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Intra regular and interior ideal in Γ-AG*- groupoids

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Abstract

Non-associative algebraic structures are of interest to consider for their remarkable properties. In this paper, we generalize the AG^*-groupoids to $\Gamma - AG^*$-groupoids and study their algebraic properties. Among other results, it is shown that every $\Gamma - AG^*$-groupoid is left alternative and a $\Gamma - AG^*$-groupoid having a left cancellative element is a $T^1 - \Gamma - AG^*$-groupoid, a $a\Gamma - AG^*$-groupoid S is a Γ-intra-regular if $S\Gamma a = S$ holds for all $a \in S$, let S be a Γ-intra-regular of $\Gamma - AG^*$-groupoid then B is a right Γ-ideal of S if $B\Gamma S = B$, if S is a Γ-intra-regular of $\Gamma - AG^*$-groupoid then $(S\Gamma B)\Gamma S = B$, where B is a Γ-interior ideal of S, in an Γ-intra-regular of $\Gamma - AG^*$-groupoid S if A is a Γ-interior ideal of S then A is a $\Gamma - bi$-ideal of S, in an Γ-intra-regular of $\Gamma - AG^*$-groupoid S if A is a Γ-interior ideal of S then A is a $\Gamma(1, 2)$-ideal of S.

Keywords: $\Gamma - AG$-groupoid, $\Gamma - AG^*$-groupoid, $T^1 - \Gamma - AG$-groupoid, Γ-left alternative, Γ-left cancellative, $\Gamma - 3$-band, Γ-interior ideal, Γ-intra-regular, $\Gamma - bi$-ideal, $\Gamma - (1, 2)$-ideal.

1. Introduction and Preliminaries

The idea of generalization of communicative semigroups was introduced in 1977 by Kazim and Naseerudin [2]. They named this structure as the left almost semigroup (LA-semigroup) in [1]. It is also called as Abel-Grassmanns groupoid (AG-groupoid) in [3]. In generalizing this notion the new structure $\Gamma - AG$- groupoid ($\Gamma - LA$-semigroup) is also defined by Shah and Rehman in [9]. Here we introduce the notion of $\Gamma - AG^*$-groupoid which is a generalization of AG^*-groupoid studied in [8], and then investigate some of their properties. Some new results on AG^*-groupoids have been recently studied by Ahmad and Mushtaqin [3, 7]. We generalize these results and investigate some

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properties of $\Gamma - AG^*$-groupoids. Following [4, 7, 9], we first recall some preliminary definitions. Let S and Γ be non-empty sets. We call S to be a Γ-Semigroup if there exists a mapping $S \times \Gamma \times S \to S$ writing (a, γ, b) by $a\gamma b$, such that S satisfies the identity $(a\gamma b)\beta c = a\gamma(b\beta c)$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$. A Γ-Semigroup with identity is called a Γ-monoid.

Let S and Γ be non-empty sets. We call S to be a $\Gamma - AG$-groupoid if there exists a mapping $S \times \Gamma \times S \to S$ writing (a, γ, b) by $a\gamma b$ such that satisfies the identity $(a\gamma b)\beta c = (c\gamma b)\beta a$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.

An element $e \in S$ is called a left identity if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$. A $\Gamma - AG$-groupoid S is called:

(i) Γ-medial if for every $a, b, c, d \in S$ and $\gamma, \beta \in \Gamma$, $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$.

(ii) Γ-paramedial if for every $a, b, c, d \in S$ and $\gamma, \beta \in \Gamma$, $(a\alpha b)\beta(c\gamma d) = (d\alpha b)\beta(c\gamma a)$.

(iii) Γ-locally associative if for every $a \in S$ and $\gamma, \beta \in \Gamma$ it satisfies $\gamma, \beta \in \Gamma$, $(a\gamma a)\beta a = a\gamma(a\beta a)$.

(iv) Γ-idempotent if for every $a \in S$ and $\gamma \in \Gamma$, $a\gamma a = a$.

In the following we recall the definitions from [3] which are applied in this paper.

Definition 1.1. A $\Gamma - AG$-groupoid S is called a $\Gamma - AG$-band if every its element is Γ-idempotent.

Definition 1.2. A $\Gamma - AG$-groupoid is called a $T^1 - \Gamma - AG$-groupoid if for every $a, b, c, d \in S$, $\gamma \in \Gamma$, $a\gamma b = c\gamma d$ implies $b\gamma a = d\gamma c$.

Definition 1.3. A $\Gamma - AG$-groupoid S is called a $\Gamma - AG - 3$-band if $a\alpha(a\beta a) = (a\gamma a)\beta a = a$, for all $a \in S$ and $\beta, \gamma \in \Gamma$.

Definition 1.4. A $\Gamma - AG$-groupoid S is called a Γ-left alternative if for all $a, b \in S$ and $\alpha, \beta \in \Gamma$, $(a\alpha a)\beta b = a\alpha(a\beta b)$.

Definition 1.5. A $\Gamma - AG$-groupoid S is called a Γ-left cancellative if for every $a, b, c \in S$ and $\gamma \in \Gamma$, $a\alpha b = a\alpha c$ implies $b = c$.

Definition 1.6. An element a of a $\Gamma - AG$-groupoid S is called a Γ-intra-regular if there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = \big(x\alpha(a\beta a)\big)\gamma y$ and S is called a Γ-intra-regular, if every element of S is an Γ-intra-regular.

Definition 1.7. A $\Gamma - AG$-subgroupoid A from a $\Gamma - AG$-groupoid S is called a Γ-interior ideal of S if $(S\Gamma A)\Gamma S \subseteq A$.

Definition 1.8. A $\Gamma - AG$-subgroupoid A from a $\Gamma - AG$-groupoid S is called a Γ-bi-ideal of S if $(A\Gamma S)\Gamma A \subseteq A$.

Definition 1.9. A $\Gamma - AG$-subgroupoid A from a $\Gamma - AG$-groupoid S is called a $\Gamma - (1, 2)$-ideal of S if $(A\Gamma S)\Gamma A^2 \subseteq A$.

We recall the three following lemmas from [4] which are applied to get some results.

Lemma 1.10. *Every $\Gamma - AG$-groupoid is Γ-medial.*

Lemma 1.11. *Every $\Gamma - AG$-groupoid with left identity is Γ-paramedial.*

Lemma 1.12. *In an $\Gamma - AG$-groupoid S with left identity, we have $a\alpha(b\beta c) = b\alpha(a\beta c)$ for every $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 1.13. *Let S be a $\Gamma - AG$-groupoid with left identity e then, $S^2_\Gamma = S\Gamma S = S$ and $S\Gamma e = e\Gamma S = S$.*

2. On $\Gamma - AG^*$-groupoids

In this section we introduce the notion of $\Gamma - AG^*$-groupoid which is a generalization of AG^*-groupoid studied in [8], and then investigate some of their properties.
An AG-groupoid S is called an AG^*-groupoid if it satisfies the identity $(ab)c = b(ac)$ for all $a, b, c \in S$.

Definition 2.1. A $\Gamma - AG$-groupoid S is called a $\Gamma - AG^*$-groupoid if it satisfies the identity $(a\gamma b)\beta c = b\gamma(a\beta c)$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.

Example 2.2. Let S be an arbitrary AG^*-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \to S$, by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a $\Gamma - AG$-groupoid (see [6]). Also for every $a, b, c \in S$ and $\gamma, \beta \in \Gamma$ we have $(a\gamma b)\beta c = (ab)\beta c = (ab)c = b(ac)$. On the other hand, $b\gamma(a\beta c) = b\gamma(ac) = b(ac)$. Hence, $(a\gamma b)\beta c = b\gamma(a\beta c)$ and then S is a $\Gamma - AG^*$-groupoid.

Example 2.3. Let S be an arbitrary $\Gamma - AG^*$-groupoid and γ a fixed element in Γ. We define $a \circ b = a\gamma b$ for every $a, b \in S$. Then $(S, \circ)$ is an AG^*-groupoid (see [6]). Also for every $a, b, c \in S$ and $\gamma \in \Gamma$, we have $(a \circ b) \circ c = (a\gamma b) \circ c = (a\gamma b)\gamma c = b\gamma(a\gamma c)$. On the other hand, $b \circ (a \circ c) = b \circ (a\gamma c) = b\gamma(a\gamma c)$. Hence, $(a \circ b) \circ c = b \circ (a \circ c)$. Therefore, S is an AG^*-groupoid.

Example 2.4. Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. If $a\gamma b$ denotes as usual multiplication of integers for $a, b \in S$ and $\gamma \in \Gamma$, then S is a $\Gamma - AG^*$-groupoid but not an AG^*-groupoid.

Example 2.5. Let S be the set of all integers of the form $4n + 1$ where n is an integer and Γ denote the set of all integers of the form $4n + 3$. If $a\gamma b$ is $a + \gamma + b$, for all $a, b \in S$ and $\gamma \in \Gamma$, then S is a $\Gamma - AG^*$-groupoid but not an AG^*-groupoid.

Note that by Examples 2.4 and 2.5, $\Gamma - AG^*$-groupoids are a generalization of AG^*-groupoids.

Lemma 2.6. *In every $\Gamma - AG^*$-groupoid, we have $b\gamma(a\beta c) = b\gamma(c\beta a)$ for every $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.*

Proof. Let S be a $\Gamma - AG^*$-groupoid. We have
$(a\gamma b)\beta c = (c\gamma b)\beta a$, (by left invertive law)
$(a\gamma b)\beta c = b\gamma(a\beta c)$, (by $\Gamma - AG^*$-groupoid)
$(a\gamma b)\beta c = b\gamma(c\beta a)$, (by $\Gamma - AG^*$-groupoid)
Then, $b\gamma(a\beta c) = b\gamma(c\beta a)$. □

Lemma 2.7 ([5]). *In a $\Gamma - AG$-groupoid S with a left identity, we have $a\alpha b = a\beta b$ for every $a, b \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 2.8. *Every $\Gamma - AG^*$-groupoid with a left identity is a commutative Γ-semigroup.*

Proof. Let S be a $\Gamma - AG^*$-groupoid with a left identity e. Then by Lemma 2.6, we have $b\gamma(a\beta c) = b\gamma(c\beta a)$ for every $a, b, c \in S$ and $\gamma, \beta \in \Gamma$. Now putting $b = e$, it follows that $e\gamma(a\beta c) = e\gamma(c\beta a)$ and then $a\beta c = c\beta a$ for every $a, c \in S$ and $\beta \in \Gamma$. Therefore, S is commutative. Now since S is commutative, we obtain,

$$\begin{aligned}(a\gamma b)\beta c &= (b\gamma a)\beta c \text{ (by commutativity)}\\ &= a\gamma(b\beta c)(by\Gamma - AG^* - groupoid).\end{aligned}$$

Hence, S is a Γ-semigroup. □

Theorem 2.9 ([5])**.** *If a $\Gamma - AG$-band S contains a left identity e, then S becomes a commutative Γ-monoid.*

Proposition 2.10 ([5])**.** *Every $T^1 - \Gamma - AG$-groupoid is Γ-paramedial.*

Proposition 2.11 ([8])**.** *If S is an AG^*-groupoid, then for every $x_1, x_2, x_3, x_4 \in S$, we have $(x_1x_2)(x_3x_4) = (x_{P(1)}x_{P(2)})(x_{P(3)}x_{P(4)})$ where P is any permutation on the set $\{1, 2, 3, 4\}$.*

Note that Proposition 2.11 can be generalized for $\Gamma - AG^*$-groupoids with left identity. As for $\Gamma - AG^*$-groupoids without left identity we have:

Proposition 2.12. *If S is a $\Gamma - AG^*$-groupoid, then for every $x_1, x_2, x_3, x_4 \in S$ and $\alpha, \beta, \gamma \in \Gamma$, $(x_1\alpha x_2)\beta(x_3\gamma x_4) = (x_1\alpha x_{P(2)})\beta(x_{P(3)}\gamma x_{P(4)})$ where P is any permutation on the set $\{2, 3, 4\}$.*

Proof. Let x_1, x_2, x_3, x_4 be arbitrary elements of S. Then we have
$(x_1\alpha x_2)\beta(x_3\gamma x_4) = (x_1\alpha x_3)\beta(x_2\gamma x_4)$ (by Γ-medial law)
$(x_1\alpha x_3)\beta(x_4\gamma x_2)$ (by Lemma 2.6)
$(x_1\alpha x_2)\beta(x_4\gamma x_3) = (x_1\alpha x_4)\beta(x_2\gamma x_3) = (x_1\alpha x_4)\beta(x_3\gamma x_2)$ (by Lemma 2.6. and Γ-medial law). □

Proposition 2.13. *Let S be a $\Gamma - AG^*$-groupoid. Then S is a commutative Γ-semigroup if any of the following holds:*

(i) $a\alpha b = c\alpha d \Rightarrow a\alpha d = b\alpha c$,

(ii) $a\alpha b = c\alpha d \Rightarrow d\alpha a = c\alpha b$,

for every $a, b, c \in S$ and $\alpha \in \Gamma$.

Proof. Since for every $a, b \in S$ and $\alpha \in \Gamma$ the equation $a\alpha b = a\alpha b$ trivially holds, an application of (i) or (ii) proves commutatively. So for every $a, b, c \in S$ and $\alpha, \gamma \in \Gamma$, we get

$$(a\alpha b)\gamma c = b\alpha(a\gamma c) = (a\gamma c)\alpha b = (b\gamma c)\alpha a = a\alpha(b\gamma c).$$

Hence, S is a commutative Γ-semigroup. □

Lemma 2.14. *If S is a $\Gamma - AG^*$-band, then $S = S^2_\Gamma$, where $S^2_\Gamma = S\Gamma S$.*

Proof. By the definition of $\Gamma - AG$-groupoid we have $S^2_\Gamma = S\Gamma S \subseteq S$. Let S be a $\Gamma - AG^*$-band. For every $x \in S$, $x = x\gamma x \in S\Gamma S$ for every $\gamma \in \Gamma$. Therefore, $S \subseteq S\Gamma S = S^2_\Gamma$. Hence, $S = S^2_\Gamma$. □

3. properties of $\Gamma - AG^*$-groupoids

In this section we investigate some important properties of $\Gamma - AG^*$-groupoids.

Proposition 3.1. *Every $\Gamma - AG^*$-groupoid is a Γ-left alternative.*

Proof. Let S be a $\Gamma - AG^*$-groupoid and let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Then by the definition of $\Gamma - AG^*$-groupoid we have

$$(a\alpha b)\beta c = b\alpha(a\beta c). \tag{3.1}$$

Now replacing b by a in (3.1), we have $(a\alpha a)\beta c = a\alpha(a\beta c)$. Hence, S is a Γ-left alternative. □

Theorem 3.2. *A $\Gamma - AG^*$-groupoid having a Γ-left cancellative element is $T^1 - \Gamma - AG$-groupoid.*

Proof. Let x be a Γ-left cancellative element of a $\Gamma-AG^*$-groupoid S and $a,b,c,d \in S$, and $\alpha,\gamma \in \Gamma$. Let $a\alpha b = c\alpha d$. Then

$$\begin{aligned} x\gamma(b\alpha a) &= (b\gamma x)\alpha a (\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= (a\gamma x)\alpha b (\text{by left invertive law}) \\ &= x\gamma(a\alpha b)(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= x\gamma(c\alpha d)(\text{by assumption}) \\ &= (c\gamma x)\alpha d(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= (d\gamma x)\alpha c(\text{by left invertive law}) \\ &= x\gamma(d\alpha c)(\text{by } \Gamma-AG^*\text{-groupoid}), \end{aligned}$$

$$\begin{aligned} &\Rightarrow x\gamma(b\alpha a) = x\gamma(d\alpha c) \\ &\Rightarrow b\alpha a = d\alpha c.(\text{by left cancellative law}). \end{aligned}$$

Hence, S is a $T^1-\Gamma-AG$-groupoid. □

Corollary 3.3. *A $\Gamma-AG^*$-groupoid S having a Γ-left cancellative element is Γ-paramedical.*

Proof. By Proposition 2.10 and Theorem 3.2, the result is immediate. □

Proposition 3.4. *Let S be a $\Gamma-AG^*$-groupoid. If $t \in S$ is a $\Gamma-3$-band, i.e. $(t\beta t)\gamma t = t\beta(t\gamma t) = t$ for every $\gamma,\beta \in \Gamma$, then t^2_γ is a Γ-idempotent, where $t^2_\gamma = t\gamma t$, $\gamma \in \Gamma$.*

Proof. Let $t \in S$ be a $\Gamma-3$-band. Then,

$$\begin{aligned} t^2_\gamma \beta t^2_\gamma = (t\gamma t)\beta(t\gamma t) &= t\gamma\big(t\beta(t\gamma t)\big)(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= t\gamma t = t^2_\gamma.(\text{by } \Gamma-3\text{-band}). \end{aligned}$$

Hence, t^2_γ is a Γ-idempotent. □

Theorem 3.5. *A $\Gamma-AG^*$-groupoid S having a Γ-left cancellative square element is a $T^1-\Gamma-AG$-groupoid.*

Proof. Suppose x is a Γ-left cancellative square element of S and $a\alpha b = c\alpha d$ for every $a,b,c,d \in S$ and $\alpha,\gamma \in \Gamma$. Then it gives the following:

$$\begin{aligned} (x\alpha x)\gamma(b\alpha a) &= \big(b\gamma(x\alpha x)\big)\alpha a(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= \big((x\gamma b)\alpha x\big)\alpha a(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= (a\alpha x)\alpha(x\gamma b)(\text{by left invertive law}) \\ &= (a\alpha b)\alpha(x\gamma x)(\text{by Proposition 2.12}) \\ &= (c\alpha d)\alpha(x\gamma x)(\text{by assumption}) \\ &= (c\alpha x)\alpha(x\gamma d)(\text{by Proposition 2.12}) \\ &= \big((x\gamma d)\alpha x\big)\alpha c(\text{by left invertive law}) \\ &= \big(d\gamma(x\alpha x)\big)\alpha c(\text{by } \Gamma-AG^*\text{-groupoid}) \\ &= (x\alpha x)\gamma(d\alpha c)(\text{by } \Gamma-AG^*\text{-groupoid}) \end{aligned}$$

i.e. $(x\alpha x)\gamma(b\alpha c) = (x\alpha x)\gamma(d\alpha c)$, so $b\alpha a = d\alpha c$. (by left cancellative law)
Hence, S is a $T^1-\Gamma-AG$-groupoid. □

Proposition 3.6. *Every $T^1-\Gamma-AG$-groupoid is Γ-paramedial.*

Proof. Let S be a $T^1-\Gamma-AG$-groupoid and let $a,b,c,d\in S$ and $\alpha,\beta,\gamma\in\Gamma$. Now we have

$$\begin{aligned}&(a\alpha b)\gamma(c\beta d)=(a\alpha c)\gamma(b\beta d)(\text{by Lemma2.6})\\ &\Rightarrow (c\beta d)\gamma(a\alpha b)=(b\beta d)\gamma(a\alpha c)(by T^1-\Gamma-AG-groupoid)\\ &\Rightarrow (c\beta d)\gamma(a\alpha b)=(b\beta a)\gamma(d\alpha c)(\text{by Lemma2.6})\\ &\Rightarrow (a\alpha b)\gamma(c\beta d)=(d\alpha c)\gamma(b\beta a)(by T^1-\Gamma-AG-groupoid)\\ &\Rightarrow (a\alpha b)\gamma(c\beta d)=(d\alpha b)\gamma(c\beta a).(\text{by Lemma2.6})\end{aligned}$$

Hence, S is Γ-paramedial. □

Corollary 3.7. *Every Γ-left cancellative $\Gamma-AG^*$-groupoid is Γ-paramedial.*

Proof. By Proposition 2.10 and Theorem 3.2, the result is immediate. □

Lemma 3.8. *Let S be a $\Gamma-AG^*$-groupoid such that $S\Gamma a=S$, hold for all $a\in S$, then $a\Gamma S=S$ and $S\Gamma S=S$.*

Proof. Since $S\Gamma a=S$ for all $a\in S$ then $a\Gamma S=S$, $S=S\Gamma S=(S\Gamma a)\Gamma S=a\Gamma(S\Gamma S)=a\Gamma S$. Hence $a\Gamma S=S$. □

Theorem 3.9. *A $\Gamma-AG^*$-groupoid is a Γ-intra-regular if $S\Gamma a=S$ holds for all $a\in S$.*

Proof. Since $S\Gamma a=S$ by lemma 3.8 $S\Gamma S=S$. Let $a\in S$, then

$$\begin{aligned}&\big((S\Gamma a)\Gamma(S\Gamma a)\big)\Gamma S(\text{by } \Gamma-AG^*\text{-groupoid})\\ &\big((S\Gamma S)\Gamma(a\Gamma a)\big)\Gamma S(\text{by mediallaw})\\ &(S\Gamma a^2_\Gamma)\Gamma S.\end{aligned}$$

Hence $a\in(S\Gamma a^2_\Gamma)\Gamma S$, i.e. S is Γ-intra-regular. □

Theorem 3.10. *Let S be a Γ-intra-regular of $\Gamma-AG^*$-groupoid, then B is a right Γ-ideal of S if $B\Gamma S=B$.*

Proof. Let B be a right Γ-ideal i.e. $B\Gamma S\subseteq B$. Now let $b\in B$, then by assumption there exists $x,y\in S$ and $\alpha,\beta,\gamma\in\Gamma$ such that

$$\begin{aligned}b&=\big(x\alpha(b\beta b)\big)\gamma y\ \ (\text{by }\Gamma\text{-intra-regular})\\ &(b\beta b)\alpha(x\gamma y)\in B\Gamma S\ \ (\text{by}\Gamma-AG^*\text{-groupoid}),\end{aligned}$$

implies $B\subseteq B\Gamma S$. Hence $B\Gamma S=B$. □

Theorem 3.11. *If S is a Γ-intra-regular of $\Gamma-AG^*$-groupoid then $(S\Gamma B)\Gamma S=B$, where B is a Γ-interior ideal of S.*

Proof. Since B is a Γ-interior ideal of S also $(S\Gamma B)\Gamma S\subseteq B$, let $b\in B$, we have

$$\begin{aligned}b&=\big(x\alpha(b\beta b)\big)\gamma y\ \ (\text{by }\Gamma\text{-intra-regular})\\ &\big((b\alpha x)\beta b\big)\gamma y\ \ (\text{by }\Gamma-AG^*\text{-groupoid}).\end{aligned}$$

Since $\beta\subseteq S$ implies $b\alpha x\in S$. Hence $b\in(S\Gamma B)\Gamma S$. □

Theorem 3.12. *In an* Γ*-intra-regular of* $\Gamma - AG^*$*-groupoid* S*, if* A *is a* Γ*-interior ideal of* S *then is a* $\Gamma(1,2)$*-ideal of* S*.*

Proof. let A be a Γ-interior ideal of, i.e. $(S\Gamma A)\Gamma S \subseteq A$. Let $\rho \in (A\Gamma S)\Gamma(A\Gamma A)$, then $\rho = (a\mu s)\theta(b\alpha c)$ for some $a, b, c \in A$, $s \in S$ and $\mu, \theta, \alpha \in \Gamma$.
Since S is Γ-intra-regular so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = \big(x\beta(a\delta a)\big)\gamma y$. Now we have

$$\begin{aligned}
\rho = (a\mu s)\theta(b\alpha c) &= \big((b\alpha c)\mu s\big)\theta a \quad \text{(by } \Gamma\text{-leftinvertive)} \\
&= \big((b\alpha c)\mu s\big)\theta\Big(\big(x\beta(a\delta a)\big)\Big)\gamma y \quad \text{(by } \Gamma\text{-intra-regular)} \\
&= \big(x\beta(a\delta a)\big)\theta\big((b\alpha c)\mu s\big)\gamma y \quad \text{(by } \Gamma - AG^*\text{-groupoid)} \\
&= \big(x\beta(a\delta a)\big)\theta\big((s\alpha c)\mu b\big)\gamma y \quad \text{(by } \Gamma\text{-leftinvertive)} \\
&= \Big(\big((s\alpha c)\theta\big(x\beta(a\delta a)\big)\big)\mu b\Big) \quad \text{(by } \Gamma - AG^*\text{-groupoid),}
\end{aligned}$$

obviously $\big((s\alpha c)\theta\big(x\beta(a\delta a)\big)\big) \in S$, as S is $\Gamma - AG^*$-subgroupoid and A is $\Gamma - AG$-subgroupoid, therefore $\rho \in (S\Gamma A)\Gamma S \subseteq A$ i.e. $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Hence A is a $\Gamma - (1,2)$-ideal of S. □

Theorem 3.13. *In a* Γ*-interior ideal of* $\Gamma - AG^*$*-subgroupoid* S *if* A *is a* Γ*-interior ideal of* S *then* A *is a* $\Gamma - bi$*-ideal of* S*.*

Proof. Let A be a Γ-interior ideal of S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $\rho \in (A\Gamma S)\Gamma A$, then $\rho = (a\mu s)\psi b$ for some $a, b \in A$, $s \in S$ and $\mu, \psi \in \Gamma$. Since S is Γ-intra-regular so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $b = \big(x(b\delta b)\big)\gamma y$. Now we have

$$\begin{aligned}
\rho &= (a\mu s)\psi\big(x\beta(b\delta b)\big)\gamma y \quad \text{(by } \Gamma\text{-leftinvertive)} \\
&= \big(x(b\delta b)\big)\psi(a\mu s)\gamma y \quad \text{(by } \Gamma - AG^*\text{-groupoid)} \\
&= \big((b\delta b)\beta\big(x\psi(a\mu s)\big)\big)\gamma y \quad \text{(by } \Gamma - AG^*\text{-groupoid)} \\
&= \Big(\Big(\big(x\psi(a\mu s)\big)\delta b\Big)\beta b\Big)\gamma y \in (S\Gamma A)\Gamma S \subseteq A \quad \text{(by } \Gamma\text{-leftinvertive).}
\end{aligned}$$

Thus $(A\Gamma S)\Gamma A \subseteq A$. Hence A is a $\Gamma - bi$-ideal of S. □

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