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Bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences

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Abstract

In this study, we define and study the generalized bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences. The Binet formulae, generating functions, some sum formulae, and interesting properties of these sequences are given.

Keywords: Jacobsthal sequence, Jacobsthal Lucas sequence, bivariate polynomial sequences, Binet formula.

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1. Introduction and Preliminaries

From the definition of Fibonacci numbers (the first known special integer sequence), there are many studies on the integer sequences because of so many applications in science and art, and etc. For instance, the ratio of two consecutive elements of Fibonacci sequence is the golden ratio, is very important number almost every area of science and art. And the other integer sequence Jacobsthal numbers are met in computer science. It is well known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers. There are a lot of identities of number sequences described in [6, 9]. From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2},\ \ j_0 = 0,\ \ j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2},\ c_0 = 2,\ c_1 = 1$ for $n \geqslant 2$, respectively in [4–6]. There are some generalizations of these integer sequences. For example, a generalization of Jacosthal sequences is given in [8] as $j_n(s,t) = sj_{n-1}(s,t) + 2tj_{n-2}(s,t),\ \ j_0(s,t) = 0,\ \ j_1(s,t) = 1$ and $c_n(s,t) = sc_{n-1}(s,t) + 2tc_{n-2}(s,t),\ c_0(s,t) = 2,\ c_1(s,t) = s$ for $n \geqslant 2$. The bivariate Fibonacci $\{F_n\,(x,y)\}$ and Lucas $\{L_n\,(x,y)\}$ polynomials sequences are defined as by using the following recurrence relation

$$F_n\,(x,y) = xF_{n-1}\,(x,y) + yF_{n-2}\,(x,y)\,, \quad (F_0\,(x,y) = 0,\ F_1\,(x,y) = 1),$$

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$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), \quad (L_0(x,y) = 2,\ L_1(x,y) = x),$$

where $x \neq 0$, $y \neq 0$, and $x^2 + 4y \neq 0$. Some identities about the bivariate Fibonacci and Lucas polynomials are obtained by Catalani in [1, 2]. And then the bivariate Pell and Pell Lucas polynomials are defined as by using the following recurrence relation

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y), \qquad (P_0(x,y) = 0,\ P_1(x,y) = 1),$$
$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y), \qquad (Q_0(x,y) = 2,\ Q_1(x,y) = 2xy),$$

where $x \neq 0$, $y \neq 0$, and $x^2y^2 + 8y \neq 0$. The Binet formula for these sequences are given in [3] as

$$P_n(x,y) = \frac{\left(xy + \sqrt{x^2y^2 + y}\right)^n - \left(xy - \sqrt{x^2y^2 + y}\right)^n}{2\sqrt{x^2y^2 + y}},$$
$$Q_n(x,y) = \left(xy + \sqrt{x^2y^2 + y}\right)^n + \left(xy - \sqrt{x^2y^2 + y}\right)^n.$$

In [7], the authors study bivariate Fibonacci and Lucas polynomials sequences and give some properties of these sequences. In [3], by using different matrices new sum formulae for bivariate Pell and Pell Lucas polynomials are obtained.

In this paper, we study the bivariate Jacobsthal $\{j_n(x,y)\}$ and bivariate Jacobsthal Lucas $\{c_n(x,y)\}$ polynomials in detail.

2. Main Results

Definition 2.1. The bivariate Jacobsthal $\{j_n(x,y)\}$ and bivariate Jacobsthal Lucas $\{c_n(x,y)\}$ polynomials are described by using using the following recurrence relations respectively as

$$j_n(x,y) = xyj_{n-1}(x,y) + 2yj_{n-2}(x,y), \quad (j_0(x,y) = 0,\ j_1(x,y) = 1), \tag{2.1}$$
$$c_n(x,y) = xyc_{n-1}(x,y) + 2yc_{n-2}(x,y), \quad (c_0(x,y) = 2,\ c_1(x,y) = xy). \tag{2.2}$$

where $x \neq 0$, $y \neq 0$and $x^2y^2 + 8y > 0$. The characteristic equation of recurrence relation (2.1) and (2.2)

$$r^2 - xyr - 2y = 0.$$

The roots of the characteristic equation are

$$\alpha = \frac{xy + \sqrt{x^2y^2 + 8y}}{2}, \qquad \beta = \frac{xy - \sqrt{x^2y^2 + 8y}}{2},$$

with the following properties

$$\alpha + \beta = xy, \qquad \alpha - \beta = \sqrt{x^2y^2 + 8y}, \qquad \alpha.\beta = -2y.$$

The first some generalized bivariate Jacobsthal polynomials are$j_0(x,y) = 0$, $j_1(x,y) = 1$, $j_2(x,y) = xy$, $j_3(x,y) = x^2y^2 + 2y$, $j_4(x,y) = x^3y^3 + 4xy^2$. The first some generalized bivariate Jacobsthal Lucas polynomials are $c_0(x,y) = 2$, $c_1(x,y) = xy$, $c_2(x,y) = x^2y^2 + 4y$, $c_3(x,y) = x^3y^3 + 6xy^2$.

The Binet formulas for these sequences are

$$j_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.3}$$
$$c_n(x,y) = \alpha^n + \beta^n. \tag{2.4}$$

The negative term for bivariate Jacobsthal and Jacobsthal Lucas polynomials are defined

$$j_{-n}(x,y) = -j_n(x,y)/(-2y)^n c_{-n}(x,y) = c_n(x,y)/(-2y)^n$$

with the roots $1/\alpha$ and $1/\beta$. For the brevity, in the rest of the paper we use the notation j_n for $j_n(x,y)$ and c_n for $c_n(x,y)$.

Theorem 2.2 (Explicit closed form)**.**

$$j_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} (xy)^{n-1-2k}(2y)^k$$

and

$$c_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (xy)^{n-2k}(2y)^k.$$

Proof. Induction on n provides the required proofs. □

Theorem 2.3 (Derivative)**.**

$$\frac{\partial c_n}{\partial x} = nyj_n,$$
$$\frac{\partial c_n}{\partial y} = nxj_n + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{kxn}{n-2k} \left(fracn-1-kk\right) (xy)^{n-2k-1}(2y)^k.$$

Proof. From the partial derivations of the explicit formula of bivariate Jacobsthal Lucas polynomials we have

$$\begin{aligned}
\frac{\partial c_n}{\partial x} =& \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (n-2k)\,(x)^{n-2k-1} 2^k y^{n-k} \\
=& \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n(n-k-1)!}{k!(n-2k-1)!} (x)^{n-2k-1} 2^k y^{n-k} \\
=& \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} n \binom{n-1-k}{k} (xy)^{n-1-2k}(2y)^k y \\
=& nyj_n,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial c_n}{\partial y} =& \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (n-2k)\, y^{n-2k-1} x^{n-2k} (2y)^k \\
&+ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{kn}{n-k} \binom{n-k}{k} (xy)^{n-2k} 2^k y^{k-1} \\
=& \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n(n-k-1)!}{yk!(n-2k-1)!} (xy)^{n-2k}(2y)^k \\
&+ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{kn}{y(n-k)} \binom{n-k}{k} (xy)^{n-2k}(2y)^k \\
=& \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} xn \binom{n-1-k}{k} (xy)^{n-2k-1}(2y)^k + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{kxn}{n-2k} \binom{n-1-k}{k} (xy)^{n-2k-1}(2y)^k \\
=& nxj_n + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{kxn}{n-2k} \binom{n-1-k}{k} (xy)^{n-2k-1}(2y)^k.
\end{aligned}$$

□

Theorem 2.4 (The Generating functions of bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences)**.** *Let* i *any positive integer and* $\left|\alpha^i t\right| < 1$ *and* $\left|\beta^i t\right| < 1$. *Then the generating functions of these sequences for different values of* i *are obtained as*

$$\sum_{n=0}^{\infty} j_{in} t^n = \frac{j_i t}{1 - c_i t + (-2y)^i t^2},$$
$$\sum_{n=0}^{\infty} c_{in} t^n = \frac{2 + t(\sqrt{x^2y^2+8y}) j_i}{1 - c_i t + (-2y)^i t^2}.$$

Proof. By using Binet formula for generalized bivariate Jacobsthal polynomial sequence, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} j_{in} t^n &= \sum_{n=0}^{\infty} \frac{\alpha^{in} - \beta^{in}}{\alpha - \beta} t^n = \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \left[\left(\alpha^i t\right)^n - \left(\beta^i t\right)^n \right] \\
&= \frac{1}{\alpha-\beta} \left[\frac{1}{1-\alpha^i t} - \frac{1}{1-\beta^i t} \right] \\
&= \frac{\left(\alpha^i - \beta^i\right) t}{(\alpha-\beta)\left(1 - t\left(\alpha^i + \beta^i\right) + t^2 (-2y)^i\right)} \\
&= \frac{j_i t}{1 - c_i t + (-2y)^i t^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{in} t^n &= \sum_{n=0}^{\infty} \left(\alpha^{in} + \beta^{in}\right) t^n = \sum_{n=0}^{\infty} \left[\left(\alpha^i t\right)^n + \left(\beta^i t\right)^n \right] \\
&= \left[\frac{1}{1-\alpha^i t} + \frac{1}{1-\beta^i t} \right] \\
&= \frac{2 - \left(\alpha^i + \beta^i\right) t}{\left(1 - t\left(\alpha^i + \beta^i\right) + t^2 (-2y)^i\right)} \\
&= \frac{2 - c_i t}{1 - c_i t + (-2y)^i t^2}.
\end{aligned}$$

□

Corollary 2.5 (The Exponential Generating Functions of Jacobsthal and Jacobsthal Lucas Sequences)**.**

$$\begin{aligned}
\sum_{n=0}^{\infty} j_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha-\beta} \frac{t^n}{n!} = \frac{1}{\sqrt{x^2y^2+8y}} \sum_{n=0}^{\infty} \frac{(\alpha t)^n - (\beta t)^n}{n!} \\
&= \frac{1}{\sqrt{x^2y^2+8y}} \left(e^{\alpha t} - e^{\beta t}\right),
\end{aligned}$$

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = \left(e^{\alpha t} + e^{\beta t}\right).$$

Lemma 2.6 (Important Relationships)**.**

- $j_n c_n = j_{2n}$,
- $c_n = j_{n+1} + 2y j_{n-1}$,
- $(x^2y^2 + 8y) j_n = c_{n+1} + 2y c_{n-1}$,

- $xyj_n + c_n = 2j_{n+1}$,
- $(x^2y^2 + 8y)j_n + xyc_n = 2c_{n+1}$,
- $\sqrt{x^2y^2 + 8y}j_n + c_n = 2\alpha^n$,
- $\sqrt{x^2y^2 + 8y}j_n - c_n = -2\beta^n$,
- $c_{n+2}^2 + 2yc_{n+1}^2 = c_{2n+4} + 2yc_{2n+2}$,
- $j_{n+1}^2 + 2yj_n^2 = \sqrt{x^2y^2 + 8y}j_{2n+1}$,
- $c_{2n} = j_n^2\left(x^2y^2 + 8y\right) + 2(-2y)^n$,
- $c_n^2 = c_{2n} + 2(-2y)^n$,
- $\left(x^2y^2 + 8y\right) j_n^2 = c_{2n} - 2(-2y)^n$,
- $c_{3n} = c_n(c_{2n} - (-2y)^n)$,
- $j_{3n} = j_n(c_{2n} + (-2y)^n)$.

Proof. All of the proofs can be seen easily by using Binet formula or mathematical induction method. □

Theorem 2.7 (Summation Formulas). *Let a, b are positive integers. For bivariate Jacobsthal polynomial sequence, we get*

$$\sum_{k=0}^{n-1} j_{ak+b} = \frac{j_b - j_{na+b} - (-2y)^a j_{b-a} j_{(n+1)i} + (-2q(x,y))^a j_{(n-1)a+b}}{1 - c_a + (-2y)^a}.$$

and for generalized bivariate Jacobsthal Lucas polynomial sequence, we get

$$\sum_{k=0}^{n-1} c_{ak+b} = \frac{(-2y)^a \left[c_{a(n-1)+b} - c_{b-a}\right] - c_{an+b} + c_b}{1 - c_a + (-2y)^a}.$$

Theorem 2.8 (D'ocagne's property). *Let $n \geqslant m$ and $n, m \in \mathbb{Z}^+$. For generalized bivariate Jacobsthal polynomial sequence, we have*

$$j_{m+1}j_n - j_m j_{n+1} = (-2y)^m j_{n-m}.$$

Let $m \geqslant n$ and $n, m \in \mathbb{Z}^+$. For bivariate Jacobsthal Lucas polynomial sequence, we have

$$c_{m+1}c_n - c_m c_{n+1} = \sqrt{x^2y^2 + 8y}(-2y)^n c_{m-n}.$$

Proof. By (2.3) and(2.4), we have

$$\begin{aligned}
j_{m+1}j_n - j_m j_{n+1} &= \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta}\frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\alpha^m - \beta^m}{\alpha - \beta}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\
&= \frac{1}{(\alpha - \beta)^2}\left[-\alpha^{m+1}\beta^n - \beta^{m+1}\alpha^n + \alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m\right] \\
&= \frac{1}{(\alpha - \beta)^2}\left[\alpha^n\beta^m(\alpha - \beta) - \alpha^m\beta^n(\alpha - \beta)\right] \\
&= \frac{1}{\alpha - \beta}\left[(\alpha\beta)^m(\alpha^{n-m} - \beta^{n-m})\right] \\
&= \frac{1}{\alpha - \beta}\left[(-2y)^m\left(\alpha^{n-m} - \beta^{n-m}\right)\right].
\end{aligned}$$

It can be proved for bivariate Jacobsthal Lucas numbers as bivariate Jacobsthal numbers. □

Theorem 2.9 (Catalan's property)**.** *Assume that* $n, r \in \mathbb{Z}^+$*. For bivariate Jacobsthal polynomial sequence, we have*

$$j_{n+r}j_{n-r} - j_n^2 = -(-2y)^{n-r}j_r^2$$

and for bivariate Jacobsthal Lucas polynomial sequence,

$$c_{n+r}c_{n-r} - c_n^2 = (-2y)^{n-r}j_r^2(x^2y^2 + 8y).$$

Theorem 2.10 (Cassini's property or Simpson property)**.** *For* $n \in \mathbb{Z}^+$*, it is obtained that*

$$j_{n+1}j_{n-1} - j_n^2 = -(-2q(x,y))^{n-1},$$

and

$$c_{n+1}c_{n-1} - c_n^2 = (-2q(x,y))^{n-1}(x^2y^2 + 8y).$$

Theorem 2.11. *For bivariate Jacobsthal polynomial sequence, the following results are denoted*

$$\begin{aligned} j_{4n+p} - (2y)^{2n}.j_p &= j_{2n}c_{2n+p}, \\ j_{4n+p} + (2y)^{2n}.j_p &= c_{2n}j_{2n+p}, \\ j_{3n+p} - (-2y)^{n}.j_{n+p} &= j_n c_{2n+p}, \\ j_{3n+p} + (-2y)^{n}.j_{n+p} &= c_n j_{2n+p}, \end{aligned}$$

where $n \geqslant 1$, $p \geqslant 0$.

Proof. It can be proved by using Binet formulas as the following theorem. □

Theorem 2.12. *For bivariate Jacobsthal Lucas polynomial sequence, the following results are satisfied*

$$\begin{aligned} c_{4n+p} - (2y)^{2n}.c_p &= (x^2y^2 + 8y)j_{2n}j_{2n+p}, \\ c_{4n+p} + (2y)^{2n}.c_p &= c_{2n}c_{2n+p}, \\ c_{3n+p} - (-2y)^{n}.c_{n+p} &= (x^2y^2 + 8y)j_n j_{2n+p}, \\ c_{3n+p} + (-2y)^{n}.c_{n+p} &= c_n c_{2n+p}, \end{aligned}$$

where $n \geqslant 1$, $p \geqslant 0$.

Theorem 2.13. *Let* $n \geqslant 0$ *any integer. Then generating function with negative indices are obtained as*

$$\begin{aligned} \sum_{k=0}^{n} j_k t^{-k} &= \frac{-1}{t^n.(t^2 - xyt - 2y)}\left[-t^{n+1} + tj_{n+1} + 2tj_n\right], \\ \sum_{k=0}^{n} c_k t^{-k} &= \frac{-1}{t^n(t^2 - xyt - 2y)}\left[tc_{n+1} + 2yc_n\right] + \frac{2t^2 - xyt}{(x^2 - xyt - 2y)}. \end{aligned}$$

Proof. By using expansion of geometric series, it is calculated as

$$\begin{aligned} \sum_{k=0}^{n} j_k t^{-k} =& \frac{1}{\alpha - \beta}\left[\frac{1 - \left(\frac{\alpha}{t}\right)^{n+1}}{1 - \frac{\alpha}{t}} - \frac{1 - \left(\frac{\beta}{t}\right)^{n+1}}{1 - \frac{\beta}{t}}\right] \\ =& \frac{1}{(\alpha - \beta)t^n}\left[\frac{t^{n+1} - \alpha^{n+1}}{t - \alpha} - \frac{t^{n+1} - \beta^{n+1}}{t - \beta}\right] \\ =& \frac{-1}{(\alpha - \beta)t^n}\left[\frac{-t^{n+1}(\alpha - \beta) + t(\alpha^{n+1} - \beta^{n+1}) + 2t(\alpha^n - \beta^n)}{t^2 - xyt - 2y}\right] \end{aligned}$$

$$=\frac{-1}{t^n.(t^2-xyt-2y)}\left[-t^{n+1}+tj_{n+1}+2tj_n\right].$$

Similarly

$$\sum_{k=0}^{n} c_k t^{-k}=\left[\frac{1-\left(\frac{\alpha}{t}\right)^{n+1}}{1-\frac{\alpha}{t}}+\frac{1-\left(\frac{\beta}{t}\right)^{n+1}}{1-\frac{\beta}{t}}\right]=\frac{1}{t^n}\left[\frac{t^{n+1}-\alpha^{n+1}}{t-\alpha}+\frac{t^{n+1}-\beta^{n+1}}{t-\beta}\right]$$
$$=\frac{-1}{t^n}\left[\frac{-2t^{n+2}+t^{n+1}(\alpha+\beta)+t(\alpha^{n+1}+\beta^{n+1})+2y(\alpha^n+\beta^n)}{t^2-xyt-2y}\right]$$
$$=\frac{-1}{t^n(t^2-xyt-2y)}\left[tc_{n+1}+2yc_n\right]+\frac{2t^2-xyt}{t^2-xyt-2y}. \qquad \square$$

Corollary 2.14. *If we take $n\to\infty$ in the above theorem, we get*

$$\sum_{i=0}^{\infty} j_i t^{-i}=\frac{t}{t^2-xyt-2y}.$$

Corollary 2.15. *If we take $n\to\infty$ in the above theorem, we get*

$$\sum_{i=0}^{\infty} c_i t^{-i}=\frac{2t^2-xyt}{t^2-xyt-2y}.$$

Theorem 2.16. *For $\left|\alpha^k(-\beta^{r-k})t\right|<1$,*

$$\sum_{i=0}^{\infty} j_i^r t^i=\sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^{r-k}}{(\sqrt{x^2y^2+8y})^r}.\frac{1}{1-\alpha^k(-\beta)^{r-k}t}.$$

Proof. By using geometric series and Binet formula, we have

$$\sum_{i=0}^{\infty} j_i^r t^i=\sum_{i=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}\left(\frac{\alpha^i}{\alpha-\beta}\right)^k\left(\frac{-\beta^i}{\alpha-\beta}\right)^{r-k}t^i$$
$$=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{(\alpha-\beta)^k}.\frac{1}{(\alpha-\beta)^{r-k}}\sum_{i=0}^{\infty}\left[\alpha^k(-\beta)^{r-k}t\right]^i$$
$$=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{(\sqrt{x^2y^2+8y})^r}.\frac{1}{1-\alpha^k(-\beta)^{r-k}t}. \qquad \square$$

Theorem 2.17. *For $\left|\alpha^k\beta^{r-k}t\right|<1$, we get*

$$\sum_{i=0}^{\infty} c_i^r t^i=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{1-\alpha^k\beta^{r-k}t}.$$

Proof. By using geometric series and Binet formula, it is calculated as

$$\sum_{i=0}^{\infty} c_i^r t^i=\sum_{i=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(\alpha^i)^k(\beta^i)^{r-k}t^i=\sum_{k=0}^{r}\binom{r}{k}\sum_{i=0}^{\infty}\left[\alpha^k\beta^{r-k}t\right]^i$$
$$=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{1-\alpha^k\beta^{r-k}t}. \qquad \square$$

Theorem 2.18. *By this theorem new relations between the roots* α, β *and bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences ar demonstrated*

$$\begin{aligned}\alpha^n &= \alpha j_n + 2y j_{n-1},\\ \beta^n &= \beta j_n + 2y j_{n-1},\\ \sqrt{x^2y^2+8y}\,\alpha^n &= \alpha c_n + 2y c_{n-1},\\ \sqrt{x^2y^2+8y}\,\beta^n &= \beta c_n + 2y c_{n-1}.\end{aligned}$$

Proof. The proof is made by using Binet formula and the product of the roots:

$$\begin{aligned}\beta j_n + 2y j_{n-1} =& \beta\frac{\alpha^n-\beta^n}{\alpha-\beta} + 2y\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\\ =& \frac{1}{\alpha-\beta}\left[\beta\left(\alpha^n-\beta^n\right)+2y\left(\alpha^{n-1}-\beta^{n-1}\right)\right]\\ =& \frac{1}{\alpha-\beta}\left(-2y\alpha^{n-1}-\beta^{n+1}+2y\alpha^{n-1}-2y\beta^{n-1}\right)\\ =& \frac{1}{\alpha-\beta}\left[-\beta^{n-1}\left(\beta^2+2y\right)\right]=\beta^n.\end{aligned}$$

Similarly

$$\begin{aligned}\alpha c_n + 2y c_{n-1} &= \alpha\left(\alpha^n+\beta^n\right)+2y\left(\alpha^{n-1}+\beta^{n-1}\right)\\ &= \alpha^{n+1}-2y\beta^{n-1}+2y\alpha^{n-1}+2y\beta^{n-1}\\ &= \alpha^{n-1}\left(\alpha^2+2y\right)\\ &= \alpha^n\left(\alpha-\beta\right)=\sqrt{x^2y^2+8y}\,\alpha^n.\end{aligned}$$

Other proofs can be done by using the same way. □

Theorem 2.19. *The square of elements of bivariate Jacobsthal sequence is obtained by the following:*

$$\sum_{i=0}^{n-1} j_i^2 = \frac{1}{x^2y^2+8y}\left(\frac{4y^2c_{2n-2}-c_{2n}-c_2+2}{1+4y^2-c_2}+2\frac{(-2y)^n-1}{2y+1}\right).$$

Proof. By the definition of Binet formulas, we have

$$\begin{aligned}\sum_{i=0}^{n-1} j_i^2 =& \sum_{i=0}^{n-1}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)^2 = \frac{1}{x^2y^2+8y}\sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}-2(-2y)^i\right)\\ =& \frac{1}{x^2y^2+8y}\left(\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}+2\frac{(-2y)^n-1}{2y+1}\right)\\ =& \frac{1}{x^2y^2+8y}\left(\frac{4y^2c_{2n-2}-c_{2n}-c_2+2}{1+4y^2-c_2}+2\frac{(-2y)^n-1}{2y+1}\right).\end{aligned}$$

□

Theorem 2.20. *The square of elements of bivariate Jacobsthal Lucas sequence is obtained by the following:*

$$\sum_{i=0}^{n-1} c_i^2 = \frac{4y^2c_{2n-2}-c_{2n}-c_2+2}{1+4y^2-c_2} - 2\frac{(-2y)^n-1}{2y+1}.$$

Proof. The proof is made by the same method as the above

$$\begin{aligned}\sum_{i=0}^{n-1} c_i^2 =& \sum_{i=0}^{n-1}\left(\alpha^i+\beta^i\right)^2=\sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}+2(-2y)^i\right)\\ =&\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(-2y)^n-1}{2y+1}\\ =&\frac{4y^2c_{2n-2}-c_{2n}-c_2+2}{1+4y^2-c_2}-2\frac{(-2y)^n-1}{2y+1}.\end{aligned}$$

□

Theorem 2.21. *By this theorem we can see another sum property, equals to* $2n$. *th element of bivariate Jacobsthal, bivariate Jacobsthal Lucas sequence respectively*

$$\sum_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(xy)^i j_i = j_{2n},$$

$$\sum_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(xy)^i c_i = c_{2n}.$$

Proof.

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(xy)^i j_i &= \frac{1}{\alpha-\beta}\left(\begin{array}{c}\sum\limits_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(\alpha xy)^i\\ -\sum\limits_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(\beta xy)^i\end{array}\right)\\ &=\frac{1}{\alpha-\beta}\left[(2y+\alpha xy)^n-(2y+\beta xy)^n\right]\\ &= j_{2n},\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(xy)^i c_i &= \left(\begin{array}{c}\sum\limits_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(\alpha xy)^i\\ +\sum\limits_{i=0}^{n}\binom{n}{i}(2y)^{n-i}(\beta xy)^i\end{array}\right)\\ &=\left[(2y+\alpha xy)^n+(2y+\beta xy)^n\right]\\ &= c_{2n}.\end{aligned}$$

□

Lemma 2.22. *For* m, n *positive integers*

$$j_{m+n+1}=j_{m+1}j_{n+1}+2yj_mj_n.$$

Theorem 2.23 (Divisibility Properties of Bivariate Jacobsthal Polynomial Sequence)**.** *Let* $n \geqslant 2$ *positive integers,*

$$j_m/j_n \Longleftrightarrow m/n.$$

Proof. $\Leftarrow$ Assume that m/n, then there exists an integer k such that $n = km$. We want to show j_m/j_n. We use induction method. For $k = 1$, it's easily seen that j_m/j_m.Suppose that j_m/j_{km}. For $k = n+1$,from the above Lemma

$$j_{(k+1)m}=j_{km+m}=j_{km}j_{m+1}+2q(x,y)j_{km-1}j_m.$$

Since j_m/j_{km} then it's easily seen that $j_m/j_{(k+1)m}$.

$\Rightarrow$ Let j_m/j_n and $m \nmid n$. So there exist integers q, r with $0 < r < m$, such that $n = mq + r$. From the above theorem

$$j_n=j_{mq+r}=j_{mq+1}j_r+2q(x,y)j_{mq}j_{r-1}.$$

Since j_m/j_{mq} then $j_m/j_{mq+1}j_r$. Since $(j_n, j_{n+1}) = y$ then j_m/j_r.This is impossible because of lower degree of j_r than j_m in x. So we have $r = 0$ and m/n. □

3. Conclusion

In this paper we have obtained a lot of interesting properties, are satisfied by generalized bivariate Jacobsthal and Jacobsthal Lucas polynomials. We give some generating functions, explicit formulas, different sum properties, divisibility properties, partial derivatives etc.

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