

Solving fuzzy Burgers equation by variational iteration method



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Abstract

In this paper, the variational iteration method (VIM) has been applied to find the fuzzy solutions of the fuzzy Burgers equations with variable coefficients and fuzzy parameters. We follow the same strategy as in Buckley and Feuring which is: (1) first check to see if the Buckley-Feuring method produces a solution, and (2) if the Buckley-Feuring method does not give a solution, then see if the Seikkala procedure generates a solution. Several examples are given to show the new theorem of the Buckley-Feuring solution and the Seikkala solution.

Keywords: Fuzzy Burgers equations, variational iteration method, fuzzy number.

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1. Introduction

Zadeh is credited with introducing the concept of fuzzy sets in 1965 as a mathematical means of describing vagueness in linguistics. The idea may be considered as a generalization of classical set theory, in the decade since Zadeh's pioneering paper on fuzzy sets [22]. Assume Burgers models have imprecise parameters, since fuzzy sets theory is a powerful tool for modeling imprecise and processing vague in mathematical models, hence the our idea is solving Burgers equation with fuzzy parameters via the same strategy as Buckley and Feuring [4] using Variational Iteration Method (VIM) [13–15]. We begin this section with definition the notation we will use in the paper and then specify the fuzzy Burgers equation we wish to solve. In the next section we present the Buckley-Feuring solution and the Seikkala solution [18]. Also, in this section we give a sufficient condition for BF-solution to exist. The last section contains two examlpes where in the first example both solutions can exist in disjoint subregion of the domain as shown in Figs 1 and 2. In the last example the BF-solution does not exist but the S-solution can exist within some subregion of the domain as shown in Figs. 3, 4, and 5. In comparison with the paper [5, 17], we investigate nonlinear partial differential equations with fuzzy parameters, fuzzy initial value and fuzzy forcing functions, we propose a new theorem for finding the exact fuzzy solutions, witch extended to the Buckley-Feuring for the proposed models.

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2. Preliminaries

We place a bar over a capital letter to denote a fuzzy number of \mathbb{R}^n . So, \bar{A} , \bar{K} , $\bar{\varepsilon}$, $\bar{\beta}$ etc. all represent fuzzy numbers of \mathbb{R}^n for some n . We write $\mu_{\bar{A}}(t)$, a number in $[0, 1]$, for the membership function of \bar{A} evaluated at $t \in \mathbb{R}^n$. An α -cut of \bar{A} is always a closed and bounded interval written as $\bar{A}[\alpha]$, is defined as $\{t \mid \mu_{\bar{A}}(t) \geq \alpha\}$ for $0 < \alpha < 1$. We separately specify $\bar{A}[0]$ as the closure of the union of all the $\bar{A}[\alpha]$ for $0 < \alpha \leq 1$.

Definition 2.1 ([6]). Let $\mathbb{R}_{\mathcal{F}} = \{\bar{A} \mid \bar{A} : \mathbb{R} \rightarrow [0, 1], \text{ satisfies (1)-(4)}\}$:

- (1) $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, \bar{A} is normal;
- (2) $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, \bar{A} is a fuzzy convex set;
- (3) $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, \bar{A} is upper semi-continuous on \mathbb{R} ;
- (4) $\bar{A}[0]$ is a compact set.

Then $\mathbb{R}_{\mathcal{F}}$ is called fuzzy number space and $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, \bar{A} is called a fuzzy number.

Definition 2.2 ([6, 16]). We represent an arbitrary fuzzy number by an ordered pair of functions $\bar{A}[\alpha] = [A_1(\alpha), A_2(\alpha)]$, $\alpha \in [0, 1]$, which satisfy the following requirements:

1. $A_1(\alpha)$ is an increasing function over $[0, 1]$;
2. $A_2(\alpha)$ is a decreasing function on $[0, 1]$;
3. $A_1(\alpha)$ and $A_2(\alpha)$ are bounded left continuous on $(0, 1]$, and right continuous at $\alpha = 0$;
4. $A_1(\alpha) \leq A_2(\alpha)$, for $0 \leq \alpha \leq 1$.

Definition 2.3. For a fuzzy set $\bar{A} = (a_1, a_2, a_3)$, ($a_1 < a_2 < a_3$), \bar{A} is called triangular fuzzy number with peak a_2 , left width $a_2 - a_1 > 0$ and right width $a_3 - a_2 > 0$, if its membership function has the following form:

$$\mu_{\bar{A}}(t) = \begin{cases} 1 - \frac{(a_2 - t)}{a_2 - a_1}, & a_1 \leq t \leq a_2, \\ 1 - \frac{(t - a_2)}{a_3 - a_2}, & a_2 \leq t \leq a_3, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4. For arbitrary fuzzy numbers $\bar{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$ and $\bar{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$ we have algebraic operations as follows:

1. $(\bar{A} + \bar{B})[\alpha] = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]$;
2. $(\bar{A} - \bar{B})[\alpha] = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)]$;
3. $k\bar{A}[\alpha] = \begin{cases} [ka_1(\alpha), ka_2(\alpha)] & k \geq 0, \\ [ka_2(\alpha), ka_1(\alpha)] & k < 0; \end{cases}$
4. $(\bar{A} \cdot \bar{B})[\alpha] = \{\min z, \max z\}$ with $z = \{a_1(\alpha) \cdot b_1(\alpha), a_1(\alpha) \cdot b_2(\alpha), a_2(\alpha) \cdot b_1(\alpha), a_2(\alpha) \cdot b_2(\alpha)\}$;
5. If $0 \notin [b_1(\alpha), b_2(\alpha)]$, $\frac{\bar{A}}{\bar{B}}[\alpha] = [(\frac{a_1}{b_1})(\alpha), (\frac{a_2}{b_2})(\alpha)]$, where

$$(\frac{a_1}{b_1})(\alpha) = \min \left\{ \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)} \right\}, \quad (\frac{a_2}{b_2})(\alpha) = \max \left\{ \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)} \right\}.$$

We adopt the general definition of a fuzzy number given in [7–9].

3. Fuzzy Burgers equations

Consider the one-dimensional Burgers equation has the form [1]

$$U_t(t, x) + P(t, x, \varepsilon, U(t, x))U_x(t, x) + Q(x, v)U_{xx}(t, x) = F(t, x, k), \quad (3.1)$$

or

$$U_t(t, x) + P(t, x, \varepsilon, U(t, x))U_x(t, x) = Q(x, \nu)U_{xx}(t, x) + F(t, x, k), \quad (3.2)$$

subject to certain initial and boundary conditions.

These initial and boundary conditions, in state one-dimensional, can come in a variety of forms such as $U(0, x) = c_1$ or $U(0, x) = g_1(x, c_2)$ or $U(M_1, x) = g_2(x, c_3, c_4), \dots$. In this work the method is applied for the Burgers equation (3.1). For Eq. (3.2), the same discussion can be made. In following lines, the components of (3.1) are enumerated:

- $I_1 = [0, M_1]$ and $I_2 = [M_2, M_3]$ are two intervals, which M_{n_1} ($n_1 = 2, 3$) is negative or positive and $M_1 > 0$;
- $F(t, x, k)$, $U(t, x)$, $P(t, x, \varepsilon, U(t, x))$ and $Q(x, \nu)$ will be continuous functions for $(t, x) \in \prod_{j=1}^2 I_j$;
- $Q(x, \nu)$ has a finite number of roots each $x \in I_2$;
- $k = (k_1, \dots, k_n)$, $c = (c_1, \dots, c_m)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_s)$, and $\nu = (\nu_1, \dots, \nu_e)$ are vectors of constants with $k_j \in J_j$, $c_i \in L_i$ and $\varepsilon_r \in H_r$ and $\nu_l \in D_l$.

Assume that Eq. (3.1) has a solution

$$U(t, x) = G(t, x, k, c, \varepsilon, \nu) \quad (3.3)$$

for G and

$$G_t(t, x, k, c, \varepsilon, \nu) + P(\varepsilon, G(t, x, k, c, \varepsilon, \nu))G_x(t, x, k, c, \varepsilon, \nu) + Q(x, \nu)G_{xx}(t, x, k, c, \varepsilon, \nu)$$

are continuous with $(t, x) \in \prod_{j=1}^2 I_j$, $k \in J = \prod_{j=1}^n J_j$, $c \in L = \prod_{i=1}^m L_i$, $\varepsilon \in H = \prod_{r=1}^s H_r$ and $\nu \in D = \prod_{l=1}^e D_l$.

Suppose the constants k_j , c_i , ε_r and ν_l are imprecise in their values. We will model this uncertainty by substituting triangular fuzzy numbers for the k_j , c_i , ε_r and ν_l . If we fuzzify (3.1), then we obtain the fuzzy Burgers equation. Using the extension principle, we compute \bar{F} , \bar{P} , and \bar{Q} from F , P , and Q , where $\bar{F}(t, x, \bar{K})$ has $\bar{K} = (\bar{k}_1, \dots, \bar{k}_n)$ and $\bar{P}(\bar{\varepsilon}, \bar{U})$ has $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_s)$ and $\bar{Q}(\bar{\nu})$ a $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_e)$ for k_j , ε_r and ν_l triangular fuzzy numbers in J_j ($0 \leq j \leq n$), H_r ($0 \leq r \leq s$) and D_l ($0 \leq l \leq e$).

The function U is changed to \bar{U} where $\bar{U} : \prod_{j=1}^2 I_j \rightarrow \mathbb{R}_{\mathcal{F}}$. That is, $\bar{U}(t, x)$ is a fuzzy function. The fuzzy Burgers equation is

$$\bar{U}_t(t, x) + \bar{P}(\bar{\varepsilon}, \bar{U}(t, x))\bar{U}_x(t, x) + \bar{Q}(\bar{\nu})\bar{U}_{xx}(t, x) = \bar{F}(t, x, \bar{K}), \quad (3.4)$$

subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form

$$\bar{U}(0, x) = \bar{C}_1 \text{ or } \bar{U}(0, x) = \bar{g}_1(x, \bar{C}_2) \text{ or } \bar{U}(M_1, x) = \bar{g}_2(x, \bar{C}_3, \bar{C}_4).$$

The \bar{g}_j is the fuzzification g_i via extension principle. Then, we wish to solve the problem given in (3.4). Finally, we fuzzify G in (3.3).

Let $\bar{Z}(t, x) = \bar{G}(t, x, \bar{K}, \bar{C}, \bar{\varepsilon}, \bar{\beta})$ where \bar{Z} is computed using the extension principal and is a fuzzy solution. In Section 5, we will discuss the concept solution with the same strategy as Buckley-Feuring for fuzzy Burgers equation.

Let

$$\bar{K}[\alpha] = \prod_{j=1}^n \bar{K}_j[\alpha], \quad \bar{\varepsilon}[\alpha] = \prod_{r=1}^s \bar{\varepsilon}_r[\alpha], \quad \bar{C}[\alpha] = \prod_{i=1}^m \bar{C}_i[\alpha], \quad \text{and } \bar{\nu}[\alpha] = \prod_{l=1}^e \bar{\nu}_l[\alpha].$$

4. The variational iteration method

To illustrate the basic concepts of VIM, consider the following general nonlinear partial differential equation:

$$\begin{cases} LU(t, x) + RU(t, x) + NU(t, x) = F(t, x, k), \\ U(t, 0) = g(t), \end{cases}$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives with respect to x , N is a nonlinear operator, also $F(t, x, k)$ is the source non-homogeneous term. According to the VIM [20, 21], we can construct the following iteration formula:

$$U_{n+1}(t, x) = U_n(t, x) + \int_0^t \lambda \{LU_n + R\tilde{U}_n + N\tilde{U}_n - F(s, x, k)\} ds, \quad (4.1)$$

where λ is general Lagrange multiplier [14], which can be identified optimally via the variational theory [15, 20], $R\tilde{U}_n$ and $N\tilde{U}_n$ are considered as restricted variation, i.e. $\delta R\tilde{U}_n = 0$, $\delta N\tilde{U}_n = 0$. First, it is required to determine the Lagrange multipliers λ that will be identified optimally via integration by parts. The approximations U_{n+1} , $n \geq 0$, calculate variation with respect to u_n , the following stationary conditions are obtained:

$$\lambda'(s) = 0, \quad 1 + \lambda(s)|_{s=t} = 0.$$

So, the Lagrange multiplier is $\lambda = -1$. Submitting the results into (4.1) leads to the following iteration formula

$$U_{n+1}(t, x) = U_n(t, x) - \int_0^t \{LU_n + RU_n + NU_n - F(s, x, k)\} ds. \quad (4.2)$$

The second term on the right is called the correction term (4.2) can be solved iteratively using $U_0(t, x) = g(x)$ as an initial approximation. Also the VIM used for system of linear and nonlinear partial differential equation in [20] can be handled to obtain Seikkala solution.

5. Buckley-Feuring solution (BFS) and Seikkala solution (SS)

Buckley-Feuring first presented the BFS in [3, 4]. They define for all t, x and $\alpha \in [0, 1]$,

$$\bar{Z}(t, x)[\alpha] = [z_1(t, x, \alpha), z_2(t, x, \alpha)], \quad \bar{F}(t, x, k)[\alpha] = [F_1(t, x, \alpha), F_2(t, x, \alpha)],$$

and to check (3.4) we must compute $\bar{P}(t, x, \bar{\varepsilon}, \bar{Z})$ and $\bar{Q}(x, \bar{v})$.

Let $W = \bar{K}[\alpha] \times \bar{C}[\alpha] \times \bar{\varepsilon}[\alpha] \times \bar{v}[\alpha]$. By definition

$$z_1(t, x, \alpha) = \min \{G(t, x, k, c, \varepsilon, v) : (k, c, \varepsilon, v) \in W\}, \quad (5.1)$$

$$z_2(t, x, \alpha) = \max \{G(t, x, k, c, \varepsilon, v) : (k, c, \varepsilon, v) \in W\}, \quad (5.2)$$

$$F_1(t, x, \alpha) = \min \{F(t, x, k) : k \in \bar{K}[\alpha]\}, \quad (5.3)$$

$$F_2(t, x, \alpha) = \max \{F(t, x, k) : k \in \bar{K}[\alpha]\}, \quad (5.4)$$

$\forall(t, x) \in \prod_{j=1}^2 I_j$ and $\alpha \in [0, 1]$. The α -cuts of $\bar{Q}(x, \bar{v})$ can be found as follows: $\forall \alpha \in [0, 1]$

$$\bar{Q}(x, \bar{v})[\alpha] = [Q_1(x, \alpha), Q_2(x, \alpha)], \quad Q_1(x, \alpha) = \min \{Q(x, v) | v \in \bar{v}[\alpha]\}, \quad Q_2(x, \alpha) = \max \{Q(x, v) | v \in \bar{v}[\alpha]\}, \quad (5.5)$$

$\forall x \in I_2$ and $\alpha \in [0, 1]$. Now for $\bar{Z}(t, x)$ to be a solution to the fuzzy initial value problem we need that $\frac{\partial}{\partial t} \bar{Z}(t, x)$, $\frac{\partial}{\partial x} \bar{Z}(t, x)$ and $\frac{\partial^2}{\partial x^2} \bar{Z}(t, x)$ exist but also (3.4) must hold. To check (3.4) we must first compute $\bar{P}(t, x, \bar{\varepsilon}, \bar{Z})$. α -cuts of $\bar{P}(t, x, \bar{\varepsilon}, \bar{Z})$ can be found as follows:

$$\bar{P}(t, x, \bar{\varepsilon}, \bar{Z})[\alpha] = [P_1(t, x, \alpha), P_2(t, x, \alpha)]$$

with

$$\begin{aligned} P_1(t, x, \alpha) &= \min \{P(t, x, \varepsilon, z) \mid \varepsilon \in \bar{\varepsilon}[\alpha], z \in \bar{Z}(t, x)[\alpha]\}, \\ P_2(t, x, \alpha) &= \max \{P(t, x, \varepsilon, z) \mid \varepsilon \in \bar{\varepsilon}[\alpha], z \in \bar{Z}(t, x)[\alpha]\}, \end{aligned} \quad (5.6)$$

$\forall (t, x) \in \prod_{j=1}^2 I_j$ and $\alpha \in [0, 1]$.

Assume that $P(t, x, \varepsilon, z) > 0$, $Q(x, v) > 0$ and the $z_i(t, x, \alpha)$ $i = 1, 2$, has continuous partial derivatives so

$$(z_i)_t + P_i(t, x, \alpha)(z_i)_x + Q_i(x, \alpha)(z_i)_{xx}$$

is continuous for all $t, x \in \prod_{j=1}^2 I_j$ and all $\alpha \in [0, 1]$. Define

$$\Gamma(t, x, \alpha) = [(z_1)_t + P_1(t, x, \alpha)(z_1)_x + Q_1(x, \alpha)(z_1)_{xx}, (z_2)_t + P_2(t, x, \alpha)(z_2)_x + Q_2(x, \alpha)(z_2)_{xx}]$$

for all $(t, x) \in \prod_{j=1}^2 I_j$ and all α . If for each fixed $(t, x) \in \prod_{j=1}^2 I_j$, $\Gamma(t, x, \alpha)$ defines the α -cut of a fuzzy number, then will be said that $\bar{Z}(t, x)$ is differentiable and is written as

$$\bar{Z}_t[\alpha] + P(t, x, \bar{\varepsilon}, \bar{Z})[\alpha]\bar{Z}_x[\alpha] + Q(x, \bar{v})[\alpha]\bar{Z}_{xx}[\alpha] = \Gamma(t, x, \alpha)$$

for all $(t, x) \in \prod_{j=1}^2 I_j$ and all α .

Sufficient conditions for $\Gamma(t, x, \alpha)$ to define α -cut of a fuzzy number are [7]:

- (i) $(z_1)_t(t, x, \alpha) + P_1(t, x, \alpha)(z_1)_x(t, x, \alpha) + Q_1(x, \alpha)(z_1)_{xx}(t, x, \alpha)$ is an increasing function of α for each $(t, x) \in \prod_{j=1}^2 I_j$;
- (ii) $(z_2)_t(t, x, \alpha) + P_2(t, x, \alpha)(z_2)_x(t, x, \alpha) + Q_2(x, \alpha)(z_2)_{xx}(t, x, \alpha)$ is a decreasing function of α for each $(t, x) \in \prod_{j=1}^2 I_j$; and
- (iii) for $(t, x) \in \prod_{j=1}^2 I_j$,

$$\begin{aligned} &(z_1)_t(t, x, 1) + P_1(t, x, 1)(z_1)_x(t, x, 1) + Q_1(x, 1)(z_1)_{xx}(t, x, 1) \\ &\leq (z_2)_t(t, x, 1) + P_2(t, x, 1)(z_2)_x(t, x, 1) + Q_2(x, 1)(z_2)_{xx}(t, x, 1). \end{aligned}$$

Now we can assume that the $z_i(t, x, \alpha)$ have continuous partial derivatives, so

$$(z_i)_t + P_i(t, x, \alpha)(z_i)_x + Q_i(x, \alpha)(z_i)_{xx}$$

is continuous on $\prod_{j=1}^2 I_j \times [0, 1]$ for $i = 1, 2$. Hence, if conditions (i)-(iii) above hold, $\bar{Z}(t, x)$ is differentiable.

$\bar{Z}(t, x)$ will be a BFS of the fuzzy Burgers equation if,

- (a) $\bar{Z}(t, x)$ is differentiable;
- (b) (3.4) holds for $\bar{U}(t, x) = \bar{Z}(t, x)$;
- (c) $\bar{Z}(t, x)$ satisfies the initial and boundary conditions.

Since there is no specified particular initial and boundary conditions, then only is checked if (3.4) holds.

We will only say that $\bar{Z}(t, x)$ is a BFS (without the initial and boundary conditions) if $\bar{Z}(t, x)$ is differentiable and

$$(\bar{Z})_t + \bar{P}(t, x, \bar{\varepsilon}, \bar{Z})(\bar{Z})_x + \bar{Q}(x, \bar{\nu})(\bar{Z})_{xx} = \bar{F}(t, x, \bar{k})$$

or the following equations must hold

$$(z_1)_t + P_1(t, x, \alpha)(z_1)_x + Q_1(x, \alpha)(z_1)_{xx} = F_1(t, x, \alpha), \quad (5.7)$$

$$(z_2)_t + P_2(t, x, \alpha)(z_2)_x + Q_2(x, \alpha)(z_2)_{xx} = F_2(t, x, \alpha), \quad (5.8)$$

for all $(t, x) \in \prod_{j=1}^2 I_j$ and all $\alpha \in [0, 1]$. If $\bar{Z}(t, x)$ is a BFS and it satisfies the initial and boundary conditions we will say that $\bar{Z}(t, x)$ is a BFS satisfying the initial and boundary conditions. If $\bar{Z}(t, x)$ is not a BFS, then we will consider the SS. At the end of this section we will present a sufficient condition for the existence of the BFS, but now let us define the SS [18]. Let

$$\bar{U}(t, x)[\alpha] = [u_1(t, x, \alpha), u_2(t, x, \alpha)].$$

Consider the system of Burgers equations

$$(u_1)_t + P_1(t, x, \alpha)(u_1)_x + Q_1(x, \alpha)(u_1)_{xx} = F_1(t, x, \alpha), \quad (5.9)$$

$$(u_2)_t + P_2(t, x, \alpha)(u_2)_x + Q_2(x, \alpha)(u_2)_{xx} = F_2(t, x, \alpha), \quad (5.10)$$

for all $(t, x) \in \prod_{j=1}^2 I_j$ and $\alpha \in [0, 1]$. We append to Eqs. (5.9) and (5.10) any initial and boundary conditions. For example, if they were $\bar{U}(0, x) = \bar{C}$, then we add

$$u_1(0, x, \alpha) = c_1(\alpha), \quad u_2(0, x, \alpha) = c_2(\alpha),$$

where $\bar{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$. Let $u_i(t, x, \alpha)$ $i = 1, 2$ solve Eqs. (5.9) and (5.10) plus initial and boundary conditions. If

$$[u_1(t, x, \alpha), u_2(t, x, \alpha)]$$

defines the α -cut of a fuzzy number for all $(t, x) \in \prod_{j=1}^2 I_j$, then $\bar{U}(t, x)$ is the SS. Clearly if the BFS satisfying the boundary conditions is $\bar{Z}(t, x)$, then $\bar{Z}(t, x)$ is also the SS, see [5, 17, Theorem 5.2]. As we shall see the SS can exist when the BFS fails to exist.

Now we will present a sufficient condition for the BFS to exist such as Buckley and Feuring. Since there are such a variety of possible initial and boundary conditions, so we will omit them from the following theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants c_i , $1 \leq i \leq m$, from the problem. Therefore, (3.3) becomes $U(t, x) = G(t, x, k, \varepsilon, \nu)$, so

$$\bar{Z}(t, x) = \bar{G}(t, x, \bar{K}, \bar{\varepsilon}, \bar{\nu}).$$

Theorem 5.1. Assume $\bar{Z}(t, x)$ is differentiable.

(a) if

$$P(t, x, \varepsilon_i, z) > 0, \quad \frac{\partial P}{\partial z} > 0 \quad \text{and} \quad \frac{\partial P}{\partial \varepsilon_i} \frac{\partial G}{\partial \varepsilon_i} > 0, \quad t, x \in I_1 \times I_2 \quad \text{for } i = 1, 2, \dots, m, \quad (5.11)$$

and if

$$Q(x, \nu_l) > 0 \quad \text{and} \quad \frac{\partial Q}{\partial \nu_l} \frac{\partial G}{\partial \nu_l} > 0, \quad x \in I_2 \quad \text{for } l = 1, 2, \dots, e, \quad (5.12)$$

and if

$$\frac{\partial G}{\partial k_j} \frac{\partial F}{\partial k_j} > 0 \quad \text{for } j = 1, 2, \dots, n, \quad (5.13)$$

then $BFS = \bar{Z}(t, x)$.

(b) If relations (5.11) does not hold for some i or relation (5.12) does not hold for some l , or relation (5.13) does not hold for some j , then $\bar{Z}(t, x)$ is not a BFS.

Proof.

(a) For simplicity assume $k_j = k$, $\varepsilon_i = \varepsilon$, $v_l = v$ and $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$, $\frac{\partial P}{\partial \varepsilon} > 0$, $\frac{\partial G}{\partial \varepsilon} > 0$, $\frac{\partial Q}{\partial v} < 0$ and $\frac{\partial G}{\partial v} < 0$. The proof for $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial P}{\partial \varepsilon} < 0$, $\frac{\partial G}{\partial \varepsilon} < 0$, $\frac{\partial Q}{\partial v} > 0$ and $\frac{\partial G}{\partial v} > 0$ is similar and omitted. Since $\frac{\partial G}{\partial k} < 0$, $\frac{\partial G}{\partial \varepsilon} > 0$ and $\frac{\partial G}{\partial v} < 0$, then from (5.1) and (5.2) we have

$$z_1(t, x, \alpha) = G(t, x, k_2(\alpha), \varepsilon_1(\alpha), v_2(\alpha)), \quad z_2(t, x, \alpha) = G(t, x, k_1(\alpha), \varepsilon_2(\alpha), v_1(\alpha)),$$

from (5.3), (5.4), and $\frac{\partial F}{\partial k} < 0$ we have

$$F_1(t, x, \alpha) = F(t, x, k_2(\alpha)), \quad F_2(t, x, \alpha) = F(t, x, k_1(\alpha)),$$

since (5.6), $\frac{\partial P}{\partial \varepsilon} > 0$ and $\frac{\partial P}{\partial z} > 0$ we have

$$P_1(t, x, \alpha) = P(x, \varepsilon_1(\alpha), z_1(t, x, \alpha)), \quad P_2(t, x, \alpha) = P(x, \varepsilon_2(\alpha), z_2(t, x, \alpha)),$$

from (5.5) and $\frac{\partial Q}{\partial v} < 0$ we have

$$Q_1(x, \alpha) = Q(x, v_2(\alpha)), \quad Q_2(x, \alpha) = Q(x, v_1(\alpha)),$$

for all $\alpha \in [0, 1]$, where

$$\bar{K}[\alpha] = [k_1(\alpha), k_2(\alpha)], \quad \bar{\varepsilon}[\alpha] = [\varepsilon_1(\alpha), \varepsilon_2(\alpha)], \quad \text{and} \quad \bar{v}[\alpha] = [v_1(\alpha), v_2(\alpha)].$$

Now $G(t, x, k, \varepsilon, v)$ solves (3.2), which means

$$G_t + P(t, x, \varepsilon, G)G_x + Q(x, v)G_{xx} = F(t, x, k)$$

for all $(t, x) \in \prod_{j=1}^2 I_j$, $k \in J$, $\varepsilon \in H$ and $v \in D$.

Suppose $\bar{Z}(t, x)$ is differentiable and $P(t, x, \varepsilon, z) > 0$ and $Q(x, v) > 0$, so

$$\begin{aligned} \partial_t z_1(t, x, \alpha) + P_1(t, x, \alpha) \partial_x z_1(t, x, \alpha) + Q_1(x, \alpha) \partial_{xx} z_1(t, x, \alpha) &= F_1(t, x, \alpha), \\ \partial_t z_2(t, x, \alpha) + P_2(t, x, \alpha) \partial_x z_2(t, x, \alpha) + Q_2(x, \alpha) \partial_{xx} z_2(t, x, \alpha) &= F_2(t, x, \alpha), \end{aligned}$$

for all $(t, x) \in \prod_{j=1}^2 I_j$ and $\alpha \in [0, 1]$.

Hence, (5.7) and (5.8) hold and $\bar{Z}(t, x)$ is a BFS.

(b) Now consider the situation where (5.11) or (5.12) or (5.13) does not hold.

Let us only look at one case where $\frac{\partial Q}{\partial v} < 0$ (assume $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial G}{\partial \varepsilon} > 0$, $\frac{\partial P}{\partial \varepsilon} > 0$, $\frac{\partial P}{\partial z} > 0$ and $\frac{\partial G}{\partial v} > 0$, $P(x, \varepsilon, z) > 0$ and $Q(x, v) > 0$). Then we have

$$\begin{aligned} z_1(t, x, \alpha) &= G(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_1(\alpha)), \\ z_2(t, x, \alpha) &= G(t, x, k_2(\alpha), \varepsilon_2(\alpha), v_2(\alpha)), \\ F_1(t, x, \alpha) &= F(t, x, k_1(\alpha)), \quad F_2(t, x, \alpha) = F(t, x, k_2(\alpha)), \\ P_1(t, x, \alpha) &= P(t, x, \varepsilon_1(\alpha), z_1(t, x, \alpha)), \quad P_2(t, x, \alpha) = P(t, x, \varepsilon_2(\alpha), z_2(t, x, \alpha)), \\ Q_1(x, \alpha) &= Q(x, v_2(\alpha)), \quad Q_2(x, \alpha) = Q(x, v_1(\alpha)), \end{aligned}$$

then we have

$$\begin{aligned}\partial_t z_1(t, x, \alpha) + P_1(t, x, \alpha) \partial_x z_1(t, x, \alpha) + Q_1(x, \alpha) \partial_{xx} z_1(t, x, \alpha) &= F_1(t, x, \alpha), \\ \partial_t z_2(t, x, \alpha) + P_2(t, x, \alpha) \partial_x z_2(t, x, \alpha) + Q_2(x, \alpha) \partial_{xx} z_2(t, x, \alpha) &= F_2(t, x, \alpha),\end{aligned}$$

which is not true, because

$$\begin{aligned}G_t(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_1(\alpha)) + P(t, x, \varepsilon_1(\alpha), G(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_1(\alpha))) G_x(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_1(\alpha)) \\ + Q(x, v_2(\alpha)) G_{xx}(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_1(\alpha)) = F(t, x, k_1(\alpha)), \\ G_t(t, x, k_2(\alpha), \varepsilon_2(\alpha), v_2(\alpha)) + P(t, x, \varepsilon_1(\alpha), G(t, x, k_1(\alpha), \varepsilon_2(\alpha), v_2(\alpha))) G_{xx}(t, x, k_2(\alpha), \varepsilon_2(\alpha), v_2(\alpha)) \\ + Q(x, v_1(\alpha)) G_{xx}(t, x, k_1(\alpha), \varepsilon_1(\alpha), v_2(\alpha)) = F(t, x, k_2(\alpha)).\end{aligned}$$

□

Lemma 5.2. Assume $\bar{Z}(t, x)$ is differentiable.

(a) if

$$P(t, x, \varepsilon_i, z) > 0, \quad \frac{\partial P}{\partial z} > 0, \quad \text{and} \quad \frac{\partial P}{\partial \varepsilon_i} \frac{\partial G}{\partial \varepsilon_i} > 0, \quad t, x \in I_1 \times I_2 \quad \text{for } i = 1, 2, \dots, m, \quad (5.14)$$

and if

$$\frac{\partial G}{\partial k_j} \frac{\partial F}{\partial k_j} > 0, \quad \text{for } j = 1, 2, \dots, n, \quad (5.15)$$

then $BFS = \bar{Z}(t, x)$.

(b) If relations (5.14) does not hold for some i , or relation (5.15) does not hold for some j , then $\bar{Z}(t, x)$ is not a BFS.

6. Examples

We consider the following examples from [1, 2, 19] and we added fuzzy parameters.

Example 6.1. We first consider the one-dimensional Burgers equation with variable coefficients as

$$U_t(t, x) + \varepsilon U(t, x) U_x(t, x) = \nu U_{xx}(t, x) + k \quad (6.1)$$

with the initial conditions

$$U(0, x) = \frac{c}{2} \left(1 - \tanh \left[\frac{cx}{4} \right] \right),$$

where $\varepsilon = \nu, x \in [0, 1], t > 0, k \in [0, J], \varepsilon \in [0, 1]$ and $c \in [0, L]$ are constants.

To solve Eq (6.1) by means of the VIM, substitute in Eq. (4.2) by

$$RU_n = -\varepsilon(U_n)_{xx}, \quad NU_n = \varepsilon U_n(U_n)_x,$$

and $f(t, x, k) = k$ and obtain the following variational iteration formula:

$$U_{n+1}(t, x) = U_n(t, x) - \int_0^t \{ (U_n(s, x))_s + \varepsilon U_n(s, x) (U_n(s, x))_x - \varepsilon (U_n(s, x))_{xx} - k \} ds. \quad (6.2)$$

Using (6.2), the approximate solutions $U_n(t, x)$ are obtained iteratively by substituting:

$$U_0(t, x) = U(0, x) = \frac{c}{2} \left(1 - \tanh \left[c \frac{x}{4} \right] \right),$$

$$U_1(t, x) = U_0(t, x) + \varepsilon \frac{c^3}{16} \operatorname{sech}^2 \left[c \frac{x}{4} \right] t + kt,$$

$$U_2(t, x) = U_1(t, x) + \varepsilon^2 \frac{c^5}{128} \tanh \left[c \frac{x}{4} \right] \operatorname{sech}^2 \left[c \frac{x}{4} \right] t^2 + \varepsilon k \frac{c^2}{16} \operatorname{sech}^2 \left[c \frac{x}{4} \right] t^2 + \varepsilon^3 \frac{c^7}{1536} \tanh \left[c \frac{x}{4} \right] \operatorname{sech}^4 \left[c \frac{x}{4} \right] t^3,$$

⋮

and so on. The approximate solution takes the form $U(t, x) \simeq U_n(t, x)$, where n is the final step, which gives the exact solution of Eq. (6.1) takes the form

$$U(t, x) = \frac{c}{2} \left(1 - \tanh \left[\frac{c}{4} \left(x - \left(\frac{\varepsilon}{2} \right) (ct + kt^2) \right) \right] \right) + kt.$$

Now we fuzzify $F(t, x, k)$, $Q(x, \varepsilon)$, and $P(t, x, \varepsilon, G)$,

$$G(t, x, k, c, \varepsilon) = \frac{c}{2} \left(1 - \tanh \left[\frac{c}{4} \left(x - \left(\frac{\varepsilon}{2} \right) (ct + kt^2) \right) \right] \right) + kt.$$

Clearly

$$\bar{F}(t, x, \bar{K}) = \bar{K}, \quad \bar{Q}(x, \bar{\varepsilon}) = \bar{\varepsilon},$$

so that

$$F_1(t, x, \alpha) = k_1(\alpha), \quad F_2(t, x, \alpha) = k_2(\alpha), \quad Q_1(x, \alpha) = \varepsilon_1(\alpha), \quad Q_2(x, \alpha) = \varepsilon_2(\alpha).$$

Also

$$\bar{G}(t, x, \bar{K}, \bar{C}, \bar{\varepsilon}) = \frac{\bar{C}}{2} \left(1 - \tanh \left[\frac{\bar{C}}{4} \left(x - \left(\frac{\bar{\varepsilon}}{2} \right) (\bar{C}t + \bar{K}t^2) \right) \right] \right) + \bar{K}t,$$

therefore

$$z_i(t, x, \alpha) = \frac{c_i}{2} \left(1 - \tanh \left[\frac{c_i}{4} \left(x - \left(\frac{\varepsilon_i}{2} \right) (c_i t + k_i t^2) \right) \right] \right) + k_i t,$$

for $i = 1, 2$ and $\bar{C} > 0$, i.e., $(\bar{C} = (c_1, c_2, c_3))$ also with $c_1 > 0$, $\bar{K}[\alpha] = [k_1(\alpha), k_2(\alpha)]$, $\bar{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$, and $\bar{\varepsilon}[\alpha] = [\varepsilon_1(\alpha), \varepsilon_2(\alpha)]$.

$\bar{Z}(t, x)$ is differentiable because $(z_i(t, x, \alpha))_t + \varepsilon_i(\alpha) z_i(t, x, \alpha) (z_i(t, x, \alpha))_x - \varepsilon_i(\alpha) (z_i(t, x, \alpha))_{xx}$ for $i = 1, 2$ are α -cuts of \bar{K} , i.e., α -cuts of a fuzzy number. Due to

$$\begin{aligned} P(t, x, \varepsilon, z) &> 0, \quad \frac{\partial P}{\partial z} > 0, \\ \frac{\partial G}{\partial k} &= \frac{c^2}{16} \varepsilon \operatorname{sech}^2 \left(\frac{c}{4} \left(x - \frac{\varepsilon}{2} (ct + kt^2) \right) \right) t^2 + t > 0, \quad \frac{\partial F}{\partial k} > 0, \\ \frac{\partial P}{\partial \varepsilon} &= z > 0, \quad \frac{\partial G}{\partial \varepsilon} = \frac{c^2}{16} \operatorname{sech}^2 \left(\frac{c}{4} \left(x - \frac{\varepsilon}{2} (ct + kt^2) \right) \right) (ct + kt^2) > 0. \end{aligned}$$

That is, $(\bar{Z})_t + \bar{\varepsilon} \bar{Z} (\bar{Z})_x = \bar{\varepsilon} (\bar{Z})_{xx} + \bar{K}$, a fuzzy number. So Theorem 5.1 implies the result that $\bar{Z}(t, x)$ is a BFS, see Fig. 1, 2, and Table 1. We easily see that

$$z_i(0, x, \alpha) = \frac{c_i}{2} \left(1 - \tanh \left[c_i \frac{x}{4} \right] \right)$$

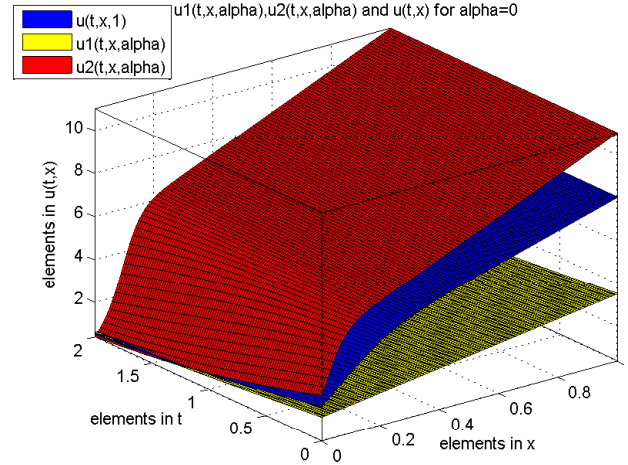
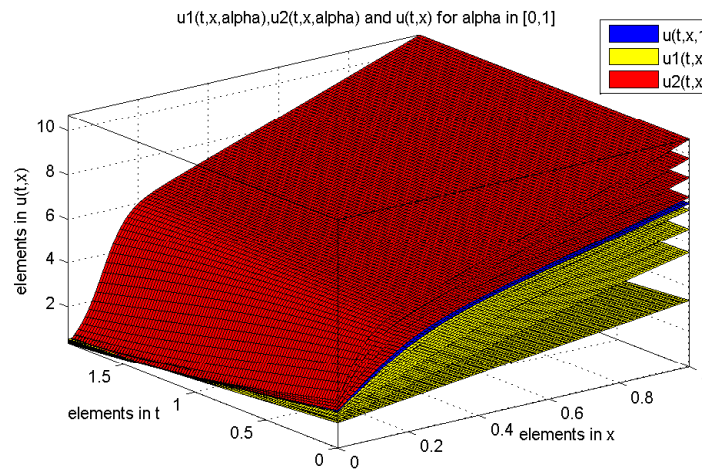
for $i = 1, 2$, so $\bar{Z}(t, x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\bar{Z}(t, x) = \frac{\bar{C}}{2} \left(1 - \tanh \left[\frac{\bar{C}}{4} \left(x - \left(\frac{\bar{\varepsilon}}{2} \right) (\bar{C}t + \bar{K}t^2) \right) \right] \right) + \bar{K}t$$

for all $t > 0$, $x \in [0, 1]$.

Table 1: Comparison of the solution $u(t, x)$ with $u_1(t, x, \alpha)$ and $u_2(t, x, \alpha)$ for $\alpha = 0$, $\bar{\varepsilon} = \bar{1}$, $\bar{c} = \bar{4}$ and $\bar{k} = \bar{2}$.

	$u(t, x)$			$u_1(t, x, \alpha)$			$u_2(t, x, \alpha)$		
$t \setminus x$	0.0099	0.0198	0.0297	0.0099	0.0198	0.0297	0.0099	0.0198	0.0297
0	2.1196	2.2404	2.3623	1.5198	1.5396	1.5594	2.8709	3.2391	3.5948
0.0198	2.0998	2.207	2.3427	1.5087	1.5285	1.5483	2.8404	3.2100	3.5679
0.0396	2.080	2.201	2.3232	1.4975	1.5173	1.5371	2.8098	3.1512	3.5407

Figure 1: The surfaces shows the fuzzy solution for $\alpha = 0$ for (6.1), when $\bar{c} = \bar{4}$, $\bar{k} = \bar{2}$ and $\bar{\varepsilon} = \bar{1}$.Figure 2: The surfaces shows the fuzzy solution for $\alpha \in [0, 1]$ for (6.1), when $\bar{c} = \bar{4}$, $\bar{k} = \bar{2}$ and $\bar{\varepsilon} = \bar{1}$.

Example 6.2. We consider a model partial differential equation described by

$$U_t(t, x) - \varepsilon U(t, x) U_x(t, x) = 2k \quad (6.3)$$

with the initial conditions

$$U(0, x) = \frac{x}{2\varepsilon},$$

where $x \in [0, 1]$, $t > 0$, $k \in]0, 2]$, $\varepsilon \in [-2, 0[$ are constants.

Similarly we can establish an iteration formula in the form

$$U_{n+1}(t, x) = U_n(t, x) - \int_0^t \left\{ (U_n(s, x))_s - \varepsilon U_n(s, x) (U_n(s, x))_x - 2k \right\} ds. \quad (6.4)$$

We begin with an initial arbitrary approximation:

$$U_0(t, x) = U(0, x) = \frac{x}{2\varepsilon}$$

and using the iteration formula (6.4), we obtain the following successive approximations

$$\begin{aligned} U_1(t, x) &= \frac{x}{2\varepsilon} + \frac{x}{4\varepsilon}t + 2kt, \\ U_2(t, x) &= \frac{x}{2\varepsilon} + \frac{x}{4\varepsilon}t + 2kt + \frac{x}{8\varepsilon}t^2 + \frac{k}{2}t^2 + \frac{k}{4}t^3 + \frac{x}{48\varepsilon}t^3 + \frac{k}{2}t + \frac{k}{6}t^3, \\ U_3(t, x) &= \frac{x}{2\varepsilon} + \frac{x}{4\varepsilon}t + 2kt + \frac{x}{8\varepsilon}t^2 + \frac{k}{2}t^2 + \frac{k}{4}t^3 + \frac{11k}{96}t^4 \\ &\quad + \frac{x}{16\varepsilon}t^3 + \frac{x}{48\varepsilon}t^4 + \frac{x}{192\varepsilon}t^5 + \frac{x}{1152\varepsilon}t^6 + \frac{x}{16128\varepsilon}t^7 + \frac{7k}{240}t^5 + \frac{k}{192}t^6 + \frac{k}{2016}t^7. \end{aligned}$$

This procedure can, in principle, be continued as far as desired, and the approximation will converge to its exact solution

$$U(t, x) = (kt^2 - 4kt - \frac{x}{\varepsilon}) / (t - 2).$$

Since $P(t, x, \varepsilon, G) > 0$, for $t \in]0, 2[\cup]4, \infty[$, $\frac{\partial P}{\partial G} > 0$, $\frac{\partial G}{\partial k} = (t^2 - 4t) / (t - 2) > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial P}{\partial \varepsilon} < 0$, $\frac{\partial G}{\partial \varepsilon} = x / \varepsilon^2 (t - 2) < 0$ for $t \in]0, 2[$. That is, $(\bar{Z})_t - \varepsilon \bar{Z}(\bar{Z})_x = 2K$, a fuzzy number.

So Lemma 5.2 implies the result that $\bar{Z}(t, x)$ is a BFS see Figs. 3, 5, and Table 2. We easily see that

$$z_i(0, x, \alpha) = \frac{x}{2\varepsilon_i(\alpha)}$$

for $i = 1, 2$, so $\bar{Z}(t, x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\bar{Z}(t, x) = (\bar{K}t^2 - 4\bar{K}t - \frac{x}{\varepsilon}) / (t - 2)$$

for all $t \in]0, 2[$, $x \in [0, 1]$.

$\frac{\partial P}{\partial \varepsilon} < 0$, $\frac{\partial G}{\partial \varepsilon} > 0$ for $t \in]4, \infty[$ or $P(t, x, \varepsilon, G) < 0$, for $t \in]2, 4[$, then there is no BFS Lemma 5.2, see Fig. 4 and Table 3. We proceed to look for a SS. We must solve for $t \in]4, \infty[$,

$$\begin{aligned} (u_1(t, x, \alpha))_t - \varepsilon_2(\alpha) u_1(t, x, \alpha) (u_1(t, x, \alpha))_x &= 2k_1(\alpha), \\ (u_2(t, x, \alpha))_t - \varepsilon_1(\alpha) u_2(t, x, \alpha) (u_2(t, x, \alpha))_x &= 2k_2(\alpha), \end{aligned}$$

subject to $u_i(0, x, \alpha) = c_i(\alpha)x / 2\varepsilon_i(\alpha)$ for $i = 1, 2$ and

$$\bar{K}[\alpha] = [k_1(\alpha), k_2(\alpha)] \quad \text{and} \quad \bar{\varepsilon}[\alpha] = [\varepsilon_1(\alpha), \varepsilon_2(\alpha)].$$

By VIM, the solution is

$$u_1(t, x, \alpha) = \left(k_1(\alpha)t^2 - 4k_1(\alpha)t - \frac{x}{\varepsilon_2(\alpha)} \right) / (t - 2), \quad u_2(t, x, \alpha) = \left(k_2(\alpha)t^2 - 4k_2(\alpha)t - \frac{x}{\varepsilon_1(\alpha)} \right) / (t - 2).$$

Now we show $[u_1(t, x, \alpha), u_2(t, x, \alpha)]$ defines α -cut of a fuzzy number.

Thus we only need to check if $\frac{\partial u_1}{\partial \alpha} > 0$ and $\frac{\partial u_2}{\partial \alpha} < 0$. Since $u_i(t, x, \alpha)$ are continuous and $u_1(t, x, 1) = u_2(t, x, 1)$, there is a region \mathfrak{R} contained in $]4, \infty[\times]0, 1]$ for which the SS exists and in $]4, \infty[\times]0, 1] - \mathfrak{R}$ there may be no SS.

Since \bar{K} and $\bar{\varepsilon}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k'_1(\alpha) = \varepsilon'_1(\alpha) = \lambda$ and $k'_2(\alpha) = \varepsilon'_2(\alpha) = -\lambda$. Then, for the SS we need

$$\frac{\partial u_1}{\partial \alpha} = \lambda \left(t^2 - 4t - x/(\varepsilon_2)^2 \right) / (t-2) > 0, \quad \frac{\partial u_2}{\partial \alpha} = -\lambda \left(t^2 - 4t - x/(\varepsilon_1)^2 \right) / (t-2) < 0.$$

Therefore inequalities hold if

$$t^2 - 4t - x/(\varepsilon_1)^2 > 0 \quad (6.5)$$

for $t \in]4, \infty[, x \in [0, 1]$. The inequality (6.5) holds if

$$0 \leq x \leq 1, \quad t \geq 2 + \sqrt{4 + \frac{x}{(\varepsilon_1)^2}} \quad \text{for all } \alpha \in [0, 1].$$

So under the above assumptions we may choose

$$\mathfrak{R} = \left\{ (t, x) \mid 0 \leq x \leq 1, \quad t \geq 2 + \sqrt{4 + \frac{x}{(\varepsilon_1)^2}} \quad \text{for all } \alpha \in [0, 1] \right\},$$

and the SS exists on \mathfrak{R} in form Eqs (6.6) see Fig. 4.

$$\bar{U}(t, x) = \left(\bar{K}t^2 - 4\bar{K}t - \frac{x}{\bar{\varepsilon}} \right) / (t-2) \quad (6.6)$$

for all $t \in]4, \infty[, x \in [0, 1]$.

Table 2: Comparison of the solution $u(t, x)$ with $u_1(t, x, \alpha)$ and $u_2(t, x, \alpha)$ for $\alpha = 0$ $t \in [0, 2]$ $\bar{\varepsilon} = -1$.

	$u(t, x)$			$u_1(t, x, \alpha)$			$u_2(t, x, \alpha)$		
$t \backslash x$	0.0495	0.1980	0.3465	0.0495	0.1980	0.3465	0.0495	0.1980	0.3465
0	0.2005	0.8356	1.5314	0.1003	0.4178	0.7657	0.3008	1.2534	2.2971
0.0099	0.1980	0.8329	1.5284	0.0986	0.4160	0.7637	0.2957	1.2479	2.2911
0.0198	0.1955	0.8301	1.5254	0.0969	0.4141	0.7617	0.2906	1.2424	2.2851

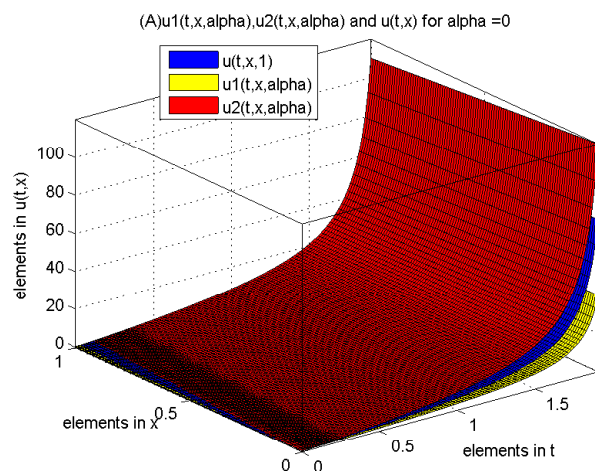
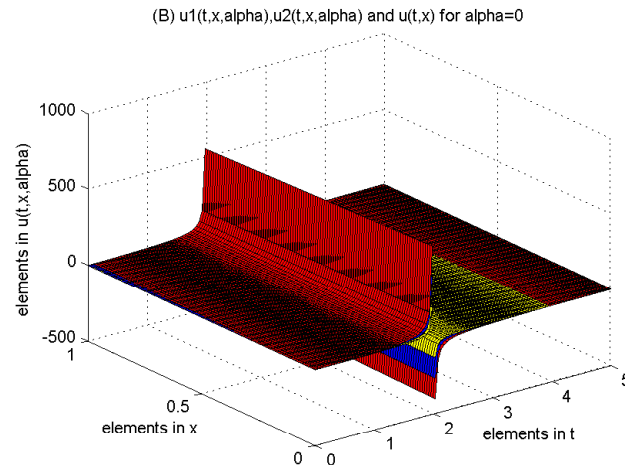
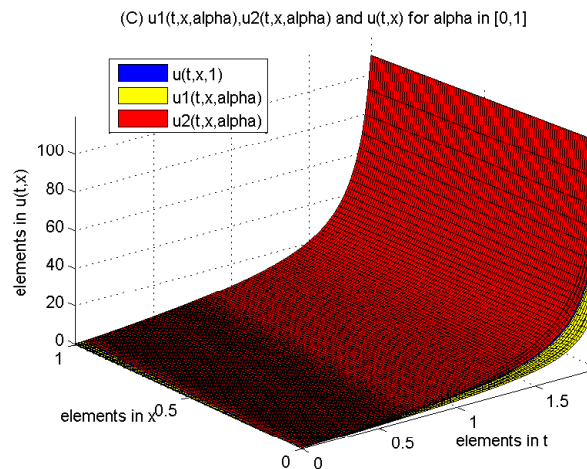


Figure 3: The surfaces shows the fuzzy solution for $\alpha = 0$ for (6.3), when $\bar{k} = \bar{2}$ and $\bar{\varepsilon} = -1$.

Table 3: Comparison of the solution $u(t, x)$ with $u_1(t, x, \alpha)$ and $u_2(t, x, \alpha)$ for $\alpha = 0$ $t \in]2, 5]$, $\bar{\varepsilon} = \overline{-1}$.

	$u(t, x)$			$u_1(t, x, \alpha)$			$u_2(t, x, \alpha)$		
$t \backslash x$	0.0495	0.1980	0.3465	0.0495	0.1980	0.3465	0.0495	0.1980	0.3465
2.0297	-264.04	-100.59	-61.70	-134.08	-50.21	-30.79	-402.24	-150.63	-92.46
2.0792	-265.94	-99.59	-61.12	-132.41	-49.58	-30.43	-397.24	-148.76	-91.30
2.1287	-263.44	-98.65	-60.55	-130.74	-48.96	-30.05	-392.24	-146.88	-90.15

Figure 4: The surfaces shows the fuzzy solution for $\alpha = 0$, $t \in [0, 5] \setminus \{2\}$ for (6.3), when $\bar{k} = \bar{2}$ and $\bar{\varepsilon} = \overline{-1}$.Figure 5: The surfaces shows the fuzzy solution for $\alpha \in [0, 1]$ for (6.3), when $\bar{k} = \bar{2}$ and $\bar{\varepsilon} = \overline{-1}$.

7. Conclusion

The proposed strategy in this paper may be useful the coefficients, initial values and forcing terms are fuzzy, for solving burgers equation. This strategy consists of two types of solutions: Buckley-Feuring solution and Seikkala solution, using the VIM. And has been successfully applied to new theorem. we proved in the first if BFsolution exists because if it does it is the same as the S-solution as shown in Fig. 1 (for $\alpha = 0$), Fig. 2 ($\forall \alpha \in [0, 1]$). However, if the BF-solution fails to exist as shown in Fig. 3 (for $\alpha = 0$, $t \in]2, 5]$, $x \in [0, 1]$) we check if the S-solution exists as shown in Fig. 4 (for $\alpha = 0$) and Fig. 5 ($\forall \alpha \in [0, 1]$, and $t \in [0, 2[, x \in [0, 1]$). When the S-solution fails to exist we offer no solution to the fuzzy

burger's equation.

For further research we propose the study for fuzzy Burgers equations, by the using the conformable differentiability concept [11, 12]. In addition, we propose to extend the results of the present paper and to combine them with the results in [10] for Hybrid fuzzy differential equations.

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