getting the general exact (numerical) solutions of some linear and non-linear MFDEs. How to apply our new method for solving systems of MFDEs such as matrix fractional time-varying descriptor system and matrix fractional control systems still need further researches.

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