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Completeness and compact generation in partially ordered sets

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Abstract

In this paper we introduce a notion of density in posets in a more general fashion. We also introduce completeness in posets and study compact generation in posets based on such completeness and density.

Keywords: U-density, U-complete poset, U-compactly generated poset, U-regular interval.

1. Introduction

We begin with the necessary definitions and terminologies in a poset P. An element x of a poset P is an upper bound of $A \subseteq P$ if $a \leq x$ for all $a \in A$. A lower bound is defined dually. The set of all upper bounds of A is denoted by A^u (read as, A upper cone), where $A^u = \{x \in P : x \leq a \text{ for every } a \in A\}$ and dually, we have the concept of a lower cone A^l of A. If P contains a finite number of elements, it is called a finite poset. A subset A of a poset P is called a chain if all the elements of A are comparable. A poset P is said to be of length n, where n is a natural number, if there is a chain in P of length n and all chains in P are of length n. A poset P is of finite length if it is of length n for some natural number n. A poset P is said to be bounded if it has the greatest (top) and the least (bottom) element denoted by 1 and 0, respectively. By $[x, y](x \leq y; x, y \in P)$ we denote an interval, that is, set of all $z \in P$ for which $x \leq z \leq y$. In a poset P we say that x is covered by y and write $x \prec y$, if $x \leq z \leq y$ implies $x = z$ or $z = y$. An element p of a poset P with 0 is called an atom if $0 \prec p$. The set of atoms of P is denoted by $A(P)$. For a non-zero element $a \in P, \omega(a)$ denotes the set of atoms contained in a, that is, $\omega(a) = \{p \in A(P) : p \prec a\}$. For the subsets A, B of a poset P, we denote the followings:

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- $\{A, B\}^L = \{x \in P : x$is a maximal element in$\{A, B\}^l\}$,
- $\{A, B\}^U = \{x \in P :$ x is a minimal element in$\{A, B\}^u\}$.

A poset P is called weakly atomic if for every pair of elements $a, b \in P$ with $a < b$, there exist elements $u, v \in P$ such that $a \leq u \prec v \leq b$. A poset P is called strongly atomic if every interval $[x, y]$ of P has an atom. Equivalently, for every interval $[x, y]$ with $x < y$, there exists $a \in P$ such that $x \prec a \leq y$.

Erne [6] studied compact generation in posets. A subset $D \subseteq P$ is called directed subset if for every $x, y \in D$, $(x, y)^u$ is non-empty in D and in this case every finite subset of D has an upper bound in D (in particular D is non-empty). A poset P is called up-complete if every directed subset $D \subseteq P$ has a join denoted by $\bigvee D$. A poset P is called chain-complete or Dedekind complete if every non-empty chain of P has a join and meet; in other words, if P and its dual are up-complete. An element x of an up-complete poset P is called compact if for every directed subset D of P with $x \leq \bigvee D$ there exists an element $y \in D$ with $x \leq y$. A poset P is called compactly generated if each element of P is a join of compact elements. The set of all compact elements of a poset P is denoted by $K(P)$. For more details see Gierz et al. [7].

In general, a subset S of a poset P is called join-dense in P, if each element of P is a join of elements from S. Equivalently, for any two elements $a, b \in P$ with $a \nleq b$, there is some $s \in S$ with $s \leq a$, $s \nleq b$. We also have the concept of meet-density which is defined dually.
Join-density plays a crucial role in poset theory to construct some important classes of posets. We mention some of these classes. Let P be a poset;

(i) if the set of all atoms is join-dense in P then P is atomistic,

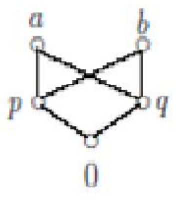
(ii) if the set of all compact elements is join-dense in P then P is compactly generated.

However, Shewale [9] has given the following definition of an atomistic poset. A Poset P with 0 is called atomistic if every $a \in P$ is such that $a \in \{p \in A(P) : p \leq a\}^U$.

2. Complete posets

We introduce a more general concept of density in posets as follows:

Definition 2.1. Let P be a poset. A subset $S \subseteq P$ is called U-dense in P, if each element of P belongs to S_1^U for some $S_1 \subseteq S$. We also have the concept of L-density which is defined dually.



P_1

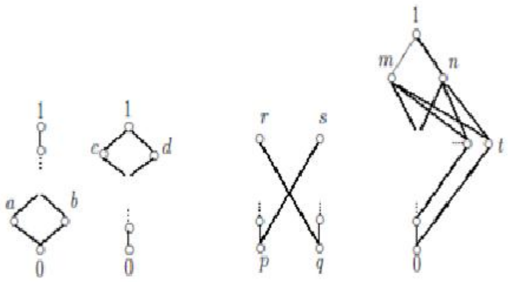
We note that every join-dense subset of a poset P is U-dense but the converse neednot be true. For instance, in the poset depicted in Figure P_1 the set of atoms is U-dense but not join-dense.

We observe that in a poset P if $A(P)$ is U-dense then P is atomistic in the sense of Shewale [9]. Henceforth, atomistic posets in the sense of Shewale is termed as U-atomistic posets.

Remark 2.2. We observe that if a subset S of a poset P is U-dense in P then for any two elements $a, b \in P$ with $a < b$, there is some $s \in S$ with $s \leq b$, $s \nleq a$. Indeed, consider $\{si : si \leq b; i \in I\} \subseteq S$. Then $b \in \{si : si \leq b; i \in I\}^U$. If $si \leq a$ for every i then $b \notin \{si : si \leq b; i \in I\}^U$, a contradiction.

We apply U-density to the set of all atoms and compact elements of a given poset to obtain and study classes of posets, namely, atomistic and compactly generated posets, respectively.
For a given pair of elements a, b of a poset we may have the set $(a, b)^u$ is nonempty but the set $(a, b)^U$ is an empty set; see the poset P_2. This observation along with U-density lead us to define completeness and further compactness and compact generation in posets.

Definition 2.3. A poset P is called conditionally U-complete if for every subset $H \subseteq P$ and for every $u \in H^u$, there exists an element $v \in H^U$ such that $v \leq u$. A poset P is called conditionally L-complete if for every subset $H \subseteq P$ and for every $l \in H^l$, there exists an element $t \in H^L$ such that $l \leq t$. A poset P is called conditionally complete if it is both conditionally U-complete and conditionally L-complete.



P_2 P_3 P_4 P_5

We observe that every U-complete poset has the top element 1 and every L-completeposet has the bottom element 0. Consequently, every complete poset is a bounded poset. We also observe that a bounded conditionally complete poset is a complete poset. If a complete poset P happens to be a lattice then our completeness coincides with the lattice completeness. There exist posets which are complete but not up-complete nor chain-complete and vice versa. The poset depicted in Figure P_5 is a complete poset which is not chain-complete. The poset P_2 is up-complete but it is not U-complete nor chain-complete. The poset P_3 is U-complete but not chain-complete nor L-complete. The poset P_4 is up-complete as well as chain-complete but it is not U-complete nor L-complete and hence it is not complete. The poset depicted in the Figure P_4 is conditionally complete. The poset depicted in Figure P_2 is conditionally L-complete but not conditionally U-complete.

Remark 2.4. We assert that the converse part of Remark 2.2 holds in U-complete posets. In fact, consider two elements $a, b \in P$ with $a < b$ such that there exists an element s of a subset S of P with $s \leq b$, $s \nleq a$. We claim that S is U-dense in P, that is, for every $x \in P$ there exists a subset S_1 of S such that $x \in S_1^U$. U-completeness assures that S_1^U is non-empty. On the contrary, assume that $x \notin S_1^U$ for some $x \in P$. Then there exists an element $y \in S_1^U$ such that $y < x$. By assumption, there exists an element $s_1 \in S_1$ such that $s_1 \leq x$, $s_1 \nleq y$. However, $s_1 \nleq y$ contradicts the fact that $s_1 \nleq y$ since $y \in S_1^U$.

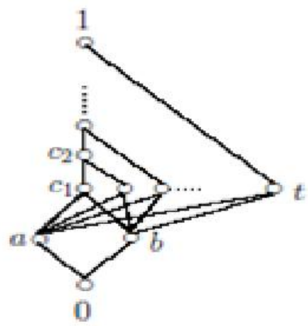
Consequently, a subset S of a U-complete poset P is U-dense in P if and only if for any two elements $a, b \in P$ with $a < b$, there is some $s \in S$ with $s \leq b$, $s \nleq a$.

3. Compact generation in posets

In this section first we define the notion of compactness in posets and then we study compact generation.

Definition 3.1. An element c of a conditionally complete poset P is called U-compact if $c \leq u$ for some $u \in X^U$, where $X \subseteq P$ implies that $c \leq u_1$ for some $u_1 \in X_1^U$, where X_1 is a finite subset of X. The set of all U-compact elements of a poset P is denoted by $C(P)$.

Definition 3.2. A complete poset P is said to be U-compactly generated if the set of all U-compact elements is U-dense in P. A poset P is said to be U-compactly atomistic if every atom of P is U-compact and the set of all atoms is U-dense in P.



P_6

Evidently, if a poset P is finite then every element is U-compact. If a poset P is both up-complete and complete then U-compactness implies compactness. Let c be a U-compact element of P and $c \leq \bigvee D$, where D is a directed subset of P. As c is U-compact and $\bigvee D = D^U$, there exists a finite subset D_1 of D such that $c \leq u$ for some $u \in D_1^U$. But $u \in D$ and so c is compact. The poset P_6 is complete as well as up-complete. The element t is not U-compact (nor compact) and it is not the join of compact elements contained in it. In fact, $t \in (a, b)^U$, where both a andb are U-compact. The poset P_6 is a U-compactly generated poset which is not compactly generated.

We investigate properties of U-compactly generated posets and its relationships with other known concepts in posets. One of the concepts which is well studied in the class of atomistic lattices is the concept of a finite element. An element a in a lattice L with 0 is called finite if either $a = 0$ or a is a join of finite number of atoms. Shewale [9] introduced the concept of a finite element in posets as follows. An element a of a poset P with 0 is called a finite element if either $a = 0$ or $a \in \{\text{finitely many atoms}\}^U$.

Remark 3.3. In every U-compactly generated poset the atoms are U-compact. For, let P be a U-compactly generated poset and let p be an atom of P. Then $p \in S^U$, where $S \subseteq \{c \in P : c \in C(P)\}$. Note that $c \leq p$ for every $c \in S$ and we must have at least one non-zero $c \in S$. Otherwise, we get $p \in S^U = \{0\}$ which is not possible. Consequently, such c is p and so p is U-compact.

Birkhoff [1] proved that in an atomistic compactly generated lattice, every compact element is a finite element. The similar fact we state for posets in the following sense.

Lemma 3.4. *In a U-atomistic U-compactly generated poset (U-compactly atomistic poset), every U-compact element is a finite element.*

Proof. Let P be a poset as described in the statement and let c be a U-compact element of P. Since P is U-atomistic, $c \in \{\omega(c)\}^U$. Now, by U-compactness of c, there exists a finite subset S of $\omega(c)$ such that $c \in S^U$. Since S essentially contains finite number of atoms, c is finite. □

Crawley and Dilworth [5] essentially proved that every compactly generated lattice is weakly atomic and here we extend this result to posets in the following sense.

Theorem 3.5. *Every U-compactly generated poset is weakly atomic.*

Proof. Let P be a U-compactly generated poset and $b < a$ in P. Then there exists a U-compact element c such that $c \leq a$, $c \nleq b$. As $b < a$, $c \leq a$ and P is complete, there exists $u \in (b, c)^U$ such that $b < u \leq a$. Consider $Q = \{x \in P : b \leq x < u; x \ngeq c\}$ which is non-empty since $b \in Q$. Note that for every chain C in Q, C^U is non-empty and so let $d \in C^U$. Clearly $b \leq d \leq u$. Also, $d \neq u$ and $d \ngeq c$. Indeed, if $d = u$ then $d \in (b, c)^U$ and we get $c \leq d$. Now, as c is U-compact, there exists a finite subset (finite chain) T of C such that $c \leq \bigvee T = T^U$ and since T is a finite chain, if x is the largest element of T, then we have $c \leq x$ and consequently $x \notin Q$, a contradiction. If $c \leq d$, by the similar arguments we get a contradiction. In nutshell, $b \leq d < u$ with $d \ngeq c$ and consequently $d \in Q$. It means that every chain in Q has an upper bound in Q and by Zorn's Lemma, Q contains a maximal element, say v. Now, $b \leq v < u \leq a$ and maximality of v ensures that there does not exist $z \in P$ with $v < z < u$. Indeed, otherwise we get $z \in (b; c)^u$ with $z < u$, a contradiction to the fact that $u \in (b, c)^U$. Therefore $b \leq v \prec u \leq a$ and P is weakly atomic. □

Next, we extend some results from Stern [10] known for lattices; see also Kalman [8]. The concept of a complement of an element in a poset is well known and studied in the literature; for more details, see Chajda [3] and Chajda and Moravkova [4].

Let P be a poset with 0 and 1. A complement of an element $a \in P$ is an element $a^{'} \in P$ if $(a, a^{'})^u = \{1\}$ and $(a, a^{'})^l = \{0\}$. A poset P with 0 and 1 is called complemented when every element of P has a complement. Let P be a poset. Let $x \in [a, b] \subseteq P$. An element $y \in [a, b]$ is called a weak relativecomplement of x in $[a, b]$ if $(x, y)^u \cap [a, b] = \{b\}$ and $(x, y)^l \cap [a, b] = \{a\}$. A poset P is called weakly relatively complemented if for every interval $[a, b]$ of P, each x in $[a, b]$ has a weak relative complement in $[a, b]$.

Proposition 3.6. *Let P be a U-compactly generated poset. If an atom $p \in P$ has a complement $p^{'}$, then there exists a dual atom $m(\geq p^{'})$ which is also a complement of p.*

Proof. Let P be a U-compactly generated poset, p be an atom and consider the set $Q = \{x \in P : (p, x)^l = \{0\}; x \geq p^{'}\}$ which is non-empty since $p^{'} \in Q$. Note that for everychain C in Q, C^U is non-empty and so let $d \in C^U$. We claim that $d \in Q$. Indeed, if $d \notin Q$ then $(d, p^{'})^l \neq \{0\}$ and so $p \leq d$. Now, P is U-compactly generated so p is U-compact and hence $p \leq \{d_1\} = C_1^U$ for some finite subset $C_1 \subseteq C$. But C_1 is a finite chain, therefore $C_1^U = \bigvee C_1$ and hence $d_1 \in C$ and consequently $(p, d)^l = \{0\}$, a contradiction to the fact that $p \leq d_1$. Thus, d is an upper bound for C and by Zorn's Lemma, there exists a maximal element in Q, say m. □

Claim 1: m is the complement of p. As $m \in Q$, $(m, p)^l = \{0\}$. As $p^{'} \leq m$, we must have $(m, p)^u = \{1\}$, otherwise $(p, p^{'})^u \neq \{1\}$ which is not possible.

Claim 2: m is a dual atom. If m is not a dual atom then there exists an element n which is not in Q such that $m < n < 1$. As $n \notin Q$, $(n,p)^l \neq \{0\}$ so $p \leq n$ and $n \in (p,m)^u = \{1\}$, that is, $n = 1$, a contradiction.

Theorem 3.7. *In an atomic U-compactly generated poset P, each atom has a complement if and only if $S^l = \{0\}$, where S is the set of all dual atoms of P.*

Proof. Let P be an atomic U-compactly generated and S be the set of all dual atoms of P. Suppose that $S^l = \{0\}$, and there exists an atom $p \in P$ which has no complement. This means $p \leq m$ for every dual atom $m \in P$ and consequently $p \in S^l$, a contradiction. Conversely, suppose that each atom has a complement. If $S^l \neq \{0\}$ then there exists a non-zero element in S^l, say a. Let p be an atom contained in a. As every dual atom d contains p, there is no dual atom which is a complement of p, a contradiction to Proposition 3.6. □

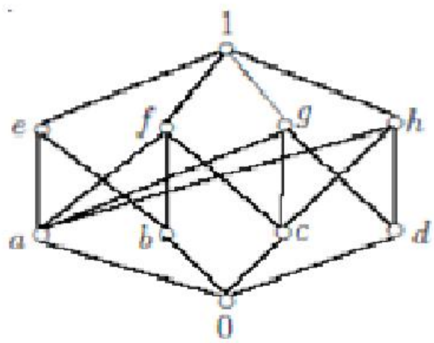
Next, we generalize some definitions and results due to Bjorner [2] and Stern [10] for lattices.

Definition 3.8. We say that an interval $[x,y]$ of a poset of finite length is U-regular, if $y \in \{\text{atoms of}[x,y]\}^U$. Dually, we have the concept of L-regular intervals. An interval $[x,y]$ is called an upperinterval if $y = 1$. It is called a lowerinterval if $x = 0$.

For every element $a \in P$, $[0,a]$ is U-regular if and only if $a \in \{\omega(a)\}^U$ and so we have the following.

Proposition 3.9. *Every lower interval of a poset P of finite length is U-regular if and only if P is U-atomistic.*

Bjorner [2] essentially proved that if L is a lattice of finite length such that all upper intervals are join-regular then L is complemented. By the following example we show that this fact fails in posets. Consider the poset depicted in Figure P_7 of which every interval is U-regular, how ever the



P_7

poset is not complemented. Also, it is known that if all upper intervals of a lattice of finite length are join-regular, then they are meet-regular too (see Stern [10]). The poset depicted in the Figure P_7 shows that although all upper intervals of this poset are U-regular but the interval $[0,1]$ is not L-regular.

Remark 3.10. The statement of Theorem 3.7 can be rephrased as: in a bounded poset P of finite length, the upper interval $[0,1]$ is L-regular if and only if each atom of P has a complement.

Corollary 3.11. *An interval $[x, y]$ of a bounded poset of finite length is L-regular if and only if each atom of $[x, y]$ has a complement in this interval.*

Proof. The statement follows immediately by applying Theorem 3.7 to the interval $[x, y]$. □

Theorem 3.12. *Let P with 0 and 1 be a poset of finite length. Consider the following statements.*

(1) P is relatively complemented.

(2) All intervals of P are L-regular.

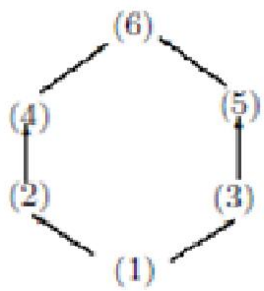
(3) All intervals of P are U-regular.

(4) P is dually U-atomistic.

(5) P is U-atomistic.

(6) P has no 3-element interval.

Then ($\rightarrow$ indicates implication)



Proof. Let P be weakly relatively complemented. Then each interval $[x, y]$ is complemented for $x, y \in P$ and in particular, each atom of $[x, y]$ has a complement in $[x, y]$. Corollary 3.11 yields now that each interval is L-regular. In particular, every upper interval $[a, 1]$, $a \in P$ is L-regular and hence P is dually U-atomistic, proving (1) $\rightarrow$ (2) $\rightarrow$ (4). (1) $\rightarrow$ (3) $\rightarrow$ (5) follows dually. (5) $\rightarrow$ (6) and (4) $\rightarrow$ (6) are evident. □

We now take a look at the definition of upper semimodular posets introduced by Shewale [9].

Definition 3.13. A poset P is called upper semimodular, briefly USM, if $l \prec a$ for some $l \in (a, b)^L$ implies that $b \prec u$ for all $u \in (a, b)^U$. A lower semimodular (LSM) poset is defined dually.

Proposition 3.14. *In a complete USM poset P the interval $[0, 1]$ is U-regular if and only if each upper interval of P is U-regular.*

Proof. Let $P = [0, 1]$ be U-regular and consider an arbitrary upper interval $[x, 1]$. Since $1 = \bigvee\{A(P)\}$ and $x < 1$, there exists an atom p such that $p \nleq x$. By upper semimodularity all the elements of $(p, x)^U$ are atoms of $[x, 1]$. It follows that the join of all atoms of $[x, 1]$, which can be written in the form of $(q, x)^U$ for every atom $q \nleq x$, must be 1 since otherwise it contradicts U-regularity of $[0, 1]$. Hence $[x, 1]$ is U-regular. Converse is true as $[0, 1]$ itself is an upper interval. □

Remark 3.15. In every bounded complemented poset P of finite length, $1 = \bigvee\{A(P)\}$ (dually, meet of dual atoms is 0). In fact, since P is complemented then in particular, each dual atom of P has a complement. The dual of Theorem 3.7 now implies that the greatest element 1 is the join of atoms.

Evidently, every strongly atomic poset is weakly atomic. The converse need not be true in general. We obtain a class of posets in which weakly atomicity and strongly atomicity are equivalent.

Proposition 3.16. *Let P be a weakly relatively complemented lower semimodular poset. Then P is strongly atomic if and only if P is weakly atomic.*

Proof. Let P be such a given poset and $b < a$ in P. Since P is U-compactly generated, P is weakly atomic. Therefore, there exist $u, v \in P$ such that $b \leq v \prec u \leq a$. As P is relatively complemented, v has a complement in $[b, u]$, say $v^{'}$. Since $v^{'} \nleq v$ and $v \prec u$, by lowersemimodularity, $l \prec v^{'}$ for all $l \in (v, v^{'})^L$. As $b \in (v, v^{'})^L$ we have $b \prec v^{'}$ and consequently P is strongly atomic. □

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References

[1] G. Birkhoff, *Lattice Theory*, American Mathematical Society, New York, (1940). 3
[2] A. Björner, *On complements in lattices of finite length*, Discrete Math., **36** (1981), 325–326. 3, 3
[3] I. Chajda, *Complemented ordered sets*, Arch. Math. (Brno), **28** (1992), 25–34. 3
[4] I. Chajda, Z. Morávková, *Relatively complemented ordered sets*, Discuss. Math. Gen. Algebra Appl., **20** (2000), 207–217. 3
[5] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice- Hall, Englewood Cliffs, (1973), 1015–1023. 3
[6] M. Erné, *Compact generation in partially ordered sets*, J. Austral. Math. Soc. Ser. A, **42** (1987), 69–83. 1
[7] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin-New York, (1980). 1
[8] J. A. Kalman, *A property of algebraic lattices whose compact elements have complements*, Algebra Universalis, **22** (1986), 100–101. 3
[9] R. S. Shewale, *Modular pairs, forbidden configurations and related aspects in partially ordered sets*, Ph. D. Thesis, University of Pune, Pune (INDIA), (2010). 1, 2, 3, 3
[10] M. Stern, *Semimodular lattices: Theory and Applications*, Cambridge University Press, Cambridge, (1999). 3, 3, 3