Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 21 (2020), 186–191
Research Article

Online: ISSN 2008-949X
Journal of Mathematics and Computer Science
Journal Homepage: www.isr-publications.com/jmcs

Variational necessary conditions for optimal control problems

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Abstract

In this work, we consider infinite spatiotemporal systems. We use a more regular control function to reach the desired target prescribed on the system domain. After presenting the existence results, we characterize the solution to our problem through variations method.

Keywords: Infinite dimensional systems, bilinear systems, optimal control problems.

2020 MSC: 93C10, 65K05, 37M05.

1. Introduction

Let $T > 0$ and $\Theta = \Omega \times]0, T[$, $\Sigma = \partial\Omega \times]0, T[$, where $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ is an open bounded domain with regular boundary $\partial\Omega$. The control space is $u \in L^2(0, T, H_0^1(\Omega))$, and our considered bi-linear equation is

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) - u(x,t)y_x(x,t) & \Theta, \\ y(x,0) = y_0(x) & \Omega, \\ y = y_x = 0 & \Sigma, \end{cases} \tag{1.1}$$

with $y_0(x) \in L^2(\Omega)$ and $W = \{y \in L^2(0, T; H_0^1(\Omega))/y_t \in L^2(0, T; H^{-2}(\Omega))\}$ represents the state space (see [5]). The system (1.1) admit a unique solution y_u in $W \cap L^\infty(0, T; L^2(\Omega))$ (see [9, 10]).

The considered problem of the system (1.1) is

$$\min_{u \in L^2(0,T,H_0^1(\Omega))} J_\varepsilon(u). \tag{1.2}$$

Let $\varepsilon > 0$, the considered minimizing function J_ε is

$$J_\varepsilon(u) = \frac{1}{2}\left\|y - y^d\right\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\varepsilon}{2}\int_\Theta \left[u_t^2(x,t) + u_x^2(x,t)\right] dxdt, \tag{1.3}$$

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doi: 10.22436/jmcs.021.03.02

Received: 2020-02-14 Revised: 2020-03-03 Accepted: 2020-03-13

with the desired profile is y^d.

For background, quadratic costs such (1.3) used by several authors like Bradly, Lenhart in [3], Bradly et all in [4], and Lenhart in ([8]). They control the coefficient term in different types of one-dimensional bilinear equations minimizing a spatiotemporal velocity. Joshi in ([6]) studies the control of velocity in a a class of bi-linear systems. Boulaaras in [2], studies a parabolic quasi-variational inequalities. Zakeri and Abchouyeh in [11], consider quadratic optimal control problem constrained by elliptic systems. Ayalaa et all in [1], gives controllability results of two dimensional bilinear systems. Kafash and Delavarkhalafi in [7] use parameterization method for optimal control problems. Zerrik and Ould Sidi in [12–14]) use quadratic control problems to steer the position of infinite-dimensional bilinear systems to the desired state on specific sub-region. Zine and Ould Sidi in [15, 16] treated quadratic control problems in the case of hyperbolic bi-linear systems.

In this work, our goal is to command the state of (1.1) to a prescribed target $y^d(x)$ by minimizing the cost (1.3), and using more regular control space $u \in L^2(0,T;H_0^1(\Omega))$. After presenting the existence results, we characterize the solution to our problem through variations method.

2. Existence results

Next theorem ensures a solution to the penalty problem (1.2).

Theorem 2.1. *There exist a pair* (u, y)*, where* y *is the output of the system*

$$\begin{cases} y_t = y_{xx} - u y_x & \Theta, \\ y(x,0) = y_0(x) & \Omega, \\ y = y_x = 0 & \Sigma, \end{cases}$$

and u *is the minimum of (1.2).*

Proof. Let the positive nonempty set $\{J_\varepsilon(u) \ \mid u \in L^2(0,T,H_0^1(\Omega))\} \subset \mathbb{R}$, and choosing $(u_n)_n$ a minimum which verifies

$$J^* = \lim_{n\to+\infty} J(u_n) = \inf_{u\in L^2(0,T,H_0^1(\Omega))} J_\varepsilon(u).$$

It follows that, the cost $J_\varepsilon(u_n)$ is bounded, consequently $\|u_n\|_{L^2(0,T,H_0^1(\Omega))} \leqslant C$, with C is a positive constant.

From results in [9], we have

$$\begin{array}{ll} u_n \rightharpoonup u & L^2(0,T,H_0^1(\Omega)), \\ y^n \rightharpoonup y & W, \\ y^n_{xx} \rightharpoonup \chi & W, \\ y^n_x \rightharpoonup \Lambda & W, \\ y^n_t \rightharpoonup \Psi & W. \end{array}$$

We Consider the equation $y^n_t(x,t) = y^n_{xx} - u_n y^n_x$, and passing to the limit we can deduce that $y_t(x,t) = \Psi$, $y \mapsto y_{xx}$, $y_{xx} = \chi$ and $u y_x = \Lambda$. Hence we obtain

$$y_t = y_{xx} - u(x,t) y_x.$$

The lower semi continuity of $J_\varepsilon(u)$, allows us to deduce that

$$\begin{aligned} J_\varepsilon(u) &= \inf_n \frac{1}{2}\left\|y^n - y^d\right\|^2_{L^2(0,T,L^2(\Omega))} + \frac{\varepsilon}{2}\int_\Theta \left[u_t^2 + u_x^2\right]_n dxdt \\ &\leqslant \lim_{n\longrightarrow\infty} J_\varepsilon(u_n) = \inf_u J_\varepsilon(u). \end{aligned}$$

Consequently u is the solution of (1.2). □

3. Characterization of solution

We want to suggest a characterization for solution of the control problem (1.2). First we study the differential of $J_\varepsilon(u)$.

Lemma 3.1. *A differential of*

$$\begin{array}{ccc} L^2(0,T,H_0^1(\Omega)) & \longrightarrow & W, \\ u & \longrightarrow & y(u), \end{array}$$

is

$$\frac{y(u+\varepsilon l)-y(l)}{\varepsilon} \rightharpoonup \psi,$$

where ψ *verifies the system*

$$\begin{cases} \psi_t = \psi_{xx} - u\psi_x - l y_x & \Theta, \\ \psi(x,0) = 0 & \Omega, \\ \psi = \psi_x = 0 & \Sigma. \end{cases} \tag{3.1}$$

With $y = y(u)$, $u \in L^2(0,T;H_0^1(\Omega))$, *and* $d(y(u))l$ *is the differential of* $u \to y(u)$ *respecting* u.

Proof. The solution of equation (3.1), verifies

$$\|\psi\|_W \leqslant k_1\|y\|_{L^\infty(0,T;H_0^1(\Omega))}\|l\|_{L^2(0,T,H_0^1(\Omega))}.$$

Also

$$\|\psi'\|_W \leqslant k_2\|y\|_{L^\infty(0,T;H_0^1(\Omega))}\|l\|_{L^2(0,T,H_0^1(\Omega))}.$$

Thus,

$$\|\psi\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leqslant k_3\|l\|_{L^2(0,T,H_0^1(\Omega))}.$$

Then , we obtain $h \in L^2(0,T;L^2(\Omega)) \to \psi \in \mathcal{C}((0,T);H_0^1(\Omega))$ is bounded, (see ([5])). Put $y_l = y(u+l)$ and $\varphi = y_l - y$, then φ is the state of

$$\begin{cases} \varphi_t(x,t) = \varphi_{xx} - u(x,t)\varphi_x(x,t) - l(x,t)(y_l)_x & \Theta, \\ \varphi(x,0) = 0 & \Omega, \\ \varphi = \varphi_x = 0 & \Sigma. \end{cases}$$

Thus

$$\|\varphi\|_{L^\infty([0,T];H_0^1(\Omega))} \leqslant k_4 \parallel l \parallel_{L^2(0,T,H_0^1(\Omega))}.$$

Let $\phi = \varphi - \psi$ which verify the system

$$\begin{cases} \phi_t = \phi_{xx} + u(x,t)\phi_x(x,t) + l(x,t)\varphi_x & \Theta, \\ \phi(x,0) = 0 & \Omega, \\ \phi = \phi_x = 0 & \Sigma, \end{cases}$$

where $\phi \in \mathcal{C}(0,T;H_0^1(\Omega))$, consequently

$$\|\phi\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leqslant k\|l\|^2_{L^2(0,T,H_0^1(\Omega))},$$

and we have

$$\|y(u+l) - y(u) - d(y(u))l\|_{\mathcal{C}(0,T;H_0^1(\Omega))} = \|\phi\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leqslant k\|l\|^2_{L^2(0,T,H_0^1(\Omega))}.$$

□

The k_i, $\{i = 1, 2, 3, 4\}$, and k are positive constants. Next, we consider the adjoint systems

$$\begin{cases} -p_t = p_{xx} + (up)_x + (y - y^d) & \Theta, \\ u_x(x, 0) = u_x(x, T) = 0 & \Omega, \\ p(x, T) = 0 & \Omega, \\ p = p_x = 0 & \Sigma. \end{cases} \tag{3.2}$$

The following lemma gives an explicit formula of the differential of $J_\varepsilon(u)$.

Lemma 3.2. *Consider $u \in L^2(0, T, H_0^1(\Omega))$ be the solution of (1.2), we obtain*

$$\lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta} = \int_\Omega \int_0^T \psi(y - y^d)dtdx + \varepsilon \int_\Omega \int_0^T \Big[(u_t l_t) + (u_x l_x)\Big] dtdx.$$

Proof. The cost $J_\varepsilon(u)$ (1.3), can be expressed by

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \int_0^T (y - y^d)^2 dtdx + \frac{\varepsilon}{2}\int_\Omega \int_0^T \Big[u_t^2 + u_x^2\Big] dtdx. \tag{3.3}$$

Put $y_\beta = y(u_\varepsilon + \beta l)$ and $y = y(u_\varepsilon)$, using (3.3) we have

$$\begin{aligned} \lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta} &= \lim_{\beta \to 0} \frac{1}{2}\int_\Omega \int_0^T \frac{(y_\beta - y^d)^2 - (y - y^d)^2}{\beta} dtdx \\ &+ \lim_{\beta \to 0} \frac{\varepsilon}{2\beta}\int_\Omega \int_0^T \Big[(u_t + \beta l_t)^2 - u_t^2 + (u_x + \beta l_x)^2 - u_x^2\Big] dtdx, \end{aligned}$$

then

$$\begin{aligned} &\lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta} \\ &= \lim_{\beta \to 0} \frac{1}{2}\int_\Omega \int_0^T \frac{(y_\beta - y)}{\beta}(y_\beta + y - 2y^d) dtdx \\ &+ \lim_{\beta \to 0} \varepsilon \int_\Omega \int_0^T \Big[(u_t l_t) + (u_x l_x)\Big](x, t) dtdx. \\ &= \int_\Omega \int_0^T \psi(x, t)(y - y^d)dtdx + \int_\Omega \int_0^T \varepsilon\Big[(u_t l_t) + (u_x l_x)\Big](x, t) dtdx. \end{aligned}$$

□

The next theorem characterizes the solution of (1.2).

Theorem 3.3. *The optimal control $u \in L^2(0, T; H_0^1(\Omega))$ of (1.2) verifies the following variational formula*

$$u_{tt} + u_{xx} + \frac{1}{\varepsilon} p y_x = 0,$$

where $y = y(u)$ is the output of (1.1) and p is the one of (3.2).

Proof. Let $l \in L^2(0, T; H_0^1(\Omega))$ and $u + \beta l \in L^2(0, T; H_0^1(\Omega))$ for $\beta > 0$. The extremal of J_ε is realized at u, then

$$0 \leqslant \lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta}.$$

Lemma (3.2) gives

$$0 \leqslant \int_\Omega \int_0^T \psi(x, t)(y - y^d)dtdx + \int_\Omega \int_0^T \varepsilon\Big[(u_t l_t) + (u_x l_x)\Big](x, t) dtdx,$$

then substituting the weak form of (3.2), we have

$$0 \leqslant \int_\Omega \int_0^T \psi(-p_t - p_{xx} - (up)_x) dtdx + \int_\Omega \int_0^T \varepsilon\Big[(u_t l_t) + (u_x l_x)\Big] dtdx.$$

Using a simple Integral by parts, we obtain

$$0 \leqslant \int_\Omega \int_0^T p(\psi_t - \psi_{xx} + u\psi_x) dtdx + \int_\Omega \int_0^T \varepsilon\Big[(u_t l_t) + (u_x l_x)\Big] dtdx.$$

The system (3.1), we obtain

$$\begin{aligned} 0 &\leqslant \int_\Omega \int_0^T -l(x,t) y_x p(x,t) dtdx \\ &+ \int_\Omega \int_0^T \varepsilon\Big[(u_t l_t) + (u_x l_x)\Big] dtdx \\ &= \int_\Omega \int_0^T \Big[-l(x,t) y_x p(x,t) + \varepsilon(u_t l_t) + \varepsilon(u_x l_x)\Big] dtdx. \end{aligned}$$

Consequently, for an arbitrary control $l = l(t) \in L^2(0,T;H_0^1(\Omega))$ we conclude the equation

$$-l y_x p - \varepsilon u_{tt} l - \varepsilon u_{xx} l = 0,$$

which allow us to drive the following variational formula

$$u_{tt} + u_{xx} + \frac{1}{\varepsilon} p y_x = 0,$$

that the solution u of (1.2) must satisfy. □

Acknowledgment

Many thanks to the anonymous referees for valuable comments and suggestions which have been included in this version of the manuscript.

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