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Positive solutions to a nonlinear eigenvalue problem

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Abstract

In this paper, the existence of positive solutions to a nonlinear eigenvalue problem is obtained by Leray-Schauder fixed point theorem.

Keywords: Existence, positive solutions, Leray-Schauder fixed point theorem.

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1. Introduction

In this paper, we consider the nonlinear eigenvalue two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(u(t)), & t \in (0,1), \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive parameter.

We will make the following assumptions:

(i) $f : [0,1) \longrightarrow R$ is continuous and $f(0) > 0$;

(ii) $h(t) \in C[0,1]$ and there exist two constants $\tau, \kappa : \tau \in [0,1],\ \kappa \in (1,\infty)$ such that $h(\tau) \neq 0$ and

$$\int_0^1 G(t,s)h^+(s)ds \geqslant \kappa[\int_0^1 G(t,s)h^-(s)ds] \tag{1.2}$$

for $t \in [0,1]$, where a^+ is the positive part of a and a^- is the negative part of a.

Next, we state the main result.

Theorem 1.1. *Let* (i) *and* (ii) *hold. Then there exists a positive number* λ^* *such that BVP* (1.1) *has at least one positive solution for* $\lambda : 0 < \lambda < \lambda^*$.

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2. Preliminaries lemmas

To prove Theorem 1.1, we need several preliminary results.

Lemma 2.1. *For* $y \in C[0,1]$, *the problem*

$$\begin{cases} u^{(4)}(t) = y(t) \quad t \in (0,1), \\ u(0) = u'(1) = 0 = u''(0) = u'''(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} (6t - 3t^2 - s^2)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ (6s - 3s^2 - t^2)t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Lemma 2.2. *If* $y \in C[0,1]$, $y \geqslant 0$, *then the unique solution u of the* (2.1) *satisfies*

$$u \geqslant 0, t \in [0,1].$$

Moreover, if $y_1(t) \geqslant y_2(t)$ *for* $t \in [0,1]$, *then the corresponding solutions* $u_1(t)$ *and* $u_2(t)$ *satisfy*

$$u_1(t) \geqslant u_2(t), \text{ for } t \in [0,1].$$

Lemma 2.3. *Let* (i) *and* (ii) *hold, then for every* $0 < \delta < 1$, *there exists a positive number* λ_1 *such that, for* $0 < \lambda < \lambda_1$, *the problem*

$$\begin{cases} u^{(4)}(t) = \lambda h^+(t) f(u(t)), \quad t \in (0,1), \\ u(0) = u'(1) = 0 = u''(0) = u'''(1) \end{cases}$$

has a positive solution u_{λ_1} *with* $|u_{\lambda_1}|_0 \longrightarrow 0$ *as* $\lambda \longrightarrow 0$, *and*

$$u_{\lambda_1} \geqslant \lambda \delta f(0) p(t), t \in [0,1], \tag{2.2}$$

where

$$p(t) = \int_0^1 G(t,s)h^+(s)ds.$$

Proof. By Lemma 2.2, we know that $p(t) \geqslant 0$ for $t \in [0,1]$. From Lemma 2.1, (2.2) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t,s)h^+(s) f u(s) ds := Au(t), \tag{2.3}$$

where $u \in C[0,1]$. Then $A : C[0,1] \longrightarrow C[0,1]$ is completely continuous and fixed points of A are solutions of (2.2). We apply the Leray-Schauder fixed point theorem to prove A has a fixed point.

Let $\varepsilon > 0$ be such that

$$f(t) \geqslant \delta f(0), \text{ for } 0 \leqslant \varepsilon.$$

Suppose that

$$\lambda < \frac{\varepsilon}{2|p|_0 f_1(\varepsilon)} := \lambda_1,$$

where

$$f_1(t) = \max_{s \in [0,t]} f(s).$$

Since

$$\lim_{t \longrightarrow 0^+} \frac{f_1(t)}{t} = +\infty,$$

it follows that there exists $\tau_\lambda \in (0,\varepsilon)$, such that

$$\frac{f_1(\tau_\lambda)}{\tau_\lambda} = \frac{1}{2\lambda|p|_0}. \tag{2.4}$$

We note that (2.4) implies

$$\tau_\lambda \longrightarrow 0 \text{ as } \lambda \longrightarrow 0.$$

Now, we consider the equations

$$u = \theta A u,\ \theta \in (0,1),$$

let $u \in C(0,1)$ and $\theta \in (0,1)$ be such that $u=\theta Au$. We claim that $|u|_0 \neq \tau_\lambda$. In fact

$$u(t) = \theta\lambda \int_0^1 G(t,s)h^+(s)fu(s)ds,$$

set

$$w(t) = \theta\lambda \int_0^1 G(t,s)h^+(s)f_1|u|_0 ds \leqslant \theta\lambda f_1(|u|_0)p(t),$$

then by Lemma 2.2 and the fact that $f(u)\leqslant f_1(|u|_0)$, we know that

$$u(t) \leqslant w(t),\ \text{for } t \in [0,1].$$

Moreover, we have

$$|u|_0 \leqslant \lambda|p|_0 f_1(|u|_0)$$

or

$$\frac{f_1(|u|_0)}{|u|_0} \geqslant \frac{1}{\lambda|p|_0}, \tag{2.5}$$

which implies that $|u|_0 \neq \tau_\lambda$. Thus by Leray-Schauder fixed point theorem, A has a fixed point u_{λ_1} with

$$|u_{\lambda_1}|_0 \leqslant \tau_\lambda < \varepsilon.$$

Therefore, combining (2.3), (2.5), and using Lemma 2.2, we have that

$$u_{\lambda_1}(t) \geqslant \lambda\delta f(0)p(t), t \in [0,1].$$ □

3. Proof of the main result

Proof of Theorem 1.1. Let

$$q(t) = \int_0^1 G(t,s)h^-(s)ds,$$

then $q(t)\geqslant 0$. By (ii), there exist positive numbers $c \in (0,1)$, $d \in (0,1)$ such that

$$q(t)|f(y)| \leqslant dp(t)f(0) \tag{3.1}$$

for $y \in [0,c]$ and $t \in [0,1]$. Fix $\delta \in (d,1)$, and let $\lambda_2 > 0$ be such that

$$|u_{\lambda_1}|_0 + \lambda\delta f(0)|p|_0 \leqslant c \tag{3.2}$$

for $\lambda < \lambda_2$, where u_{λ_1} is given by Lemma 2.3, and

$$|f(x) - f(y)| \leqslant f(0)(\frac{\delta - d}{2}) \tag{3.3}$$

for $x \in [-c,c]$, $y \in [-c,c]$ with $|x-y| \leqslant \lambda_2\delta f(0)|p|_0$.

Let $\lambda < \lambda_2$, we look for a solution u_λ of the form $u_\lambda + \nu_\lambda$. Here ν_λ solves

$$\begin{cases} u^{(4)}(t) = \lambda h^+(t)(f(u_{\lambda_1} + \nu) - f(u_{\lambda_1})) - \lambda h^-(t) f(u_{\lambda_1} + \nu), & t \in (0,1), \\ u(0) = u'(1) = 0 = u''(0) = u'''(1) = 0. \end{cases}$$

For each $\omega \in C[0,1]$, let $\nu = T(\omega)$ be the solution of

$$\begin{cases} u^{(4)}(t) = \lambda h^+(t)(f(u_{\lambda_1} + \omega) - f(u_{\lambda_1})) - \lambda h^-(t) f(u_{\lambda_1} + \omega), & t \in (0,1), \\ u(0) = u'(1) = 0 = u''(0) = u'''(1) = 0, \end{cases}$$

then $T : C[O,1] \longrightarrow C[0,1]$ is completely continuous. Let $\nu \in C[0,1]$ and $\theta \in C[0,1]$ be such that $\nu=\theta T\nu$. Then we have

$$\begin{cases} u^{(4)}(t) = \theta\lambda h^+(t)(f(u_{\lambda_1} + \nu) - f(u_{\lambda_1})) - \theta\lambda h^-(t) f(u_{\lambda_1} + \nu), & t \in (0,1), \\ u(0) = u'(1) = 0 = u''(0) = u'''(1) = 0. \end{cases}$$

We claim that $|\nu|_0 \neq \lambda\delta f(0)|p_0|$. Suppose to the contrary that $|\nu|_0 \neq \delta f(0)|p_0|$. Then by (3.2) and (3.3), we obtain

$$|u_{\lambda_1} + \nu|_0 \leqslant |u_{\lambda_1}|_0| + |\nu|_0 \leqslant c$$

and

$$|f(u_{\lambda_1} + \nu) - f(u_{\lambda_1})|_0 \leqslant f(0)(\frac{\delta - d}{2}). \tag{3.4}$$

Using (3.1), (3.4), Lemmas 2.1, and 2.2, we have

$$|\nu(t)| \leqslant \lambda(\frac{\delta - d}{2}) f(0) p(t) + \lambda d f(0) p(t) = \lambda(\frac{\delta + d}{2}) f(0) p(t). \tag{3.5}$$

In particular,

$$|\nu|_0 \leqslant \lambda(\frac{\delta + d}{2}) f(0) p_0 < \lambda\delta f(0)|p|_0$$

is a contradiction, and the claim is proved. Thus by Leray-Schauder fixed point theorem, T has a fixed point ν_λ with

$$|\nu_\lambda|_0 \leqslant \lambda\delta f(0)|p|_0.$$

Using (2.2) and (3.5), we obtain

$$u_\lambda \geqslant u_{\lambda_1} - |\nu_\lambda| \geqslant \lambda\delta f(0) p(t) - \lambda(\frac{\delta + d}{2}) f(0) p(t)$$

and

$$\lambda\delta f(0) p(t) - \lambda(\frac{\delta + d}{2}) f(0) p(t) = \lambda(\frac{\delta - d}{2}) f(0) p(t).$$

Therefore,

$$u_\lambda \geqslant \lambda(\frac{\delta - d}{2}) f(0) p(t) \geqslant 0,$$

i.e., u_λ is a positive solution of (1.1). The proof is completed. □

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