Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 21 (2020), 164–175
Research Article

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On averaging methods for general parabolic partial differential equation

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Abstract

The averaging method of the quantitative and the qualitative analysis of the parabolic partial differential equations appears as an exciting field of the investigation. The aim of this paper is to generalize some known results due to Krol on the averaging methods and use them to solve the fractional parabolic partial differential equations and a special case of these equations is studied. We treat some different cases related to the averaging method.

Keywords: Averaging method, fractional parabolic partial differential equation, Existence and uniqueness of solutions.

2020 MSC: 34A07, 34A60, 35A05, 03E72, 34C29, 34K05.

1. Introduction

The averaging method is an important computational technique. The investigation in the field of the averaging method of the qualitative and the quantitative analysis of the parabolic partial differential equations is more exciting field to be studied. We study the fractional parabolic partial differential equation in this paper using the technique of the averaging method of the linear operator. In Section 2, we discuss the averaging of the linear operator where we generalize some known results due to Krol [13]. We will consider the following fractional parabolic partial differential equation in the form

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big[\frac{\partial u(x,t)}{\partial t}-\sum_{|q|=2m} a_q(x)D^q u(x,t)\Big]=\varepsilon\sum_{|q|<2m} b_q(x,t)D^q u(x,t), \tag{1.1}$$

$$u(x,0)=\varphi(x),\ \frac{\partial u(x,0)}{\partial t}=\psi(x), \tag{1.2}$$

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doi: 10.22436/jmcs.021.02.08

Received: 2019-06-27 *Revised:* 2020-02-29 *Accepted:* 2020-03-03

where $0 < \alpha \leqslant 1$, $\varepsilon > 0$, $\mathfrak{R}^n$is the $n-$dimensional Euclidean space, $D^q = D_1^{q_1}...D_n^{q_n}$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1, ..., n$, $q = (q_1, ..., q_n)$ is an n-dimensional multi index, $|q| = q_1 + ... + q_n$. It is supposed that the linear partial differential operator $\sum_{|q|=2m} a_q(x)D^q$ is uniformly elliptic on $\mathfrak{R}^n$. In other words, it is supposed that all the coefficients $a_q(x)$, $|q| = 2m$, are bounded continuous on $\mathfrak{R}^n$ and that there exists a positive number λ such that for all $x \in \mathfrak{R}^n$ and all $\xi \neq (0, ..., 0)$, $(\xi^q = \xi_1^{q_1}, ..., \xi_n^{q_n}$, $|\xi|^2 = \xi_1^2 + ... + \xi_n^2)$,

$$(-1)^{m+1} \sum_{|q|=2m} a_q(x)\xi^q \geqslant \lambda|\xi|^{2m}.$$

We assume also that all the coefficients $a_q(x)$, $|q| = 2m$, $\varphi(x)$ are bounded continuous with bounded derivatives on $\mathfrak{R}^n$, all the coefficients $b_q(x, t)$, $|q| < 2m$ are bounded continuous with bounded derivatives on $(\mathfrak{R}^n \times [0, T])$, $a_q(x)$ satisfies a Hölder condition on $\mathfrak{R}^n$ and $b_q(x, t)$ satisfies a Hölder condition on $(\mathfrak{R}^n \times [0, T])$.

Suppose that $L_2(\mathfrak{R}^n)$ is the set of all square integrable functions on $\mathfrak{R}^n$. Notice that

$$u(x, t) = \int_{\mathfrak{R}^n} G(x, y, t)\varphi(y)dy, \tag{1.3}$$

will represent the solution of the Cauchy problem

$$\frac{\partial u(x, t)}{\partial t} = \sum_{|q|=2m} a_q(x)D^q u(x, t), \tag{1.4}$$

$$u(x, 0) = \varphi(x), \tag{1.5}$$

where G is the fundamental solution of the Cauchy problem (1.4), (1.5). The fundamental solution G satisfies the following properties [2, 3]

$$|D^q G(x, y, t)| \leqslant Kt^{c_1} exp(-c_2\rho), \tag{1.6}$$

where K, c_2 are positive constants, $t > 0$ and

$$\rho = \sum_{i=1}^{n} |x_i - y_i|^{\frac{2m}{(2m-1)}} t^{\frac{-1}{(2m-1)}}, \quad c_1 = -\frac{n + |q|}{2m}.$$

By using (1.3) and (1.6), we have

$$\| D^q u \| \leqslant \frac{N}{t^\gamma} \| \varphi \|,$$

where $0 < \gamma < 1$, N is a positive constant, $|q| < 2m$ and $\| \,.\, \|$ is the norm in $L_2(\mathfrak{R}^n)$ [3, 4, 11]. Let

$$v(x, t) = \frac{\partial u(x, t)}{\partial t} - \sum_{|q|=2m} a_q(x)D^q u(x, t), \tag{1.7}$$

and

$$L(x, t, D) = \sum_{|q|<2m} b_q(x, t)D^q. \tag{1.8}$$

By using the equations (1.7), (1.8) repeatedly in the equation (1.1) we have

$$\frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = \varepsilon L(x, t, D)u(x, t), \tag{1.9}$$

$$v(x, 0) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x). \tag{1.10}$$

In Section 3, we discuss a special case for the problem (1.1) when $\alpha = 1$. In Section 4, we treat some different cases related to the averaging method. Compare with [1, 5–10, 12, 15].

2. Averaging a linear operator

The solution of (1.7) is given formally by

$$u(x,t) = \int_{\mathfrak{R}^n} G(x,y,t)\varphi(y)dy + \int_0^t \int_{\mathfrak{R}^n} G(x,y,t-\theta)\nu(y,\theta)dyd\theta, \tag{2.1}$$

$$\begin{aligned}\nu(x,t) =& \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x)\\ &+ \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\int_{\mathfrak{R}^n}(t-s)^{\alpha-1}L(x,s,D)G(x,y,s)\varphi(y)dyds\\ &+ \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\int_0^s\int_{\mathfrak{R}^n}(t-s)^{\alpha-1}L(x,s,D)G(x,y,s-\theta)\nu(y,\theta)dyd\theta ds,\end{aligned}$$

where Γ is Gamma function. Let

$$\begin{aligned}S(x,t) &= \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\int_{\mathfrak{R}^n}(t-s)^{\alpha-1}L(x,s,D)G(x,y,s)\varphi(y)dyds,\\ V(x,t) &= \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\int_0^s\int_{\mathfrak{R}^n}(t-s)^{\alpha-1}L(x,s,D)G(x,y,s-\theta)\nu(y,\theta)dyd\theta ds.\end{aligned}$$

Then we have

$$\nu(x,t) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x) + S(x,t) + V(x,t).$$

By averaging the coefficients $b_q(x,t)$ over t, we can average the operator $L(x,t,D)$,

$$\bar{b_q}(x) = \frac{1}{T}\int_0^T b_q(x,t)dt$$

for all (x,t), $x \in \mathfrak{R}^n$ producing the averaged operator $\bar{L}(x,D)$, all the coefficients $\bar{b_q}(x)$, $|q| < 2m$ are bounded continuous with bounded derivatives on $\mathfrak{R}^n$.

Like as an approximating problems for (1.1), (1.2) and (1.9), (1.10), we take

$$\frac{\partial^\alpha}{\partial t^\alpha}[\frac{\partial u^*(x,t)}{\partial t} - \sum_{|q|=2m} a_q(x)D^q u^*(x,t)] = \varepsilon\bar{L}(x,D)u^*(x,t), \tag{2.2}$$

$$u^*(x,0) = \varphi(x), \quad \frac{\partial u^*(x,0)}{\partial t} = \psi(x), \tag{2.3}$$

$$\frac{\partial^\alpha \nu^*(x,t)}{\partial t^\alpha} = \varepsilon\bar{L}(x,D)u^*(x,t),$$

where

$$\nu^*(x,0) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x),$$

we have

$$u^*(x,t) = \int_{\mathfrak{R}^n} G(x,y,t)\varphi(y)dy + \int_0^t\int_{\mathfrak{R}^n} G(x,y,t-\theta)\nu^*(y,\theta)dyd\theta, \tag{2.4}$$

$$\nu^*(x,t) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x) + S^*(x,t) + V^*(x,t), \tag{2.5}$$

where

$$S^*(x,t) = \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \int_{\mathfrak{R}^n} (t-s)^{\alpha-1} \bar{L}(x,D) G(x,y,s) \varphi(y) dy ds,$$
$$V^*(x,t) = \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \int_0^s \int_{\mathfrak{R}^n} (t-s)^{\alpha-1} \bar{L}(x,D) G(x,y,s-\theta) v^*(y,\theta) dy d\theta ds,$$

another straightforward analysis displays the existence and uniqueness of the solutions of problems (1.1), (1.2) and (2.2), 2.3 on the time-scale $\frac{1}{\varepsilon}$.

We consider the domain $Q = \mathfrak{R}^n \times [0,T]$. The norm $\| \,.\, \|_\infty$ is defined by the supremum norm on Q and denoted by $\| u(x,t) \|_\infty = \sup\limits_Q |u(x,t)|$.

Theorem 2.1. *Let $u(x,t)$ be the solution of the initial value problem (1.1), (1.2) and $u^*(x,t)$ be the solution of the initial value problem (2.2), (2.3), then we have the estimate $\| u(x,t) - u^*(x,t) \|_\infty = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.*

Proof. Consider the near-identity transformation:

$$\hat{v}(x,t) = v^*(x,t) + \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds v^*(x,t). \tag{2.6}$$

It can be proved that the derivatives of v^* are bounded [6, 10]. So we get

$$\| \hat{v} - v^* \|_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$

By differentiating the near-identity transformation (2.6) and using the equations (2.5), (2.6), we have

$$\begin{aligned}
\frac{\partial \hat{v}(x,t)}{\partial t} =& \frac{\partial v^*(x,t)}{\partial t} + \varepsilon (L(x,t,D) - \bar{L}(x,D)) v^*(x,t) \\
&+ \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial v^*(x,t)}{\partial t} \\
=& \varepsilon L(x,t,D) \hat{v}(x,t) + \frac{\partial S^*(x,t)}{\partial t} + \frac{\partial V^*(x,t)}{\partial t} \\
&- \varepsilon \bar{L}(x,D) v^*(x,t) + \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial S^*(x,t)}{\partial t} \\
&+ \varepsilon [\int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial V^*(x,t)}{\partial t} - \varepsilon L(x,t,D) \int_0^t (L(x,s,D) \\
&- \bar{L}(x,D)) ds v^*(x,t)] \\
=& \varepsilon L(x,t,D) \hat{v}(x,t) + \frac{\partial S^*(x,t)}{\partial t} + \frac{\partial V^*(x,t)}{\partial t} \\
&+ \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial S^*(x,t)}{\partial t} \\
&- \varepsilon \bar{L}(x,D) (\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S^*(x,t) + V^*(x,t)) \\
&+ \varepsilon [\int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial V^*(x,t)}{\partial t} \\
&- \varepsilon L(x,t,D) \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds (\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S^*(x,t) + V^*(x,t))],
\end{aligned}$$

with initial value $\hat{v}(x,0) = \psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)$. Let

$$\frac{\partial}{\partial t} - \varepsilon L(x,t,D) = \mathcal{L},$$

we obtain

$$\mathcal{L}(\hat{v}(x,t) - v^*(x,t)) = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$

Moreover $\hat{v}(x,0) - v^*(x,0) = 0$. To end the proof we use the barrier functions see [14]. We introduce the barrier function:

$$\begin{aligned}
B(x,t) =& \varepsilon \parallel M(x,t) \parallel_\infty t + \parallel J(x,t) \parallel_\infty t \\
&+ \varepsilon \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t \\
&+ \frac{1}{2}\varepsilon^2 \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t^2 + \frac{1}{2}\varepsilon \parallel L(x,t,D) J(x,t) \parallel_\infty t^2,
\end{aligned}$$

where

$$\begin{aligned}
M(x,t) =& \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial V^*(x,t)}{\partial t} \\
&- \varepsilon L(x,t,D) \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds [\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S^*(x,t) + V^*(x,t)] \\
&+ L(x,t,D)[S(x,t) + V(x,t)] - \bar{L}(x,D)[S^*(x,t) + V^*(x,t)] \\
&+ \int_0^t (L(x,s,D) - \bar{L}(x,D)) ds \frac{\partial S^*(x,t)}{\partial t},
\end{aligned}$$

and

$$J(x,t) = \frac{\partial S^*(x,t)}{\partial t} - \frac{\partial S(x,t)}{\partial t} + \frac{\partial V^*(x,t)}{\partial t} - \frac{\partial V(x,t)}{\partial t},$$

and the functions (we omit the arguments)

$$Z_1(x,t) = \hat{v}(x,t) - v(x,t) - B(x,t), \; Z_2(x,t) = \hat{v}(x,t) - v(x,t) + B(x,t).$$

We get

$$\begin{aligned}
\mathcal{L} Z_1(x,t) =& (\frac{\partial}{\partial t} - \varepsilon L(x,t,D))[\hat{v}(x,t) - v(x,t) - B(x,t)] \\
=& \varepsilon M(x,t) - \varepsilon \parallel M(x,t) \parallel_\infty + J(x,t) - \parallel J(x,t) \parallel_\infty \\
&+ \varepsilon [L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \\
&- \varepsilon \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty \\
&+ \varepsilon^2 L(x,t,D) \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^2 \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t \\
&+ \varepsilon L(x,t,D) \parallel J(x,t) \parallel_\infty t - \varepsilon \parallel L(x,t,D) J(x,t) \parallel_\infty t \\
&+ \varepsilon^2 L(x,t,D) \parallel M(x,t) \parallel_\infty t \\
&+ \frac{1}{2}\varepsilon^2 L(x,t,D) \parallel L(x,t,D) J(x,t) \parallel_\infty t^2 \\
&+ \frac{1}{2}\varepsilon^3 L(x,t,D) \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x) \\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t^2 \\
\leqslant & 0,
\end{aligned}
$$

$Z_1(x,0) = 0$ similarly, $\mathcal{L}\ Z_2(x,t) \geqslant 0,\ Z_2(x,0) = 0$. Also, $Z_1(x,t)$ and $Z_2(x,t)$ are bounded, resulting in $Z_1(x,t) \leqslant 0$ and $Z_2(x,t) \geqslant 0$, we have

$$-B(x,t) \leqslant \hat{v}(x,t) - v(x,t) \leqslant B(x,t),$$

so we can estimate

$$\parallel \hat{v}(x,t) - v(x,t) \parallel_\infty \leqslant \parallel B(x,t) \parallel_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. Now we can use the triangle inequality to have

$$
\begin{aligned}
\parallel v(x,t) - v^*(x,t) \parallel_\infty \leqslant & \parallel \hat{v}(x,t) - v^*(x,t) \parallel_\infty + \parallel \hat{v}(x,t) - v(x,t) \parallel_\infty \\
= & O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon},
\end{aligned}
\tag{2.7}
$$

by using (2.1), (2.4) and (2.7) we obtain

$$
\begin{aligned}
\parallel u(x,t) - u^*(x,t) \parallel_\infty \leqslant & \int_0^t \int_{\mathfrak{R}^n} |G(x,y,t,\theta)| \parallel v(y,\theta) - v^*(y,\theta) \parallel_\infty dy d\theta \\
= & O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.
\end{aligned}
$$

□

3. A special case

We treat the special case for the problem (1.1), (1.2) when $\alpha = 1$

$$\frac{\partial}{\partial t}\Big[\frac{\partial u(x,t)}{\partial t} - \sum_{|q|=2m} a_q(x) D^q u(x,t)\Big] = \varepsilon L(x,t,D) u(x,t), \tag{3.1}$$

$$u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \tag{3.2}$$

by using the equation (1.7), we have

$$\frac{\partial v(x,t)}{\partial t} = \varepsilon L(x,t,D) u(x,t), \tag{3.3}$$

$$v(x,0) = \psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x), \tag{3.4}$$

by using the equation (2.1), we get

$$\begin{aligned} v(x,t) =&\psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x) \\ &+ \varepsilon \int_0^t \int_{\mathfrak{R}^n} L(x,s,D)G(x,y,s)\varphi(y)dyds \\ &+ \varepsilon \int_0^t \int_0^s \int_{\mathfrak{R}^n} L(x,s,D)G(x,y,s-\theta)v(y,\theta)dyd\theta ds, \end{aligned}$$

let

$$\begin{aligned} S_1(x,t) &= \varepsilon \int_0^t \int_{\mathfrak{R}^n} L(x,s,D)G(x,y,s)\varphi(y)dyds, \\ V_1(x,t) &= \varepsilon \int_0^t \int_0^s \int_{\mathfrak{R}^n} L(x,s,D)G(x,y,s-\theta)v(y,\theta)dyd\theta ds, \end{aligned}$$

we have

$$v(x,t) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x) + S_1(x,t) + V_1(x,t),$$

like as an approximating problems for (3.1), (3.2) and (3.3), (3.4), we take

$$\frac{\partial}{\partial t}[\frac{\partial u^*(x,t)}{\partial t} - \sum_{|q|=2m} a_q(x)D^q u^*(x,t)] = \varepsilon\bar{L}(x,D)u^*(x,t), \tag{3.5}$$

$$u^*(x,0) = \varphi(x), \quad \frac{\partial u^*(x,0)}{\partial t} = \psi(x), \tag{3.6}$$

$$\frac{\partial v^*(x,t)}{\partial t} = \varepsilon\bar{L}(x,D)u^*(x,t),$$

where

$$v^*(x,0) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x),$$

we have

$$v^*(x,t) = \psi(x) - \sum_{|q|=2m} a_q(x)D^q\varphi(x) + S_1^*(x,t) + V_1^*(x,t), \tag{3.7}$$

where

$$\begin{aligned} S_1^*(x,t) &= \varepsilon \int_0^t \int_{\mathfrak{R}^n} \bar{L}(x,D)G(x,y,s)\varphi(y)dyds, \\ V_1^*(x,t) &= \varepsilon \int_0^t \int_0^s \int_{\mathfrak{R}^n} \bar{L}(x,D)G(x,y,s-\theta)v^*(y,\theta)dyd\theta ds, \end{aligned}$$

another straightforward analysis displays the existence and uniqueness of the solutions of problems (3.1), (3.2) and (3.5), (3.6) on the time-scale $\frac{1}{\varepsilon}$.

Theorem 3.1. *Let* $u(x,t)$ *be the solution of the initial value problem* (3.1), (3.2) *and* $u^*(x,t)$ *be the solution of the initial value problem* (3.5), (3.6), *then we have the estimate* $\| u(x,t) - u^*(x,t) \|_\infty = O(\varepsilon)$ *on the time-scale* $\frac{1}{\varepsilon}$.

Proof. By using the near-identity transformation (2.6), we get

$$\| \hat{v}(x,t) - v^*(x,t) \|_\infty = O(\varepsilon) \quad \text{on the time-scale } \frac{1}{\varepsilon}.$$

Differentiation of the near-identity transformation (2.6) and using the equations (2.6), (3.7), we have

$$\begin{aligned}\frac{\partial \hat{v}(x,t)}{\partial t} =& \varepsilon L(x,t,D)\hat{v}(x,t) + \frac{\partial S_1^*(x,t)}{\partial t} + \frac{\partial V_1^*(x,t)}{\partial t}\\ &+ \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D))ds \frac{\partial S_1^*(x,t)}{\partial t}\\ &- \varepsilon \bar{L}(x,D)(\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S_1^*(x,t) + V_1^*(x,t))\\ &+ \varepsilon[\int_0^t (L(x,s,D) - \bar{L}(x,D))ds \frac{\partial V_1^*(x,t)}{\partial t}\\ &- \varepsilon L(x,t,D) \int_0^t (L(x,s,D) - \bar{L}(x,D))ds(\psi(x)\\ &- \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S_1^*(x,t) + V_1^*(x,t))],\end{aligned}$$

with initial value $\hat{v}(x,0) = \psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)$. We obtain

$$\mathcal{L}(\hat{v}(x,t) - v^*(x,t)) = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$

Moreover $\hat{v}(x,0) - v^*(x,0) = 0$. Consider the barrier function:

$$\begin{aligned}B_1(x,t) =& \varepsilon \parallel M_1(x,t) \parallel_\infty t + \parallel J_1(x,t) \parallel_\infty t\\ &+ \varepsilon \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t\\ &+ \frac{1}{2}\varepsilon^2 \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x)\\ &- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t^2 + \frac{1}{2}\varepsilon \parallel L(x,t,D) J_1(x,t) \parallel_\infty t^2,\end{aligned}$$

where

$$\begin{aligned}M_1(x,t) =& \int_0^t (L(x,s,D) - \bar{L}(x,D))ds \frac{\partial V_1^*(x,t)}{\partial t}\\ &- \varepsilon L(x,t,D) \int_0^t (L(x,s,D) - \bar{L}(x,D))ds[\psi(x)\\ &- \sum_{|q|=2m} a_q(x) D^q \varphi(x) + S_1^*(x,t) + V_1^*(x,t)]\\ &+ L(x,t,D)[S_1(x,t) + V_1(x,t)] - \bar{L}(x,D)[S_1^*(x,t) + V_1^*(x,t)]\\ &+ \int_0^t (L(x,s,D) - \bar{L}(x,D))ds \frac{\partial S_1^*(x,t)}{\partial t},\end{aligned}$$

and

$$J_1(x,t) = \frac{\partial S_1^*(x,t)}{\partial t} - \frac{\partial S_1(x,t)}{\partial t} + \frac{\partial V_1^*(x,t)}{\partial t} - \frac{\partial V_1(x,t)}{\partial t},$$

and the functions (we omit the arguments)

$$Z_3(x,t) = \hat{v}(x,t) - v(x,t) - B_1(x,t),\ Z_4(x,t) = \hat{v}(x,t) - v(x,t) + B_1(x,t).$$

We get

$$\begin{aligned}
\mathcal{L}Z_3(x,t) =&(\frac{\partial}{\partial t} - \varepsilon L(x,t,D))[\hat{v}(x,t) - v(x,t) - B_1(x,t)]\\
=&\varepsilon M_1(x,t) - \varepsilon \parallel M_1(x,t) \parallel_\infty + J_1(x,t) - \parallel J_1(x,t) \parallel_\infty\\
&+ \varepsilon[L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)]\\
&- \varepsilon \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x) - \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty\\
&+ \varepsilon^2 L(x,t,D) \parallel [L(x,t,D) - \bar{L}(x,D)][\psi(x)\\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t\\
&- \varepsilon^2 \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x)\\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t\\
&+ \varepsilon L(x,t,D) \parallel J_1(x,t) \parallel_\infty t - \varepsilon \parallel L(x,t,D) J_1(x,t) \parallel_\infty t\\
&+ \varepsilon^2 L(x,t,D) \parallel M_1(x,t) \parallel_\infty t\\
&+ \frac{1}{2}\varepsilon^2 L(x,t,D) \parallel L(x,t,D) J_1(x,t) \parallel_\infty t^2\\
&+ \frac{1}{2}\varepsilon^3 L(x,t,D) \parallel L(x,t,D)[L(x,t,D) - \bar{L}(x,D)][\psi(x)\\
&- \sum_{|q|=2m} a_q(x) D^q \varphi(x)] \parallel_\infty t^2\\
\leqslant& 0,
\end{aligned}$$

$Z_3(x,0) = 0$ similarly, $\mathcal{L}\ Z_4(x,t) \geqslant 0$, $Z_4(x,0) = 0$. And $Z_3(x,t)$ and $Z_4(x,t)$ are bounded, resulting in $Z_3(x,t) \leqslant 0$ and $Z_4(x,t) \geqslant 0$, so we can estimate

$$\parallel \hat{v}(x,t) - v(x,t) \parallel_\infty \leqslant \parallel B_1(x,t) \parallel_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We can use the triangle inequality to have

$$\parallel v(x,t) - v^*(x,t) \parallel_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}, \tag{3.8}$$

by using (2.1), (2.4) and (3.8) repeatedly, we obtain

$$\parallel u(x,t) - u^*(x,t) \parallel_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$ □

4. Averaging of some parabolic equations

Consider the partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon L(x,t,D) u(x,t), \tag{4.1}$$

$$u(x,0) = \gamma(x). \tag{4.2}$$

Let

$$L(x,t,D) = L_1(x,t,D) + L_2(x,t,D),$$

where

$$L_1(x,t,D) = \sum_{i=1}^{n} a_i(x,t)\frac{\partial}{\partial x_i},$$
$$L_2(x,t,D) = \sum_{i,j=1}^{n} b_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j},$$

in which $L_2(x,t,D)$ is a uniformly elliptic operator on the domain Q, the coefficients $a_i(x,t)$, $b_{ij}(x,t)$ and γ are continuous and bounded with bounded derivatives and $\frac{\partial}{\partial x_i}u(x,t)$, $\frac{\partial^2}{\partial x_i \partial x_j}u(x,t)$ are bounded on $x \in \mathfrak{R}^n$, $0 \leqslant t \leqslant T$.

By averaging the coefficients $a_i(x,t)$, $b_{ij}(x,t)$ over t, we can average the operator $L(x,t,D)$,

$$\bar{a}_i(x) = \frac{1}{T}\int_0^T a_i(x,t)dt, \quad \bar{b_{ij}}(x) = \frac{1}{T}\int_0^T b_{ij}(x,t)dt,$$

for all (x,t), $x \in \mathfrak{R}^n$, $i = (1,...,n)$ and $j = (1,...,n)$ producing the averaged operator $\bar{L}(x,D)$, the coefficients $\bar{a}_i(x)$, $\bar{b_{ij}}(x)$ are continuous and bounded with bounded derivatives.

Like as an approximating problem for (4.1), (4.2), we take

$$\frac{\partial u^*(x,t)}{\partial t} = \varepsilon \bar{L}(x,D)u^*(x,t) \tag{4.3}$$
$$u^*(x,0) = \gamma(x), \tag{4.4}$$

another straightforward analysis displays the existence and uniqueness of the solutions of the problems (4.1), (4.2) and (4.3), (4.4) on the time-scale $\frac{1}{\varepsilon}$.

Theorem 4.1. *Let $u(x,t)$ be the solution of the initial value problem (4.1), (4.2) and $u^*(x,t)$ the solution of the initial value problem (4.3), (4.4), then we have the estimate $\| u(x,t) - u^*(x,t) \|_\infty = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.*

Proof. Consider the near-identity transformation:

$$\hat{u}(x,t) = u^*(x,t) + \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D))ds u^*(x,t). \tag{4.5}$$

Suppose that the derivatives of $u^*(x,t)$ are bounded, we get

$$\| \hat{u}(x,t) - u^*(x,t) \|_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$

Differentiation of the near-identity transformation (4.5) and using equations (4.3), (4.5), we get

$$\begin{aligned}
\frac{\partial \hat{u}(x,t)}{\partial t} =& \frac{\partial u^*(x,t)}{\partial t} + \varepsilon(L(x,t,D) - \bar{L}(x,D))u^*(x,t) \\
&+ \varepsilon \int_0^t (L(x,s,D) - \bar{L}(x,D))ds \frac{\partial u^*(x,t)}{\partial t} \\
=& \varepsilon L(x,t,D)\hat{u}(x,t) \\
&+ \varepsilon^2 \int_0^t [(L(x,s,D) - \bar{L}(x,D))\bar{L}(x,D) \\
&- L(x,t,D)(L(x,s,D) - \bar{L}(x,D))]ds u^*(x,t) \\
=& \varepsilon L(x,t,D)\hat{u}(x,t) + \varepsilon^2 \mathcal{M}(x,t,D)u^*,
\end{aligned}$$

where

$$\mathcal{M}(x,t,D) = \int_0^t [(L(x,s,D) - \bar{L}(x,D))\bar{L}(x,D) - L(x,t,D)(L(x,s,D) - \bar{L}(x,D))]ds,$$

with initial value $\hat{u}(x,0) = \gamma(x)$, $\hat{u}(x,t)$ satisfies the problem (4.1), (4.2) to order ε^2. We obtain

$$\mathcal{L}(\hat{u}(x,t) - u^*(x,t)) = \varepsilon^2\mathcal{M}(x,t,D)u^* = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. Moreover $\hat{u}(x,0) - u^*(x,0) = 0$. We use barrier functions.

Let $c = \| \mathcal{M}\ (x,t,D)u^*(x,t) \|_\infty$, we introduce the barrier function:

$$B_2(x,t) = \varepsilon^2 ct,$$

and the functions (we omit the arguments)

$$Z_5(x,t) = \hat{u}(x,t) - u(x,t) - B_2(x,t),\ Z_6(x,t) = \hat{u}(x,t) - u(x,t) + B_2(x,t).$$

We get

$$\mathcal{L}Z_5(x,t) = \varepsilon^2\mathcal{M}(x,t,D)u^*(x,t) - \varepsilon^2 c \leqslant 0,\ Z_5(x,0) = 0,$$
$$\mathcal{L}Z_6(x,t) = \varepsilon^2\mathcal{M}(x,t,D)u^*(x,t) + \varepsilon^2 c \geqslant 0,\ Z_6(x,0) = 0.$$

$Z_5(x,t)$ and $Z_6(x,t)$ are bounded, resulting in $Z_5(x,t) \leqslant 0$ and $Z_6(x,t) \geqslant 0$, it follows that

$$-B_2(x,t) \leqslant \hat{u}(x,t) - u(x,t) \leqslant B_2(x,t),$$

$$-\varepsilon^2 ct \leqslant \hat{u}(x,t) - u(x,t) \leqslant \varepsilon^2 ct,$$

so we can estimate

$$\| \hat{u}(x,t) - u(x,t) \|_\infty \leqslant \| B_2(x,t) \|_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We use the triangle inequality to have

$$\| u(x,t) - u^*(x,t) \|_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}. \qquad \square$$

5. Conclusion

This paper is focused on generalizing some known results due to Krol on the averaging methods to solve the fractional parabolic partial differential equation. As a special case Cauchy problems are solved for the fractional parabolic partial differential equation and treat some different cases due to Krol on the averaging methods.

Acknowledgment

The authors would like to thank the chaif editor and the anonymous referees very much for his-her valuable comments and suggestions.

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