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Generalized essential maps and coincidence type theory for compact multifunctions

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Abstract

In this paper we discuss generalized essential maps. By establishing a very simple result we are able to present a variety of topological transversality theorems in a general setting

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1. Introduction

The topological transversality theorem [4] for continuous compact maps states that for continuous compact maps F and G with $F \cong G$ then F is essential if and only if G is essential. The essential map theory was extended to set valued maps and to d–essential maps [6–8]. In this paper we consider admissible maps (see below) and we establish a very general topological transversality theorem. To do this we first present a very simple result which we will then use to establish topological transversality theorems in a variety of settings.

Let X, Y be metric spaces and Γ paracompact. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic (with respect to the Čech cohomology functor),
(ii). p is a perfect map i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all admissible pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p, q) \sim (p', q')$ if there a homeomorphism $f : \Gamma \to \Gamma'$ such that $p' \circ f = p$ and $q' \circ f = q$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

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or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. Note if $(p,q), (p_1,q_1) \in D(X,Y)$ (where $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma' \overset{q_1}{\to} Y$) and $(p,q) \sim (p_1,q_1)$ then it is easy to see that for $x \in X$ we have $q_1(p_1^{-1}(x)) = q(p^{-1}(x))$. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. Let $\phi \in M(X,Y)$ and (p,q) a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q(p^{-1}(X))$. Note $\phi(X)$ does not depend on the representative of ϕ. Now $\phi \in M(X,Y)$ is called compact provided the set $\phi(X)$ is relatively compact in Y. We say a map ϕ is admissible or determined by a morphism $\{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\}$ provided $\phi(x) = q\,p^{-1}(x)$ for any $x \in X$ and we write $\phi \in Adm(X,Y)$ (note ϕ is upper semicontinuous) i.e., $Adm(X,Y)$ denotes the class of all admissible set-valued maps $\phi : X \to 2^Y$ (note a set-valued map $\phi : X \to 2^Y$ is admissible if it is represented by an admissible pair). Let U be open in X and let $F, G \in Adm_{\partial U}(\overline{U},X)$ (i.e., $F, G \in Adm(\overline{U},X)$ with $x \notin F(x)$, $x \notin G(x)$ for $x \in \partial U$) be compact maps. We say $F \cong G$ (compactly) in $Adm_{\partial U}(\overline{U},X)$ if there exists a (compact) admissible $\Psi : \overline{U} \times [0,1] \to 2^X$ with $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$, $\Psi_0 = F$ and $\Psi_1 = G$ (here $\Psi_t(x) = \Psi(x,t)$). Note $\cong$ (compactly) in $Adm_{\partial U}(\overline{U},X)$ is an equivalence relation; see [3, Section 46], [5, Section 5]. Suppose $F \in Adm_{\partial U}(\overline{U},X)$ is a compact map and $f : \overline{U} \to X$ is a single valued continuous compact map with $x \neq f(x)$ for $x \in \partial U$. For a condition (clearly satisfied if f is the zero map) to guarantee that $F \cong f$ (compactly) in $Adm_{\partial U}(\overline{U},X)$ see [3, (Section 46), Proposition 46.3].

2. Topological Transversality Theorem

We will consider classes **A** and **B** of maps. Let E be a completely regular space and U an open subset of E.

Definition 2.1. We say $F \in A(\overline{U},E)$ if $F \in \mathbf{A}(\overline{U},E)$ and $F : \overline{U} \to K(E)$ is a upper semicontinuous (u.s.c.) compact map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Remark 2.2. Examples of $F \in \mathbf{A}(\overline{U},E)$ might be that F has convex values or F has acyclic values or F is admissible (as described in Section 1).

In this paper we fix a $\Phi \in B(\overline{U},E)$ (i.e., $\Phi \in \mathbf{B}(\overline{U},E)$ and $\Phi : \overline{U} \to K(E)$ is a u.s.c. map).

Definition 2.3. We say $F \in A_{\partial U}(\overline{U},E)$ if $F \in A(\overline{U},E)$ and $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. Let $F, G \in A_{\partial U}(\overline{U},E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U},E)$ if there exists a u.s.c. compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi \in \mathbf{A}(\overline{U} \times [0,1],E)$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $\Psi_t(x) = \Psi(x,t)$), $\Psi_0 = F$ and $\Psi_1 = G$. In addition here we always assume for any map $\Theta \in \mathbf{A}(\overline{U} \times [0,1],E)$ and any maps $g \in \mathbf{C}(\overline{U},\overline{U} \times [0,1])$ and $f \in \mathbf{C}(\overline{U} \times [0,1],\overline{U} \times [0,1])$ then $\Theta \circ g \in \mathbf{A}(\overline{U},E)$ and $\Theta \circ f \in \mathbf{A}(\overline{U} \times [0,1],E)$; here **C** denotes the class of single valued continuous functions.

Remark 2.5.

(a). In our results below alternatively we could use the following definition for $\cong$ in $A_{\partial U}(\overline{U},E)$: $F \cong G$ in $A_{\partial U}(\overline{U},E)$ if there exists a u.s.c. compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,,\eta(\,.\,)) \in \mathbf{A}(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $\Psi_t(x) = \Psi(x,t)$), $\Psi_0 = F$ and $\Psi_1 = G$. [Note the additional assumption in Definition 2.4 is not needed here].

(b). Throughout the paper we assume $\cong$ in $A_{\partial U}(\overline{U},E)$ is a reflexive, symmetric relation.

Remark 2.6. Let $F \in A_{\partial U}(\overline{U},E)$. We say F is Φ–essential in $A_{\partial U}(\overline{U},E)$ if for every map $J \in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U},E)$ there exists a $x \in U$ with $\Phi(x) \cap J(x) \neq \emptyset$.

We now present a simple result which will more or less immediately yield a very general topological transversality theorem.

Theorem 2.7. *Let E be a completely regular topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$. Also suppose*

$$\begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \text{ and} \\ J \cong F \text{ in } A_{\partial U}(\overline{U}, E) \text{ we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E). \end{cases} \tag{2.1}$$

Then F is essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. From (2.1) there exists a u.s.c. compact map $H^J : \overline{U} \times [0, 1] \to K(E)$ with $H^J \in \mathbf{A}(\overline{U} \times [0, 1], E)$, $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ (here $H^J_t(x) = H^J(x, t)$), $H^J_0 = G$ and $H^J_1 = J$. Let

$$K = \left\{x \in \overline{U} : \ \Phi(x) \cap H^J(x, t) \neq \emptyset \ \text{ for some } \ t \in [0, 1]\right\}$$

and

$$D = \left\{(x, t) \in \overline{U} \times [0, 1] : \ \Phi(x) \cap H^J(x, t) \neq \emptyset\right\}.$$

Now $D \neq \emptyset$ (note G is Φ–essential in $A_{\partial U}(\overline{U}, E)$) and D is closed (note Φ and H^J are u.s.c.) and so D is compact (note H^J is a compact map). Let $\pi : \overline{U} \times [0, 1] \to \overline{U}$ be the projection. Now $K = \pi(D)$ is closed (see Kuratowski's theorem [2, pp 126]) and so in fact compact (recall projections are continuous). Also note $K \cap \partial U = \emptyset$ (since $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$) so since E is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Define the map R by $R(x) = H^J(x, \mu(x))$. Now $R \in A_{\partial U}(\overline{U}, E)$ (note $H^J(x, \mu(x)) = H^J \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0, 1]$ is given by $g(x) = (x, \mu(x))$) with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x, 0) = G(x)$ and so $R(x) \cap \Phi(x) = G(x) \cap \Phi(x)$). We now show $R \cong G$ in $A_{\partial U}(\overline{U}, E)$. To see this let $Q : \overline{U} \times [0, 1] \to K(E)$ be given by $Q(x, t) = H^J(x, t\,\mu(x)) = H^J \circ f(x, t)$ where $f : \overline{U} \times [0, 1] \to \overline{U} \times [0, 1]$ is given by $f(x, t) = (x, t\,\mu(x))$. Note $Q \in \mathbf{A}(\overline{U} \times [0, 1], E)$, $Q_0 = G$, $Q_1 = R$ and $\Phi(x) \cap Q_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ (since if $t \in (0, 1)$ and $x \in \partial U$ then $\Phi(x) \cap H^J(x, t\,\mu(x)) = \Phi(x) \cap H^J_{t\,\mu(x)}(x)$ so $x \in K$ and as a result $\mu(x) = 1$ i.e., $\Phi(x) \cap H^J(x, t\,\mu(x)) = \Phi(x) \cap H^J(x, t)$). Thus $R \cong G$ in $A_{\partial U}(\overline{U}, E)$. Since G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $\Phi(x) \cap R(x) \neq \emptyset$ (i.e., $\Phi(x) \cap H^J_{\mu(x)}(x) \neq \emptyset$). Thus $x \in K$, $\mu(x) = 1$ and so $\emptyset \neq \Phi(x) \cap H^J_1(x) = \Phi(x) \cap J(x)$. □

Remark 2.8.

(i). In the proof of Theorem 2.7 it is simple to adjust the proof if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5 if we note $R(\,.\,) = H^J(\,.\,, \mu(\,.\,))$ and $Q(\,.\,, \nu(\,.\,)) = H^J(\,.\,, \nu(\,.\,)\,\mu(\,.\,)) = H^J(\,.\,, w(\,.\,))$ (with $w(\,.\,) = \nu(\,.\,)\,\mu(\,.\,)$) for any continuous $\nu : \overline{U} \to [0, 1]$ with $\nu(\partial U) = 0$ (note $w : \overline{U} \to [0, 1]$ is continuous and $w(\partial U) = 0$).

(ii). One could replace u.s.c. in the definition of $A(\overline{U}, E)$, $B(\overline{U}, E)$, Definition 2.4 and Remark 2.5 with any condition that guarantees that K in the proof of Theorem 2.7 is closed; this is all that is needed if E is normal. If E is Tychonoff and not normal the one can also replace the compactness of the map in $A(\overline{U}, E)$, Definition 2.4 and Remark 2.5 with any condition that guarantees that K in the proof of Theorem 2.7 is compact.

Example 2.9. Theorem 2.7 immediately yields a general Leray–Schauder type alternative for coincidences. Let E be a completely metrizable locally convex space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ and $\Phi(x) \cap [t\,F(x) + (1 - t)\,G(x)] = \emptyset$ for $x \in \partial U$ and $t \in (0, 1)$. For any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ suppose $H^J \in \mathbf{A}(\overline{U} \times [0, 1], E)$ where $H^J(x, t) = t\,J(x) + (1 - t)\,G(x)$ [Also here we assume for any map $\Theta \in \mathbf{A}(\overline{U} \times [0, 1], E)$ and any maps $g \in \mathbf{C}(\overline{U}, \overline{U} \times [0, 1])$ and $f \in \mathbf{C}(\overline{U} \times [0, 1], \overline{U} \times [0, 1])$ then $\Theta \circ g \in \mathbf{A}(\overline{U}, E)$ and $\Theta \circ f \in \mathbf{A}(\overline{U} \times [0, 1], E)$]. Then F is Φ–essential in $A_{\partial U}(\overline{U}, E)$.

The proof follows from Theorem 2.7 since topological vector spaces are completely regular and note if $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ then with $H^J(x,t) = t\,J(x) + (1-t)\,G(x)$ note $H^J_0 = G$, $H^J_1 = J$, $H^J : \overline{U} \times [0,1] \to K(E)$ is a u.s.c. compact (see [1, Theorem 4.18]) map, $H^J \in \mathbf{A}(\overline{U} \times [0,1], E)$ and $\Phi(x) \cap H^J_t(x) = \emptyset$ for $x \in \partial U$ and $t \in (0,1)$ (if $x \in \partial U$ and $t \in (0,1)$ then since $J|_{\partial U} = F|_{\partial U}$ we note that $\Phi(x) \cap H^J_t(x) = \Phi(x) \cap [t\,F(x) + (1-t)\,G(x)]$) so as a result $G \cong J$ (Definition 2.4) in $A_{\partial U}(\overline{U}, E)$ (i.e., (2.1) holds). [Note E being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space E which has the property that the closed convex hull of a compact set in E is compact. In fact it is easy to see, if we argue differently, that all we need to assume is that E is a topological vector space].

We now present the topological transversality theorem in a general setting. Assume

$$\cong \text{ in } A_{\partial U}(\overline{U}, E) \text{ is an equivalence relation.} \tag{2.2}$$

Theorem 2.10. *Let* E *be a completely regular topological space,* U *an open subset of* E *and assume* (2.2) *holds. Suppose* F *and* G *are two maps in* $A_{\partial U}(\overline{U}, E)$ *with* $F \cong G$ *in* $A_{\partial U}(\overline{U}, E)$*. Then* F *is* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$ *if and only if* G *is* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$*.*

Proof. Assume G is Φ–essential in $A_{\partial U}(\overline{U}, E)$. To show F is Φ–essential in $A_{\partial U}(\overline{U}, E)$ let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ then (2.2) guarantees that $G \cong J$ in $A_{\partial U}(\overline{U}, E)$ i.e., (2.1) holds. Then Theorem 2.7 guarantees that F is Φ–essential in $A_{\partial U}(\overline{U}, E)$. A similar argument shows that if F is Φ–essential in $A_{\partial U}(\overline{U}, E)$ then G is Φ–essential in $A_{\partial U}(\overline{U}, E)$. □

Assume (2.2) holds. If F and G are maps in $A_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$ is $F \cong G$ in $A_{\partial U}(\overline{U}, E)$? We will discuss this now.

We assume the following conditions:

$$E \text{ is a (Hausdorff) topological vector space and } U \text{ is convex} \tag{2.3}$$

$$\text{there exists a retraction } r : \overline{U} \to \partial U \tag{2.4}$$

and

$$\begin{cases} \text{for any map } \Theta \in \mathbf{A}(\overline{U}, E) \text{ and } f \in \mathbf{C}(\overline{U} \times [0,1], \overline{U}) \\ \text{then } \Theta \circ f \in \mathbf{A}(\overline{U} \times [0,1], E). \end{cases} \tag{2.5}$$

Remark 2.11. Note topological vector spaces are completely regular. Also if E is an infinite dimensional Banach space and U is convex then (2.4) holds. Also note if **A** is closed under composition then (2.5) holds.

Let r be in (2.4) and let F and G be maps in $A_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$. Consider the map $F^\star$ given by $F^\star(x) = F(r(x))$ for $x \in \overline{U}$. Note $F^\star(x) = G(r(x))$ for $x \in \overline{U}$ since $F|_{\partial U} = G|_{\partial U}$. Let

$$H(x,\lambda) = G(2\lambda\, r(x) + (1-2\lambda)\, x) = G \circ j\,(x,\lambda) \quad \text{for } (x,\lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

(here $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ (note $\overline{U}$ is convex) is given by $j(x,\lambda) = 2\lambda\, r(x) + (1-2\lambda)\, x$). Now $H : \overline{U} \times \left[0, \frac{1}{2}\right] \to K(E)$ is a u.s.c. compact map. Also from (2.5) note $H \in \mathbf{A}(\overline{U} \times \left[0, \frac{1}{2}\right], E)$ with $\Phi(x) \cap H_\lambda(x) = \emptyset$ for $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ (note if $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ then since $r(x) = x$ we have $\Phi(x) \cap H_\lambda(x) = \Phi(x) \cap G(x)$). Thus $G \cong F^\star$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4). Similarly with

$$Q(x,\lambda) = \Phi((2-2\lambda)\, r(x) + (2\lambda - 1)\, x) \quad \text{for } (x,\lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

we see that $F^\star \cong F$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4). Combining gives $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4).

In this situation we could replace Definition 2.6 with:

Definition 2.12. Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $\Phi(x) \cap J(x) \neq \emptyset$.

Now from Theorem 2.7 (in fact here the argument would be shorter since the map Q is not needed and the assumption $\Theta \circ f \in \mathbf{A}(\overline{U} \times [0,1], E)$ is not needed in Definition 2.4) and Theorem 2.10 we have:

Theorem 2.13. *Let E be a topological vector space, U an open convex subset of E and assume (2.2), (2.4) and (2.5) hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (as in Definition 2.4). Then F is Φ–essential (Definition 2.12) in $A_{\partial U}(\overline{U}, E)$ if and only if G is Φ–essential (Definition 2.12) in $A_{\partial U}(\overline{U}, E)$.*

Remark 2.14.

(i). Suppose (2.4) and (2.5) hold and in addition assume

$$\begin{cases} \text{for any map } \Theta \in \mathbf{A}(\overline{U}, E) \text{ then } \Theta(\,.\,,\eta(\,.\,)) = \Theta \circ f(\,.\,,\eta(\,.\,)) \in \mathbf{A}(\overline{U}, E) \\ \text{for any continuous function } \eta : \overline{U} \to [0,1] \text{ with } \eta(\partial U) = 0 \text{ where} \\ f(x,t) = t\, r(x) + (1-t)\, x,\ t \in [0,1],\ x \in \overline{U}. \end{cases} \tag{2.6}$$

Let F and G be maps in $A_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$. It is simple to adjust the proof above (use (2.6) instead of (2.5)) to establish $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (as in Remark 2.5). As a result we get immediately Theorem 2.13 (with (2.5) replaced by (2.6) and $\cong$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4) replaced by $\cong$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5)).

(ii). Let F and G be maps in $A_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$. Assume the following conditions:

$$E \text{ is a completely metrizable locally convex space} \tag{2.7}$$

$$\Phi(x) \cap [t\, F(x) + (1-t)\, G(x)] = \emptyset \text{ for } x \in \partial U \text{ and } t \in (0,1) \tag{2.8}$$

and

$$\begin{cases} \eta(\,.\,)\, F(\,.\,) + (1-\eta(\,.\,))\, G(\,.\,) \in \mathbf{A}(\overline{U}, E) \text{ for any} \\ \text{continuous function } \eta : \overline{U} \to [0,1] \text{ with } \eta(\partial U) = 0. \end{cases} \tag{2.9}$$

Let $H(x,\lambda) = \lambda\, F(x) + (1-\lambda)\, G(x)$ for $(x,\lambda) \in \overline{U} \times [0,1]$. Note $H : \overline{U} \times [0,1] \to K(E)$ is a u.s.c. compact (see [1, Theorem 4.18]) map and by (2.9) note $H(\,.\,,\eta(\,.\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$, and from (2.8) note $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in \partial U$ and $t \in (0,1)$ so as a result $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5). [Note (2.7) can be replaced by any topological vector space E which has the property that the closed convex hull of a compact set in E is compact]. As a result in this setting we get immediately Theorem 2.13 (with (2.3), (2.4), (2.5) replaced by (2.7), (2.8), (2.9) and $\cong$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4) replaced by $\cong$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5)).

Now we present an example of a Φ–essential (Definition 2.12) map.

Example 2.15. Let E be a (Hausdorff) topological space, U an open subset of E, $\Phi \in B(E,E)$ (i.e., $\Phi \in \mathbf{B}(E,E)$ and $\Phi : E \to K(E)$ is a u.s.c. map) and $F \in A_{\partial U}(\overline{U}, E)$. Assume the following conditions hold:

$$\text{there exists a } x \in \overline{U} \text{ with } \Phi(x) \cap \{0\} \neq \emptyset \tag{2.10}$$

$$\text{there exists a retraction } r : E \to \overline{U} \tag{2.11}$$

$$\Phi(x) \cap \lambda\, F(x) = \emptyset \text{ for } x \in \partial U \text{ and } \lambda \in (0,1) \tag{2.12}$$

$$\begin{cases} \text{for any continuous map } \mu : E \to [0,1] \text{ with } \mu(E \setminus U) = 0 \\ \text{and any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \\ \text{there exists a } w \in E \text{ with } \Phi(w) \cap \mu(w)\, J(r(w)) \neq \emptyset \end{cases} \tag{2.13}$$

and

$$\text{there is no } z \in E \setminus U \text{ with } \Phi(z) \cap \{0\} \neq \emptyset. \tag{2.14}$$

Then F is Φ–essential (Definition 2.12) in $A_{\partial U}(\overline{U},E)$.

To see this let $J \in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$. Now let

$$K = \left\{x \in \overline{U} : \ \Phi(x) \cap \lambda\, J(x) \neq \emptyset \ \text{ for some } \ \lambda \in [0,1]\right\}.$$

Now $K \neq \emptyset$ (see (2.10)) is compact and $K \subseteq \overline{U}$. In fact $K \subseteq U$ from (2.12) (note if $x \in \partial U$ and $x \in K$ then for some $\lambda \in [0,1]$ we have $\emptyset \neq \Phi(x) \cap \lambda\, J(x) = \Phi(x) \cap \lambda\, F(x)$, a contradiction). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(K) = 1$. Let r be as in (2.11) and (2.13) guarantees that there exists a $x \in E$ with $\Phi(x) \cap \mu(x)\, J(r(x)) \neq \emptyset$. If $x \in E \setminus U$ then $\mu(x) = 0$ so $\Phi(x) \cap \{0\} \neq \emptyset$, and this contradicts (2.14). Thus $x \in U$ so $\Phi(x) \cap \mu(x)\, J(x) \neq \emptyset$, so $x \in K$, $\mu(x) = 1$ and consequently $\Phi(x) \cap J(x) \neq \emptyset$.

Remark 2.16. It is very easy to extend the above ideas to the (L,T) Φ–essential maps in [6].

Now we consider a generalization of Φ–essential maps, namely the d–Φ–essential maps. Let E be a completely regular topological space and U an open subset of E. For any map $F \in A(\overline{U},E)$ write $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\,(B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.15}$$

be any map with values in the nonempty set Ω where $B = \left\{(x,\Phi(x)) : \ x \in \overline{U}\right\}$.

Definition 2.17. Let $F \in A_{\partial U}(\overline{U},E)$ and write $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–Φ–essential if for every map $J \in A_{\partial U}(\overline{U},E)$ (write $J^\star = I \times J$) with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U},E)$ we have that $d\left((F^\star)^{-1}\,(B)\right) = d\left((J^\star)^{-1}\,(B)\right) \neq d(\emptyset)$.

Remark 2.18. If $F^\star$ is d–Φ–essential then

$$\emptyset \neq (F^\star)^{-1}\,(B) = \{x \in \overline{U} : \ (x,F(x)) \cap (x,\Phi(x)) \neq \emptyset\},$$

so there exists a $x \in U$ with $(x,\Phi(x)) \cap (x,F(x)) \neq \emptyset$ (i.e., $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 2.19. *Let E be a completely regular topological space, U an open subset of E, $B = \left\{(x,\Phi(x)) : \ x \in \overline{U}\right\}$, d is defined in (2.15), $F \in A_{\partial U}(\overline{U},E)$ and $G \in A_{\partial U}(\overline{U},E)$ (write $F^\star = I \times F$ and $G^\star = I \times G$). Suppose $G^\star$ is d–Φ–essential and*

$$\begin{cases} \textit{for any map } \ J \in A_{\partial U}(\overline{U},E) \ \textit{ with } \ J|_{\partial U} = F|_{\partial U} \ \textit{ and} \\ J \cong F \ \textit{ in } \ A_{\partial U}(\overline{U},E) \ \textit{we have} \ \ G \cong J \ \textit{ in } \ A_{\partial U}(\overline{U},E) \\ \textit{and } \ d\left((F^\star)^{-1}\,(B)\right) = d\left((G^\star)^{-1}\,(B)\right). \end{cases} \tag{2.16}$$

Then $F^\star$ is d–Φ–essential.

Proof. Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U},E)$ is as in Definition 2.4. Consider any map $J \in A_{\partial U}(\overline{U},E)$ (write $J^\star = I \times J$) with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U},E)$. From (2.16) there exists a u.s.c. compact map $H^J : \overline{U} \times [0,1] \to K(E)$ with $H^J \in \mathbf{A}(\overline{U} \times [0,1],E)$, $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$, $H^J_1 = J$ and $d\left((F^\star)^{-1}\,(B)\right) = d\left((G^\star)^{-1}\,(B)\right)$. Let $(H^J)^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $(H^J)^\star(x,t) = (x, H^J(x,t))$ and let

$$K = \left\{x \in \overline{U} : \ (x,\Phi(x)) \cap (H^J)^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $K \neq \emptyset$ is closed, compact and $K \cap \partial U = \emptyset$ so since E is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x,\mu(x))$ and write $R^\star = I \times R$. Now as in Theorem 2.7, $R \in A_{\partial U}(\overline{U},E)$ with $R|_{\partial U} = G|_{\partial U}$ and $R \cong G$ in $A_{\partial U}(\overline{U},E)$. Since $G^\star$ is d–Φ–essential then

$$d\left((G^\star)^{-1}\,(B)\right) = d\left((R^\star)^{-1}\,(B)\right) \neq d(\emptyset). \tag{2.17}$$

Now since $\mu(K) = 1$ we have

$$\begin{aligned}(R^\star)^{-1}(B) &= \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H^J(x, \mu(x))) \neq \emptyset\} \\ &= \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H^J(x, 1)) \neq \emptyset\} = (J^\star)^{-1}(B),\end{aligned}$$

so from (2.17) we have $d\left((G^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now combine with the above and we have $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

Note again it is simple to adjust the proof in Theorem 2.19 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5.

Theorem 2.20. *Let E be a completely regular topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d is defined in (2.15) and assume (2.2) holds. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ (write $F^\star = I \times F$ and $G^\star = I \times G$) and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then $F^\star$ is d–Φ–essential if and only if $G^\star$ is d–Φ–essential.*

Proof. Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Assume $G^\star$ is d–Φ–essential. Let $J \in A_{\partial U}(\overline{U}, E)$ (write $J^\star = I \times J$) with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. If we show (2.16) then $F^\star$ is d–Φ–essential from Theorem 2.19. Now (2.2) implies that $G \cong J$ in $A_{\partial U}(\overline{U}, E)$. To complete (2.16) we need to show $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. We will follow the argument in Theorem 2.19. Note since $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ let $H : \overline{U} \times [0,1] \to K(E)$ be a u.s.c. compact map with $H \in \mathbf{A}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = G$ and $H_1 = F$. Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $H^\star(x,t) = (x, H(x,t))$ and let

$$K = \{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Now $K \neq \emptyset$ and there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H(x, \mu(x))$ and write $R^\star = I \times R$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ and $R \cong G$ in $A_{\partial U}(\overline{U}, E)$ so since $G^\star$ is d–Φ–essential then $d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now since $\mu(K) = 1$ we have

$$\begin{aligned}(R^\star)^{-1}(B) &= \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\} \\ &= \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\} = (F^\star)^{-1}(B),\end{aligned}$$

so $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. □

Note again it is simple to adjust the proof in Theorem 2.20 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5.

Remark 2.21. It is very easy to extend the above ideas to the (L, T) d–Φ–essential maps in [7].

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