Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci. 16 (2016), 77–80
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

The Heun equation and generalized $\mathrm{Sl}(2)$ algebra

J. Sadeghi[a,*], A. Vaezi[b], F. Larijani[a]

[a]Department of Physics, Islamic Azad University, Ayatollah Amoli Branch, P. O. Box 678, Amol, Iran.

[b]Department of Mathematics, University of Mazandaran, P. O. Box 95447, Babolsar, Iran.

Abstract

In this paper, first we introduce the Heun equation. In order to solve such equation we show the generators of generalized sl(2). Second, we arrange the Heun equation in terms of new operators formed of generalized $sl(2)$ generators and it's commutator relation. Here, instead of $J^+(r)$, $J^-(r)$ and J^0 we use the $P^+(r)$, $P^-(r)$ and $P^0(r)$ as operators of generalized $sl(2)$ algebra. This correspondence gives us opportunity to arrange the parameters α and β in $P^0(r)$. Also, the commutator of such operators leads us to have generalized $sl(2)$ algebra. Also, we obtain the Casimir operators and show that it corresponds to P^+, P^- and some constants. These operators lead to deform the structure of generalized $sl(2)$ algebra in the Heun equation. Finally, we investigate the condition for exactly and quasi-exactly solvable system with constraint on the corresponding operators P^+ and P^-.

Keywords: Heun equation, generalized $sl(2)$ algebra, commutative relation.

1. Introduction

We deal with the Heun equation and show this equation can be factorized in terms of some operators. As we know the initially, Ref.s [3, 4, 5, 7] discussed the confluent and double-confluent Heun equations with some applications. These equations also are known as generalized spherical wave equation [1, 2, 9, 10, 11]. On the other hand, the Fuchsian equation generally will be as [6, 8],

$$Y''(x) + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \sum_{i=1}^{k} \frac{\epsilon_i}{x-a_i})Y'(x) + \frac{\alpha\beta x^k + \sum_{i=1}^{k} p_i x^{k-1}}{x(x-1)\prod_{i=1}^{k}(x-a_i)} Y(x) = 0.$$

*Corresponding author
Email addresses: pouriya@ipm.ir (J. Sadeghi), ar.vaezi@yahoo.com (A. Vaezi)

Received 2015-08-15

The two indices at each singularity a_i are $(0, 1-\epsilon_i)$ and $(0, 1-\gamma)$, $(0, 1-\delta)$, (α, β) at $x = 0$, $x = 1$, and $x = \infty$, respectively, taking into account the Fuchsian relation:

$$\alpha + \beta + 1 = \gamma + \delta + \sum_{i=1}^{k} \epsilon_i.$$

Where $\epsilon_i \equiv 0$ $(i = \overline{1, k})$ gives the hypergeometric equation and $k = 1$ the Heun equations. So, the first extension of Heuns equation with $k = 2$ can be written as,

$$Y''(x) + [\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon_1}{x-a} + \frac{\epsilon_2}{x-b}]Y'(x) + \frac{\alpha\beta x^2 + p_1 x + p_2}{x(x-1)(x-a)(x-b)} Y(x) = 0. \quad (1.1)$$

Now, in this paper we investigate equation (1.1) in generalized $sl(2)$ algebra point of view. So, we define the general form of Schrodinger equation and write Heun differential equation in terms of functions $f_1(x)$, $f_2(x)$ and $f_3(x)$. Here, we note that the corresponding equation can explain some solution model and its stability. Also, we introduce the usual generators of generalized $sl(2)$ algebra as J^+, J^- and J^0. By comparing equation (2.1) with Heun differential equation we obtain the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$. In that case the above functions help us to connect Heun differential equation with generalized $sl(2)$ operators and achieve P^+, P^- and P^0 correspond to such equation. Finally, we apply these operators and realize the commutation relation between P^+ and P^-. This relation gives us motivation to calculate parameters α and β in P^0 correspond to Heun differential equation. For this reason we start with the Heun differential equation connecting to generalized $sl(2)$ operators.

2. The Heun equation and generalized $sl(2)$ algebra

Here we first try to introduce the following Schrodinger equation,

$$H\Psi = E\Psi, \qquad H = -\frac{d^2}{dx^2} + V(x).$$

In order to connect such relation, we have to choose suitable change of variables and appropriate point canonical transformations. All these help us to avoid the singularity in such equations (2.6), (2.7). Generally, one can write such differential equations in terms of $f_1(x)$, $f_2(x)$ and $f_3(x)$ which is given by the following equation,

$$\left[f_1(x)\frac{d^2}{dx^2} + f_2(x)\frac{d}{dx} + f_3(x)\right]\Psi(x) = 0, \quad (2.1)$$

where three corresponding functions will be polynomial and corresponding to differential equation. On the other hand, the quasi-exactly solvable problems and their connection by generalized $sl(2)$ algebra with finite dimensional representation corresponding to spin j will be the following generators,

$$J^+ = x^2\frac{d}{dx} - 2jx, \qquad J^0 = x\frac{d}{dx} - j, \qquad J^- = \frac{d}{dx},$$

which is satisfied in the following commutation relations and will be formed of closed algebra.

$$[J^0, J^+] = J^+, \qquad [J^0, J^-] = -J^-, \qquad [J^+, J^-] = -2J^0.$$

In order to connect Heun differential equation with generalized $sl(2)$ algebra we have to modify the

J^+, J^0 and J^- in form of P^+, P^0 and P^- which is also satisfied by the following commutation relations and also will be formed of closed algebra,

$$[P^0, P^+] = P^+, \qquad [P^0, P^-] = -P^-, \qquad [P^+, P^-] = -2P^0. \tag{2.2}$$

In order to obtain the corresponding operators from equation (1.1), we have arranged such equation inform of equation (2.1). In that case we have rewrite $f_1(x)$, $f_2(x)$ and $f_3(x)$ as,

$$\begin{aligned} f_1(x) &= a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4, \\ f_2(x) &= a_5x^3 + a_6x^2 + a_7x + a_8, \\ f_3(x) &= a_9x^2 + a_{10}x + a_{11}. \end{aligned} \tag{2.3}$$

We put equation (2.3) in (2.1) and compare with equation (1.1), one can obtain a_i, $(i = 1 \cdots, 11)$ as,

$$\begin{aligned} &a_0 = 1, \qquad a_1 = -(a+b+1), \qquad a_2 = ab + (a+b), \qquad a_3 = -ab, \qquad a_4 = 0, \\ &a_5 = \gamma + \delta + \epsilon_1 + \epsilon_2, \qquad a_6 = -(a+b+1)\gamma + \delta(a+b) + \epsilon_1(b+1) + \epsilon_2(a+1), \\ &a_7 = \gamma(a + (a+b)b) + \delta ab + \epsilon_1 b + \epsilon_2 b, \qquad a_8 = -\gamma ab, \qquad a_9 = \alpha\beta, \qquad a_{10} = \rho_1, \\ &a_{11} = \rho_2. \end{aligned} \tag{2.4}$$

In order to construct the generalized $sl(2)$ algebra for the Heun differential equation, we have to choose suitable order in equation (2.3) and make corresponding operators as $P^+(x)$, $P^0(x)$, $P^-(x)$ and $F\big(x\frac{d}{dx}\big)$ which are given by the following expressions.

$$\begin{aligned} P^+ &= a_0x^4\frac{d^2}{dx^2} + a_5x^3\frac{d}{dx} + a - 9x^2, \\ P^0 &= ax\frac{d}{dx} - \beta, \\ P^- &= a_2x^2\frac{d^2}{dx^2} + a_7x\frac{d}{dx} + a11, \\ F\big(x\frac{d}{dx}\big) &= a_1x^3\frac{d^2}{dx^2} + a_6x^2\frac{d}{dx} + a_{10}x. \end{aligned} \tag{2.5}$$

Again by using equation (2.4) into equation (2.5) on can obtain the following expressions.

$$\begin{aligned} P^+ &= x^4\frac{d^2}{dx^2} + (\gamma + \delta + \epsilon_1 + \epsilon_2)x^3\frac{d}{dx} + \alpha\beta x^2, \\ P^- &= \big(ab + (a+b)\big)x^2\frac{d^2}{dx^2} + \big[\gamma\big(a + (a+b)b\big) + \delta ab + \epsilon_1 b + \epsilon_2 b\big]x\frac{d}{dx} + \rho_2, \\ P^0 &= \alpha x\frac{d}{dx} - \beta. \end{aligned} \tag{2.6}$$

And

$$F\big(x\frac{d}{dx}\big) = -(a+b+1)x^3\frac{d^2}{dx^2} + \big[\delta(b+a) - (a+b+1)\gamma + \epsilon_1(b+1) + \epsilon_2(a+1)\big]x^2\frac{d}{dx} + \rho x. \tag{2.7}$$

The commutation relation (2.2) leads us to fix $\alpha = \frac{1}{2}$, $\beta = 1$. So, the algebraic structure can be inferred from the following closed form.

$$[P^0, P^+] = P^+, \qquad [P^0, P^-] = -P^-, \qquad [P^+, P^-] = FP^0.$$

So, the Heun equation (1.1) satisfies the deformation of generalized $sl(2)$ structure or algebra. Here, the eigen value of Casimir operator will be as,

$$C = P^- P^+ + a_6 A_7.$$

As we know the general differential equation (1.1) can be cast in terms of P^+, P^- and P^0 operators if $ab = 0$, so we have the following equation,

$$\big[p^+ + F(P^0) + P^-\big] Y(x) = 0. \tag{2.8}$$

For this purpose, the differential equation (1.1) should be cast in the form (2.8) provided that the condition $F\big(x\dfrac{d}{dx}\big)x^\lambda = 0$. This yields

$$\lambda_+ = 0, \qquad \lambda_- = \frac{1}{a+b+c}.$$

In that case the corresponding wave function $Y(x)$ will be as,

$$Y(x) = C_{\lambda_-} \sum_{m=0}^{\infty} (-1)^m \Big[\frac{1}{D(D+\lambda_-)}(P^+ + P^-)\Big]^m = \lambda_-.$$

Finally, we can say that the above expression is exactly solvable only when $P^- = 0$ and is quasi - exactly solvable under certain condition when P^+ and P^- are present.

3. Conclusion

In this paper, we showed that the Heun equation will be form of generalized $sl(2)$ generators as P^+, P^- and P^0. We checked the commutation relation between three operators and obtained the corresponding parameters as α and β in P^0 generators. The commutation relation between P^+, P^- and shown that we have generalized generalized $sl(2)$ algebra. Also this paper we have shown that the Heun equation (1.1) satisfied by generalized $sl(2)$ algebra with some deformation structure. Finally we obtained Casimir operators and investigated the condition for exactly and quasi- exactly solvable system with constraint on the corresponding operators P^+ and P^-.

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